The Era of Gravitational Wave Astronomy

POST-NEWTONIAN MODELLING OF INSPIRALLING COMPACT BINARIES

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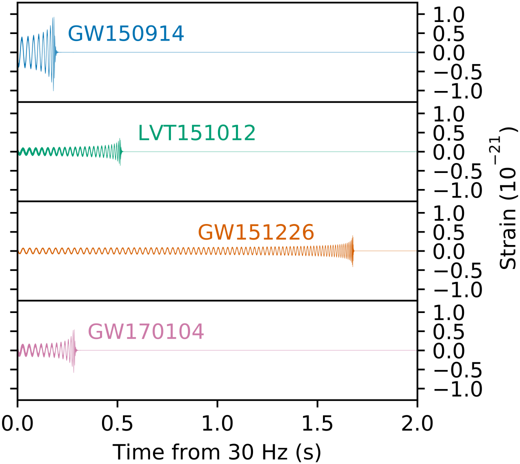
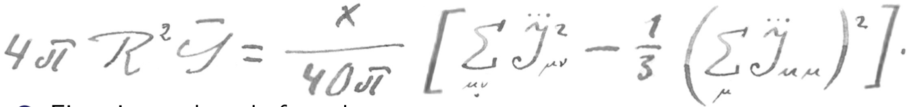
Gravitational wave events [LIGO/VIRGO collaboration 2016, 2017]



- For BH binaries the detectors are mostly sensitive to the merger phase and a few cycles are observed before coalescence
- For NS binaries the detectors will be sensitive to the inspiral phase prior the merger and tens of thousands of cycles will be observable

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]



1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathsf{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

$$h_{ij}^{\mathsf{TT}} = \frac{2G}{c^4 D}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{D}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathsf{TT}} + \mathcal{O}\left(\frac{1}{D^2}\right)$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\mathsf{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

The quadrupole formula works for GW150914 !

[see also the talk by Bruce Allen]

1. The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi} \left[\frac{256}{5} \frac{G\mathcal{M}^{5/3}}{c^5} (t_\mathsf{f} - t) \right]^{-3/8}$$

2. Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96} \frac{c^5}{G\pi^{8/3}} f^{-11/3} \dot{f} \right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

3. The GW amplitude is predicted to be

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot} \right)^{5/6} \left(\frac{100\,\mathsf{Mpc}}{D} \right) \left(\frac{100\,\mathsf{Hz}}{f_\mathsf{merger}} \right)^{-1/6} \sim 1.6 \times 10^{-21}$$

4. The distance $D = 400\,\mathsf{Mpc}$ is measured from the signal itself

Total energy radiated by GW150914

1. The ADM energy of space-time is constant and reads (at any time t)

$$E_{\mathrm{ADM}} = (m_1 + m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5} \int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

2. Initially $E_{\mathrm{ADM}} = (m_1 + m_2)c^2$ while finally (at time t_{f})

$$E_{\mathrm{ADM}} = M_{\mathrm{f}} c^2 + \frac{G}{5c^5} \int_{-\infty}^{t_{\mathrm{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t')$$

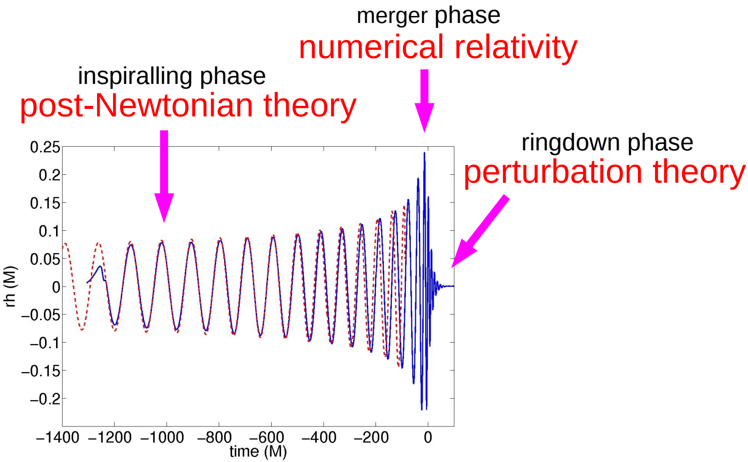
3. The total energy radiated in GW is

$$\boxed{\Delta E^{\mathrm{GW}} = (m_1 + m_2 - M_{\mathrm{f}})c^2 = \frac{G}{5c^5} \int_{-\infty}^{t_{\mathrm{f}}} \mathrm{d}t' \left(Q_{ij}^{(3)}\right)^2 (t') = \frac{G m_1 m_2}{2 r_{\mathrm{f}}}}$$

4. The total power released is

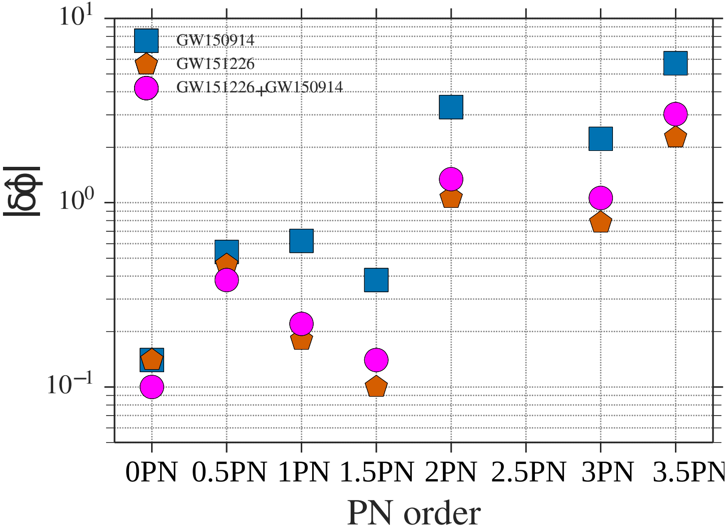
$$\boxed{\mathcal{P}^{\mathrm{GW}} \sim \frac{3 M_{\odot} c^2}{0.2\,\mathrm{s}} \sim 10^{49}\,\mathrm{W} \sim 10^{-3}\,\frac{c^5}{G}}$$

The gravitational chirp of compact binaries

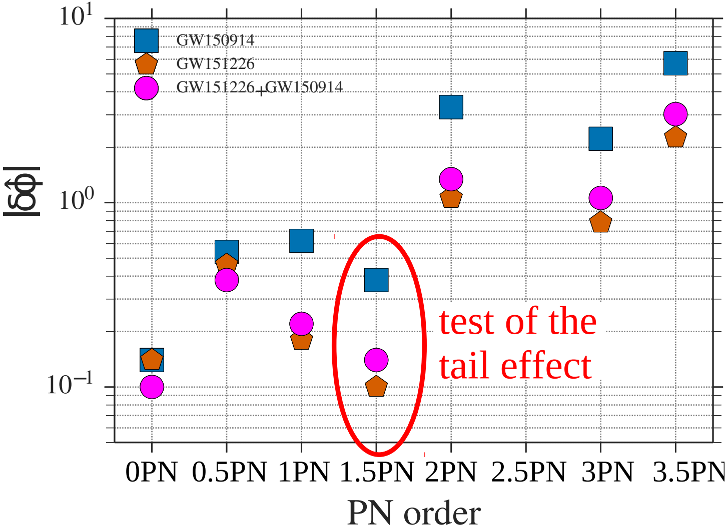


Effective methods such as EOB that interpolate between the PN and NR are also very important notably for the data analysis [see the talk by Thibault Damour]

Measurement of PN parameters [see the talk by Chris Van den Broek]



Measuring GW tails [Blanchet & Sathyaprakash 1994, 1995]



Methods to compute PN equations of motion

1. ADM Hamiltonian canonical formalism [Ohta *et al.* 1973; Schäfer 1985]
2. EOM in harmonic coordinates [Damour & Deruelle 1985; Blanchet & Faye 1998, 2000]
3. Extended fluid balls [Grishchuk & Kopeikin 1986]
4. Surface-integral approach [Itoh, Futamase & Asada 2000]
5. Effective-field theory (EFT) [Goldberger & Rothstein 2006; Foffa & Sturani 2011]

- EOM derived in a general frame for arbitrary orbits
- Dimensional regularization is applied for UV divergences[1]
- Radiation-reaction dissipative effects added separately by matching
- Spin effects can be computed within a pole-dipole approximation
- Tidal effects incorporated at leading 5PN and sub-leading 6PN orders

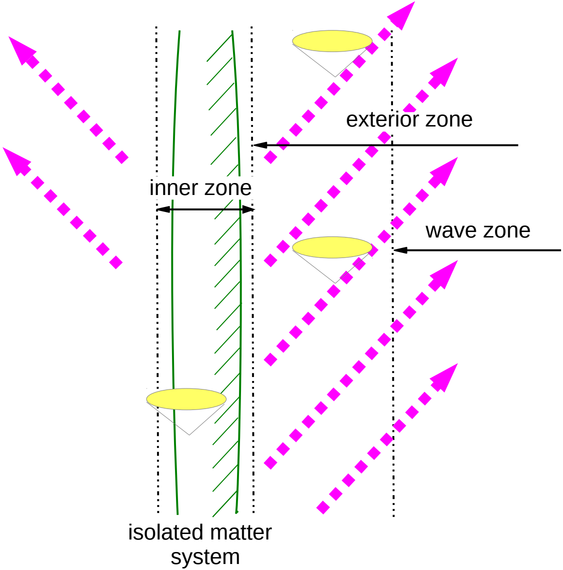
[1] Except in the surface-integral approach

Methods to compute PN radiation field

1. Multipolar-post-Minkowskian (MPM) & PN [Blanchet-Damour-Iyer 1986, . . . , 1998]
2. Direct iteration of the relaxed field equations (DIRE) [Will-Wiseman-Pati 1996, . . .]
3. Effective-field theory (EFT) [Hari Dass & Soni 1982; Goldberger & Ross 2010]

- Involves a machinery of tails and related non-linear effects
- Uses dimensional regularization to treat point-particle singularities
- Phase evolution relies on balance equations valid in adiabatic approximation
- Spin effects are incorporated within a pole-dipole approximation
- Provides polarization waveforms for DA & spin-weighted spherical harmonics decomposition for NR

Isolated matter system in general relativity



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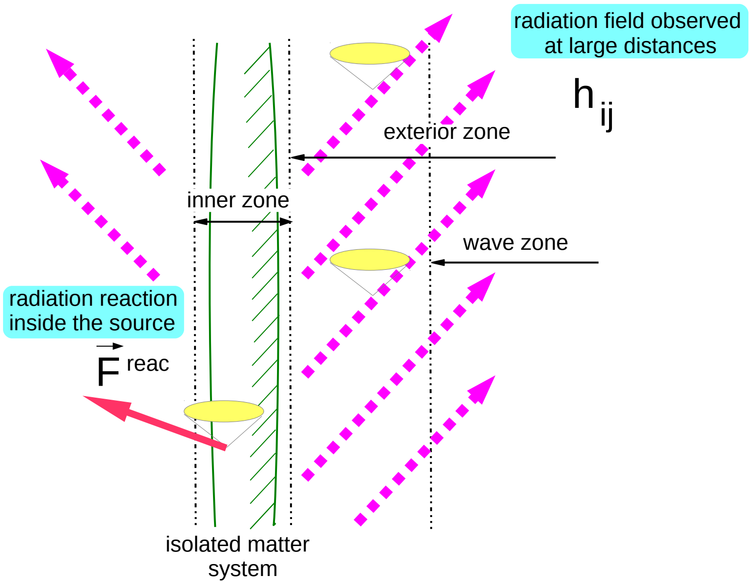
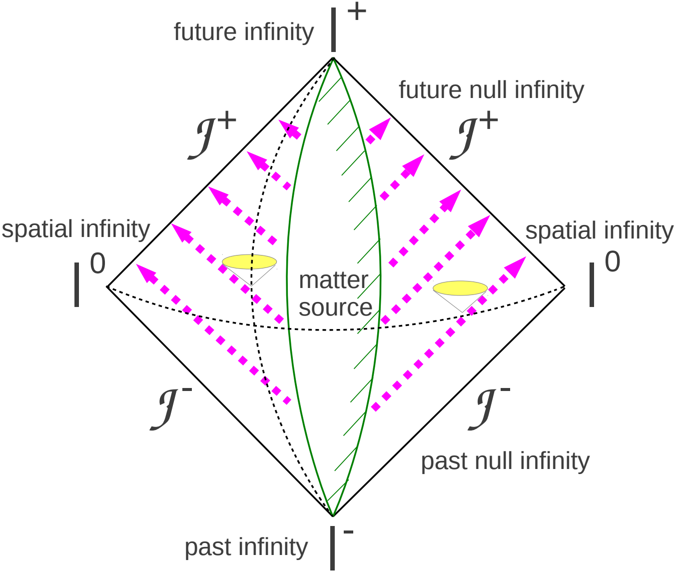


Asymptotic structure of radiating space-time

[Bondi-Sachs-Penrose formalism 1960s]



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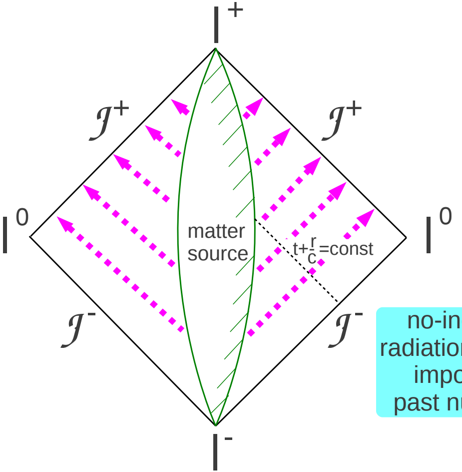
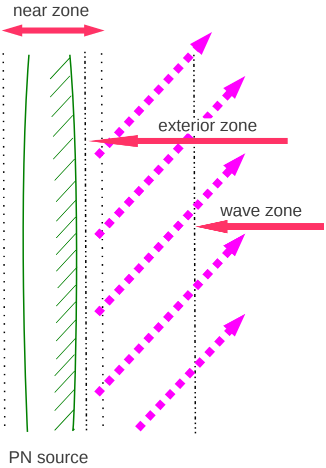


no-incoming radiation condition imposed at past null infinity

$$\lim_{\substack{r\to+\infty \\ t+\frac{r}{c}=\text{const}}} \left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(r h^{\alpha\beta}\right)=0$$

The MPM-PN formalism [Blanchet-Damour-Iyer formalism 1980-90s]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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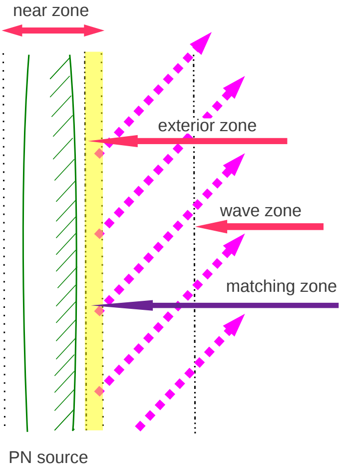

The matching equation [Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

This is a variant of the theory of matched asymptotic expansions

$$\text{match} \quad \begin{cases} \text{the multipole expansion} \quad \mathcal{M}(h^{\alpha\beta}) \equiv h^{\alpha\beta}_{\text{MPM}} \\ \qquad\qquad \text{with} \\ \text{the PN expansion} \quad \bar{h}^{\alpha\beta} \equiv h^{\alpha\beta}_{\text{PN}} \end{cases}$$

$$\boxed{\overline{\mathcal{M}(h^{\alpha\beta})} = \mathcal{M}(\bar{h}^{\alpha\beta})}$$

- Left side is the NZ expansion ($r \to 0$) of the exterior MPM field
- Right side is the FZ expansion ($r \to \infty$) of the inner PN field

1. The matching equation has been implemented at any post-Minkowskian order in the exterior field and any PN order in the inner field
2. It gives a unique (formal) multipolar-post-Newtonian solution valid everywhere inside and outside the source
3. The solution recovers the [Bondi-Sachs-Penrose] formalism at $\mathscr{I}^+$

General solution for the multipolar field [Blanchet 1995, 1998]

$$\mathcal{M}(h^{\mu\nu}) = \text{FP}\Box^{-1}_{\text{ret}}\mathcal{M}(\Lambda^{\mu\nu}) + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{M_L^{\mu\nu}(t-r/c)}{r}\right\}}_{\text{homogeneous retarded solution}}$$

where $$M_L^{\mu\nu}(t) = \text{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\;\underbrace{\bar{\tau}^{\mu\nu}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}$$

- The FP procedure plays the role of an UV regularization in the non-linearity term but an IR regularization in the multipole moments
- From this one obtains the multipole moments of the source at any PN order solving the wave generation problem
- This is a formal PN solution *i.e.* a set of rules for generating the PN series regardless of the exact mathematical nature of this series
- The formalism is equivalent to the DIRE formalism [Will-Wiseman-Pati]

General solution for the inner PN field

[Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2004]

$$\bar{h}^{\mu\nu} = \mathsf{FP}\Box^{-1}_{\text{ret}}\bar{\tau}^{\mu\nu} + \underbrace{\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{R_L^{\mu\nu}(t-r/c) - R_L^{\mu\nu}(t+r/c)}{r}\right\}}_{\text{homogeneous antisymmetric solution}}$$

where $$R_L^{\mu\nu}(t) = \mathsf{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L \int_1^\infty \mathrm{d}z\,\gamma_\ell(z)\;\underbrace{\mathcal{M}(\tau^{\mu\nu})(\mathbf{x}, t - zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

- The radiation reaction effects starting at 2.5PN order appropriate to an isolated system are determined to any order
- In particular nonlinear radiation reaction effects associated with tails are contained in the second term and start at 4PN order

Summary of known PN orders

Method	Equations of motion	Energy flux	Waveform
Multipolar-post-Minkowskian & post-Newtonian (MPM-PN)	4PN non-spin 3.5PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin 4PN (NNL) SO 3PN (NL) SS 3.5PN (NL) SSS	3.5PN non-spin 1.5PN (L) SO 2PN (L) SS
Canonical ADM Hamiltonian	4PN non-spin 3.5PN (NNL) SO 4PN (NNL) SS 3.5PN (NL) SSS		
Effective Field Theory (EFT)	3PN non-spin 2.5PN (NL) SO 4PN (NNL) SS	2PN non-spin 3PN (NL) SS	
Direct Integration of Relaxed Equations (DIRE)	2.5PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS	2PN non-spin 1.5PN (L) SO 2PN (L) SS
Surface Integral	3PN non-spin		

- The 4.5PN non-spin coefficient in the energy flux has been computed with MPM-PN [see the talk by Tanguy Marchand]
- Many works devoted to spins:
    - Spin effects (SO, SS, SSS) are known in EOM up to 4PN order
    - SO effects are known in radiation field up to 4PN
    - SS in radiation field known to 3PN

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Surface Integral	3PN non-spin		

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  - SS in radiation field known to 3PN

THE 4PN EQUATIONS OF MOTION

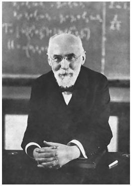
Based on collaborations with

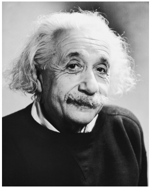
Laura Bernard, **Alejandro Bohé**, **Guillaume Faye** & **Sylvain Marsat**

[PRD **93**, 084037 (2016); PRD **95**, 044026 (2017); PRD submitted (2017)]

The 1PN equations of motion

[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]



$$
\begin{aligned}
\frac{\mathrm{d}^2 \boldsymbol{r}_A}{\mathrm{d}t^2} = &-\sum_{B\neq A} \frac{G m_B}{r_{AB}^2} \boldsymbol{n}_{AB} \Bigg[1 - 4 \sum_{C\neq A} \frac{G m_C}{c^2 r_{AC}} - \sum_{D\neq B} \frac{G m_D}{c^2 r_{BD}} \left(1 - \frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2} \right) \\
&\qquad + \frac{1}{c^2} \left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2} (\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2 \right) \Bigg] \\
&+ \sum_{B\neq A} \frac{G m_B}{c^2 r_{AB}^2} \boldsymbol{v}_{AB} [\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2} \sum_{B\neq A} \sum_{D\neq B} \frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3} \boldsymbol{n}_{BD}
\end{aligned}
$$

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}\left[\cdots\right]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}\left[\cdots\right]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\cdots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\cdots\right]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\cdots\right]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

2PN:

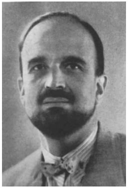
[Otha, Okamura, Kimura & Hiida 1973, 1974; Damour & Schäfer 1985]	ADM Hamiltonian
[Damour & Deruelle 1981; Damour 1983]	Harmonic coordinates
[Kopeikin 1985; Grishchuk & Kopeikin 1986]	Extended fluid balls
[Blanchet, Faye & Ponsot 1998]	Direct PN iteration
[Itoh, Futamase & Asada 2001]	Surface integral method

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{G m_2}{r_{12}^2} n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \cdots\right] n_{12}^i + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}\left[\cdots\right]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}\left[\cdots\right]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\cdots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\cdots\right]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\cdots\right]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye & Marsat 2015, 2016, 2017]	Fokker Lagrangian
	[Foffa & Sturani 2012, 2013] (partial)	Effective field theory

Fokker action of N particles [Fokker 1929]



1. Gauge-fixed Einstein-Hilbert action for N point particles

$$S_{\text{g.f.}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu}_{\text{Gauge-fixing term}}\Big]$$

$$-\sum_A \underbrace{m_A c^2 \int \mathrm{d}t\,\sqrt{-(g_{\mu\nu})_A\,v_A^\mu v_A^\nu/c^2}}_{N\text{ point particles}}$$

2. Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

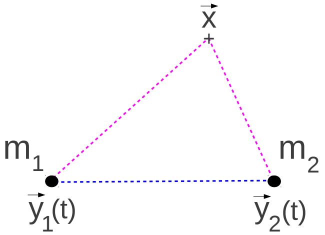
$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots)$$

3. The PN equations of motion of the N particles (self-gravitating system) are

$$\boxed{\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{y}_A} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{y}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

Dimensional regularization for UV divergences

[t'Hooft & Veltman 1972; Bollini & Giambiagi 1972; Breitenlohner & Maison 1977]



In the Newtonian approximation

$$U^{(3)} = \frac{Gm_1}{|\mathbf{x} - \mathbf{y}_1(t)|} + \frac{Gm_2}{|\mathbf{x} - \mathbf{y}_2(t)|}$$

1. Einstein's field equations are solved in d spatial dimensions (with $d \in \mathbb{C}$)

$$U^{(d)} = \frac{2(d-2)\tilde{k}}{d-1}\left(\frac{Gm_1}{|\mathbf{x} - \mathbf{y}_1|^{d-2}} + \frac{Gm_2}{|\mathbf{x} - \mathbf{y}_2|^{d-2}}\right) \quad \text{with} \quad \tilde{k} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}$$

2. Computations are performed when $\Re(d)$ is a large negative number, and the result is analytically continued for any $d \in \mathbb{C}$ except for isolated poles
3. Poles $\propto (d-3)^{-1}$ appear at 3PN order in the Fokker action and are absorbed in a renormalization of the worldlines of the particles

Fokker action in the PN approximation

We face the problem of the near-zone limitation of the PN expansion

- **Lemma 1:** The Fokker action can be split into a PN (near-zone) term plus a contribution involving the multipole (far-zone) expansion

$$S_{\mathsf{F}}^{g} = \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^4 x \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g + \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^4 x \left(\frac{r}{r_0}\right)^B \mathcal{M}(\mathcal{L}_g)$$

- **Lemma 2:** The multipole contribution is zero for any "instantaneous" term thus only "hereditary" terms contribute to this term and they appear at least at 5.5PN order

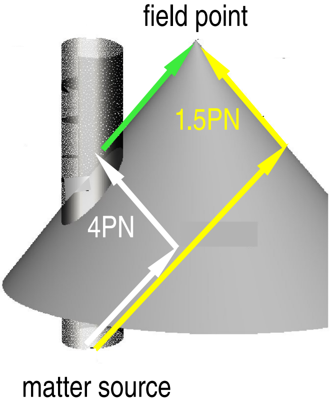
$$\boxed{S_{\mathsf{F}}^{g} = \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^4 x \left(\frac{r}{r_0}\right)^B \overline{\mathcal{L}}_g}$$

- The constant r_0 will play the role of an IR cut-off scale
- IR divergences appear at the 4PN order

Gravitational wave tail effect at the 4PN order

[Blanchet & Damour 1988; Blanchet 1993]

- At the 4PN order there is an imprint of gravitational wave tails in the local (near-zone) dynamics of the source
- This leads to a non-local-in-time contribution in the Fokker action
- This corresponds to a 1.5PN modification of the radiation field beyond the quadrupole approximation



$$S_F^{\text{tail}} = \frac{G^2 M}{5c^8} \underset{s_0}{\text{Pf}} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')$$

where the Hadamard partie finie (Pf) is parametrized by an arbitrary constant s_0

Problem of the IR ambiguity parameter

1. Using dimensional regularization one can properly regularize the UV divergences and renormalize the UV poles
2. The result depends on two constants
   - r_0 the IR cut-off scale in the Einstein-Hilbert part of the action
   - s_0 the Hadamard regularization scale coming from the tail effect
3. Modulo unphysical shifts these combine into a single parameter

$$\alpha = \ln\left(\frac{r_0}{s_0}\right)$$

   which is left undetermined at this stage
4. This parameter is equivalent to the constant C in the 4PN ADM Hamiltonian [Damour, Jaranowski & Schäfer 2014]
5. It is fixed by computing the conserved energy of circular orbits and comparing with gravitational self-force (GSF) results

Conserved energy for a non-local Hamiltonian

1. Because of the tail effect at 4PN order the Lagrangian or Hamiltonian becomes non-local in time

$$H\left[\mathbf{x},\mathbf{p}\right] = H_0\left(\mathbf{x},\mathbf{p}\right) + \underbrace{H_{\text{tail}}\left[\mathbf{x},\mathbf{p}\right]}_{\text{non-local piece at 4PN}}$$

2. Hamilton's equations involve functional derivatives

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\delta H}{\delta p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\delta H}{\delta x^i}$$

3. The conserved energy is not given by the Hamiltonian on-shell but $E = H + \Delta H^{\text{AC}} + \Delta H^{\text{DC}}$ where the AC term averages to zero and

$$\boxed{\Delta H^{\text{DC}} = -\frac{2GM}{c^3}\mathcal{F}^{\text{GW}} = -\frac{2G^2M}{5c^5}\left\langle\left(I_{ij}^{(3)}\right)^2\right\rangle}$$

4. On the other hand [DJS] perform a non-local shift to transform the Hamiltonian into a local one, and both procedure are equivalent

Conserved energy for circular orbits at 4PN order

- The 4PN energy for circular orbits in the small mass ratio limit is known from GSF of the redshift variable [Le Tiec, Blanchet & Whiting 2012; Bini & Damour 2013]
- This permits to fix the ambiguity parameter α and to complete the 4PN equations of motion

$$
\begin{aligned}
E^{\mathsf{4PN}} = -\frac{\mu c^2 x}{2}\Bigg\{&1 + \left(-\frac{3}{4}-\frac{\nu}{12}\right)x + \left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{\nu^2}{24}\right)x^2\\
&+\left(-\frac{675}{64}+\left[\frac{34445}{576}-\frac{205}{96}\pi^2\right]\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3\\
&+\left(-\frac{3969}{128}+\left[-\frac{123671}{5760}+\frac{9037}{1536}\pi^2+\frac{896}{15}\gamma_\mathsf{E}+\frac{448}{15}\ln(16x)\right]\nu\right.\\
&\qquad\left.+\left[-\frac{498449}{3456}+\frac{3157}{576}\pi^2\right]\nu^2+\frac{301}{1728}\nu^3+\frac{77}{31104}\nu^4\right)x^4\Bigg\}
\end{aligned}
$$

Periastron advance for circular orbits at 4PN order

The periastron advanced (or relativistic precession) constitutes a second invariant which is also known in the limit of circular orbits from GSF calculations

$$\begin{aligned}
K^{\mathsf{4PN}} &= 1 + 3x + \left(\frac{27}{2} - 7\nu\right) x^2 \\
&\quad + \left(\frac{135}{2} + \left[-\frac{649}{4} + \frac{123}{32}\pi^2\right]\nu + 7\nu^2\right) x^3 \\
&\quad + \left(\frac{2835}{8} + \left[-\frac{275941}{360} + \frac{48007}{3072}\pi^2 - \frac{1256}{15}\ln x \right.\right. \\
&\qquad \left. - \frac{592}{15}\ln 2 - \frac{1458}{5}\ln 3 - \frac{2512}{15}\gamma_{\mathsf{E}}\right]\nu \\
&\qquad \left. + \left[\frac{5861}{12} - \frac{451}{32}\pi^2\right]\nu^2 - \frac{98}{27}\nu^3\right) x^4
\end{aligned}$$

Problem of the second ambiguity parameter

- The initial calculation of the Fokker action was based on the Hadamard regularization (HR) to treat the IR divergences (FP procedure when $B \to 0$)
- Computing the periastron advance for circular orbits it did not agree with GSF calculations (offending coefficient $-\frac{275941}{360}$)
- We found that the problem was due to the HR and conjectured that a different regularization would give (modulo shifts)

$$L = L^{\text{HR}} + \underbrace{\frac{G^4 m\, m_1^2 m_2^2}{c^8 r_{12}^4}\Big(\delta_1 (n_{12}v_{12})^2 + \delta_2 v_{12}^2\Big)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

- One combination of the two parameters δ_1 and δ_2 is equivalent to the previous ambiguity parameter α
- Matching with GSF results for the energy and periastron we have

$$\delta_1 = -\frac{2179}{315} \qquad \delta_2 = \frac{192}{35}$$

Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I^{\mathsf{HR}}_{\mathcal{R}} = \mathop{\mathrm{FP}}_{B=0} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I^{\mathsf{DR}}_{\mathcal{R}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}}\, F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d-3$)

$$\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi^{(\varepsilon)}_{3,q}(\mathbf{n}) + \mathcal{O}\left(\varepsilon\right)$$

Computing the tail effect in d dimensions

1. We solve the wave equation in $d+1$ space-time dimensions

$$\begin{aligned} \Box h(\mathbf{x},t) &= \Lambda(\mathbf{x},t) \\ h(\mathbf{x},t) &= -\frac{\tilde{k}}{4\pi}\int_1^{+\infty} \mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z) \int \mathrm{d}^d\mathbf{x}'\,\frac{\Lambda(\mathbf{x}',t-z|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|^{d-2}} \end{aligned}$$

2. We identify an homogeneous piece in the post-Newtonian or near zone expansion when $r\to 0$ of that solution

$$\overline{h} = \Box^{-1}_{\text{ret}}\overline{\Lambda} + \overline{h}^{\text{asym}}$$

$$\boxed{\overline{h}^{\text{asym}} \propto \sum_{j=0}^{+\infty} \Delta^{-j}\hat{x}_L \int_1^{+\infty} \mathrm{d}z\,\gamma_{\frac{1-d}{2}-\ell}(z) \int_0^{+\infty} \mathrm{d}r'\,r'^{-\ell+1}\,\Lambda_L^{(2j)}(r',t-zr')}$$

- Such homogeneous solution is of the anti-symmetric type (half-retarded minus half-advanced) and is regular when $r\to 0$
- It contains the 4PN tail effect as determined from the general solution of the matching equation

Computing the tail effect in d dimensions

1. We apply the previous formula to the computation of the interaction between the static mass monopole M and the varying mass quadrupole $I_{ij}(t)$
2. In a particular gauge the 4PN tail effect is entirely described by a single scalar potential in the 00 component of the metric

$$g_{00}^{\text{tail}} = -\frac{8G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\sqrt{\bar{q}}\,\tau}{2\ell_0}\right)\underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}}+\kappa\right]I_{ij}^{(7)}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

3. The conservative part of the 4PN tail effect corresponds in the action

$$\boxed{S_g^{\text{tail}} = \frac{G^2M}{5c^8}\ \underset{s_0^{\text{DR}}}{\text{Pf}}\iint\frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|}I_{ij}^{(3)}(t)\,I_{ij}^{(3)}(t')}$$

$$\text{with}\quad \ln s_0^{\text{DR}} = \ln\left(\frac{2\ell_0}{c\sqrt{\bar{q}}}\right)+\frac{1}{2\varepsilon}-\kappa$$

[see also Galley, Leibovich, Porto & Ross 2016]

Computation of the second ambiguity parameter

- The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action [in agreement with Porto & Rothstein 2017]
- The parameter κ is equivalent to the first ambiguity parameter α and to the constant C in the Hamiltonian formalism [DJS]
- Finally we obtain exactly the conjectured form of the ambiguity terms with

$$\delta_1 = \frac{1733}{1575} - \frac{176}{15}\kappa \qquad \delta_2 = -\frac{1712}{525} + \frac{64}{5}\kappa$$

- The unique choice to get at once the energy and periastron advance is

$$\kappa = \frac{41}{60}$$

- More work is needed to compute κ from first principles (i.e. without resorting to external GSF calculations)

4PN FIRST LAW OF COMPACT BINARIES

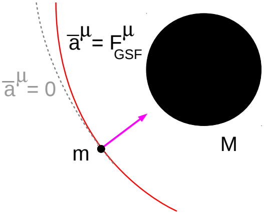
Based on a collaboration with

Alexandre Le Tiec [CQG to appear (2017)]

Problem of the gravitational self-force (GSF)

[Mino, Sasaki & Tanaka 1997; Quinn & Wald 1997; Detweiler & Whiting 2003]

- A particle is moving on a background space-time of a massive black hole
- Its stress-energy tensor modifies the background gravitational field
- Because of the back-reaction the motion of the particle deviates from a background geodesic hence the gravitational self force

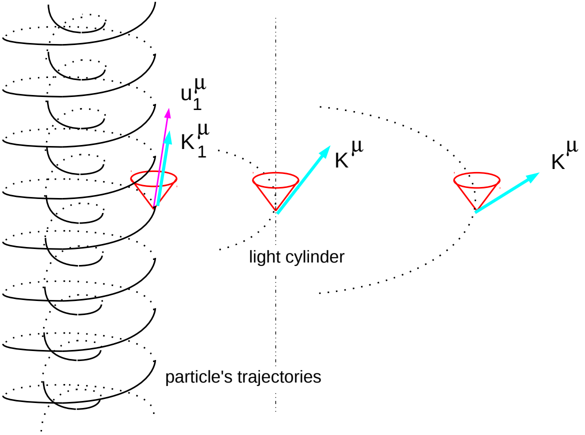


$$\boxed{\bar{a}^\mu = F^\mu_{\mathsf{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)}$$

The GSF is computed to high accuracy by

- numerical methods [Sago, Barack & Detweiler 2008; Shah, Friedmann & Whiting 2014]
- analytical ones [Bini & Damour 2013, 2014; Bini, Damour & Geralico 2016]

The redshift observable [Detweiler 2008; Barack & Sago 2011]



$$K_1^\mu = z_1 u_1^\mu$$

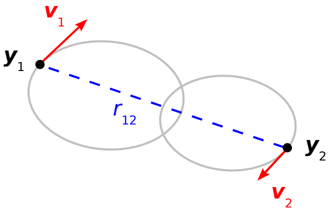
For eccentric orbits one must consider the averaged redshift $\langle z_1 \rangle = \frac{1}{P}\int_0^P \mathrm{d}t\, z_1(t)$

Post-Newtonian calculation of the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010, 2011; Blanchet, Faye & Whiting 2014, 2015]

In a coordinate system such that $K^\mu \partial_\mu = \partial_t + \omega\, \partial_\varphi$ we have

$$z_1 = \frac{1}{u_1^t} = \left(- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \right)^{1/2}$$



One needs a self-field regularization

- Hadamard "partie finie" regularization is extremely useful in practical calculations but yields (UV and IR) ambiguity parameters at high PN orders
- Dimensional regularization is an extremely powerful regularization which seems to be free of ambiguities at any PN order

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\mathrm{SF}} &= -y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
&+ \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_{\mathrm{E}} - \frac{64}{5}\ln(16y)\right) y^5 \\
&- \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
&+ \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
&- \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
&+ \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tails (and tail-of-tails)

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1. Integral PN terms such as 3PN permit checking dimensional regularization
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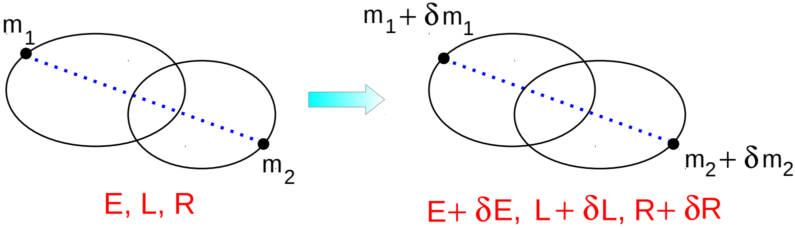
First law of compact binary mechanics

[Friedman, Uryū & Shibata 2002; Le Tiec, Blanchet & Whiting 2012; Le Tiec 2015]



$$\boxed{\delta E = \omega\,\delta L + n\,\delta R + \langle z_1\rangle\,\delta m_1 + \langle z_2\rangle\,\delta m_2}$$

- E, L : ADM energy and angular momentum
- $\boxed{R = \frac{1}{2\pi}\oint p_r \mathrm{d}r}$: radial action integral
- n, ω : radial and azimuthal frequencies

First law valid versus non-local dynamics

1. The basic variable computed by GSF techniques is the averaged redshift $\langle z_a \rangle$ in the test-mass limit $m_1/m_2 \to 0$
2. The first law permits to derive from $\langle z_a \rangle$ the binary's conserved energy E and periastron advance K for circular orbits

$$K = \frac{\omega}{n}$$

3. These results are then used to fix the ambiguity parameters in the 4PN equations of motion [DJS, BBBFM]
4. However the first law has been derived from a local Hamiltonian but at 4PN order the dynamics becomes non-local due to the tail term

Are we still allowed to use the first law in standard form
for the non-local dynamics at the 4PN order ?

Derivation of the first law at 4PN order

1. At 4PN order the dynamics becomes non-local due to the tail term

$$H = H_0(r, p_r, p_\varphi; m_a) + H_{\text{tail}}[r, \varphi, p_r, p_\varphi; m_a]$$

with
$$\boxed{H_{\text{tail}} = -\frac{m}{5} I_{ij}^{(3)}(t) \int_{-\infty}^{+\infty} \frac{\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t')}$$

2. For the non-local dynamics H and p_φ are no longer conserved but instead

$$\begin{aligned} E &= H + \Delta H^{\text{DC}} + \Delta H^{\text{AC}} \\ L &= p_\varphi + \Delta p_\varphi^{\text{DC}} + \Delta p_\varphi^{\text{AC}} \end{aligned}$$

where H^{AC} and p_φ^{AC} (given by Fourier series) average to zero and

$$\boxed{\Delta H^{\text{DC}} = -2m\, \mathcal{F}^{\text{GW}} \qquad \Delta p_\varphi^{\text{DC}} = -2m\, \mathcal{G}^{\text{GW}}}$$

Derivation of the first law at 4PN order

1. We perform an unconstrained variation of the Hamiltonian

$$\begin{aligned}\delta H &= \dot{\varphi}\delta p_\varphi - \dot{p}_\varphi\delta\varphi + \dot{r}\delta p_r - \dot{p}_r\delta r + \frac{2m}{5}\left(I^{(3)}_{ij}\right)^2\frac{\delta n}{n}\\ &+ \sum_a z_a\delta m_a + \Delta\end{aligned}$$

   where Δ is a complicated double Fourier series but such that $\boxed{\langle\Delta\rangle = 0}$

2. By averaging we obtain

$$\begin{aligned}\langle\dot{r}\,\delta p_r - \dot{p}_r\,\delta r\rangle &= n\,\delta R\\ \langle\dot{\varphi}\,\delta p_\varphi - \dot{p}_\varphi\,\delta\varphi\rangle &= \omega\,\delta L + \omega\,\delta\big(2m\,\mathcal{G}^{\mathsf{GW}}\big) - n\,\delta\left(\frac{1}{2\pi}\oint\Delta p_\varphi^{\mathsf{AC}}\,\mathrm{d}\varphi\right)\end{aligned}$$

Derivation of the first law at 4PN order

1. Combining all the terms we obtain a first law in standard form

$$\boxed{\delta E = \omega\,\delta L + n\,\delta\mathcal{R} + \sum_a \langle z_a\rangle\,\delta m_a}$$

but where the radial action integral gets corrected at 4PN order

$$\boxed{\mathcal{R} = R + 2m\left(\mathcal{G}^{\mathsf{GW}} - \frac{\mathcal{F}^{\mathsf{GW}}}{\omega}\right) - \frac{1}{2\pi}\oint \Delta p_\varphi^{\mathsf{AC}}\,\mathrm{d}\varphi}$$

2. By performing a non-local shift of canonical variables to transform the non-local Hamiltonian into a local one [DJS] one would get the ordinary first law with an ordinary radial action integral

$$\mathcal{R}(E,L,m_a) = R^{\mathsf{loc}}(E,L,m_a) = \frac{1}{2\pi}\oint \mathrm{d}r^{\mathsf{loc}}\,p_r^{\mathsf{loc}}(r^{\mathsf{loc}},E,L,m_a)$$

3. With the 4PN first law we fully confirm E^{4PN} and K^{4PN} in the test-mass limit