Towards an advanced wave extraction algorithm in numerical relativity

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Current problems with wave extraction in NR

- Extraction (in most cases) takes place at finite radius rather than null infinity
- Gauge ambiguity in the determination of the correct tetrad to project the Weyl tensor components.
- Numerical integration of the Weyl scalar Ψ_4 to obtain the strain

Systematic errors coming from wave extraction are starting to become the dominant source of errors. The need of a mathematically rigorous wave extraction technique is becoming an urgent topic in numerical relativity.

Wave extraction with the Newman-Penrose formalism

The Newman-Penrose formalism is used in numerical relativity to obtain gauge invariant information about gravitational waves.

$$\Psi_4 = \frac{\partial^2 h_+^{TT}}{\partial t^2} + i\frac{\partial^2 h_\times^{TT}}{\partial t^2}$$

The Kinnersley tetrad guarantees that $\Psi_4 = 0$ for Kerr.

- The chosen tetrad must converge to the Kinnersley tetrad in the background limit.
- Calculating Ψ_4 using tetrads evaluated numerically normally brings unwanted gauge effects into the final waveform.
- Our objective is to eliminate the numerical evaluation of the tetrad (no more Gram-Schmidt!).

Relevant tetrads in the NP formalism for a general (Petrov type I) space-time

1) Symmetric transverse tetrad (STT)

$$\Psi_1 = \Psi_3 = 0 \qquad\qquad \Psi_0 = \Psi_4$$

2) Quasi-Kinnersley tetrad (QKT)

$$\Psi_1 = \Psi_3 = 0 \qquad\qquad \epsilon = 0$$

- QKT is the "right tetrad", it guarantees the convergence the Kinnersley tetrad in the Petrov type D limit.
- STT in a convenient tetrad because of its symmetric properties, but <u>not good</u> for numerical applications.
- STT and QKT are related by a spin/boost (type III) tetrad transformation (complex parameter $\mathcal{B}$).

The transformation STT $\rightarrow$ QKT

The spin coefficient ϵ is fundamental

$$\epsilon^{QKT} = \frac{1}{|\mathcal{B}|}\left(\epsilon^{STT} - \frac{1}{2}\ell^a \nabla_a \ln \mathcal{B}\right)$$

Imposing $\epsilon^{QKT} = 0$ gives the condition for the spin-boost parameter $\mathcal{B}$

$$\ell^a \nabla_a \ln \mathcal{B} = 2\epsilon^{STT}$$

The derivative of $\mathcal{B}$ along the other null vectors can be obtained from the spin coefficients γ, α and β.

In order to calculate $\mathcal{B}$ we need to know ϵ^{STT}, γ^{STT}, α^{STT}, β^{STT}.

Different approaches to Einstein's equations in vacuum

Coord. approach	Newman-Penrose	NP in STT
$g_{\mu\nu}$	$\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu$	$\Sigma_{\mu\nu}, \Sigma^+_{\mu\nu}, \Sigma^-_{\mu\nu}$
Γ_{abc}	$\rho, \mu, \tau, \pi, \sigma, \lambda, \nu, \kappa, \epsilon, \gamma, \beta, \alpha$	A_μ, B_μ, C_μ
C_{abcd}	$\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$	Ψ_2, Ψ_4

- In STT the only remaining degrees of freedom are Ψ_2 and Ψ_4

$$\begin{aligned} \Psi_2^{STT} &= -\frac{I^{\frac{1}{2}}}{2\sqrt{3}} \left(\Theta + \Theta^{-1}\right), \\ \Psi_4^{STT} &= \frac{I^{\frac{1}{2}}}{2i} \left(\Theta - \Theta^{-1}\right), \end{aligned}$$

- I is one of the two curvature invariants and $\Theta = f(I, J)$.

Numerical implementation

The calculation of curvature invariants in numerical codes is very simple

$$W_{ab} = E_{ab} + iB_{ab} = -^{(3)}R_{ab} + K_a{}^c K_{cb} - KK_{ab} - i\epsilon_a{}^{cd} D_c K_{db},$$

and then I and J are simply given by

$$\begin{aligned} I &= \frac{1}{2} W_{ab} W^{ab}, \\ J &= \frac{1}{6} W_{ac} W^c{}_b W^{ab}, \end{aligned}$$

while Θ is given by ($\Theta \to 1$ for Kerr)

$$\Theta = \sqrt{\frac{3}{I}} \left[-J + \sqrt{J^2 - (I/3)^3} \right]^{\frac{1}{3}}.$$

The Bianchi identities

The Bianchi identities ($\nabla_a C^{*a}{}_{bcd} = 0$) written as functions of the variables introduced within our approach give

$$
\begin{aligned}
A_a &= -\frac{iK}{\sqrt{3}} B_a - \frac{1}{6}\nabla_a \ln I - \frac{K}{3}\nabla_a \ln \Theta \\
C_a &= -\frac{i\left(K + 3K^{-1}\right)}{2\sqrt{3}} B_a + \frac{1}{6}\nabla_a \ln I + \left(\frac{3K^{-1} - K}{6}\right)\nabla_a \ln \Theta .
\end{aligned}
$$

where

$$K = \frac{\Theta - \Theta^{-1}}{\Theta + \Theta^{-1}}$$

It turns out that the Bianchi identities can be used as simple relations to derive the two vectors A_a and C_a once B_a is known. But what about the third vector? Can we find a third potential?

A quadratic function of the Weyl tensor

We introduce following function of the self-dual Weyl tensor

$$D^*_{abcd} = \nabla_\mu \nabla^\mu C^*_{abcd}.$$

Using the Bianchi identities it is possible to show that D^*_{abcd} is given by

$$D^*_{abcd} = 16\, I\, I_{abcd} - \frac{3}{2}\, C^*_{abef}\, C^{*ef}{}_{cd},$$

where I_{abcd} is the identity tensor: $I_{abcd} = \frac{1}{4}\left(g_{ac}g_{bd} - g_{ad}g_{bc} + i\epsilon_{abcd}\right)$.

The tensor D^*_{abcd} has the same symmetries of the self-dual Weyl tensor, included its trace-free property.

On the divergence of D^*_{abcd}

Analogously to $\nabla_a C^{*a}{}_{bcd} = 0$, the divergence of D^*_{abcd} must satisfy

$$\nabla_a D^{*a}{}_{bcd} = \mathcal{S}_a \, C^{*a}{}_{bcd} + \mathcal{T}_a \, D^{*a}{}_{bcd}.$$

$\mathcal{T}_a$ and $\mathcal{S}_a$ are tetrad invariant vectors given by

$$\mathcal{T}_a = \nabla_a \ln \left[I^{\frac{1}{2}} \, \left(\Theta^3 + \Theta^{-3}\right)^{\frac{1}{3}} \right] - \frac{I^{-\frac{1}{2}}}{\sqrt{3}\,(\Theta^3 + \Theta^{-3})} \, \mathcal{S}_a.$$

$$\mathcal{S}_a = f(\nabla_a I, \nabla_a \Theta) + D^{*\,bcd}_{a} \nabla_e D^{*e}{}_{bcd}.$$

These two vectors naturally introduce a third tetrad invariant quantity that cannot be expressed as a function of I and J. Gradient of a third scalar?

Solution for A_a, B_a and C_a

Considering the Bianchi identities and the divergence of D^*_{abcd} one obtains

$$
\begin{aligned}
A_a &= \frac{\mathcal{E}_A}{12}\left[\tilde{S}_a + \nabla_a \ln\left(\frac{K}{\mathcal{E}_A}\right)\right] - \frac{1}{6}\nabla_a \ln I, \\
B_a &= \frac{i\,\mathcal{E}_B}{4\sqrt{3}}\left[\tilde{S}_a + \nabla_a \ln\left(\frac{K}{\mathcal{E}_B}\right)\right], \\
C_a &= \frac{\mathcal{E}_C}{6}\left[\tilde{S}_a + \nabla_a \ln\left(\frac{K}{\mathcal{E}_C}\right)\right] + \frac{1}{6}\nabla_a \ln I.
\end{aligned}
$$

where $\mathcal{E}_A = \left(\Theta - \Theta^{-1}\right)^2$, $\mathcal{E}_B = \Theta^2 - \Theta^{-2}$ and $\mathcal{E}_C = \Theta^2 + \Theta^{-2} + 1$.

$$
K = \frac{\Theta^3 - \Theta^{-3}}{\left(\Theta^3 + \Theta^{-3}\right)^{\frac{1}{3}}}.
$$

The Petrov type D limit

In the limit of Petrov type D the three vectors tend to

$$
\begin{aligned}
A_a &= \frac{1}{6}\,\nabla_a \ln\, I, \\
B_a &= 0, \\
C_a &= -\frac{1}{6}\nabla_a \ln\, I - \frac{I^{-\frac{1}{2}}}{6\sqrt{3}}\,\mathcal{S}_a.
\end{aligned}
$$

$$
\begin{aligned}
&\rho, \mu, \tau, \pi \\
&\lambda, \sigma, \nu, \kappa \\
&\epsilon, \gamma, \beta, \alpha
\end{aligned}
$$

- These values are consistent with the known expressions for the spin coefficients in Kerr.
- The value of C_a calculated in the Kerr space-time confirms that $\mathcal{S}_a = \nabla_a \Phi$! (at least in this limit)
- Knowing C_a in STT and in QKT allows to calculate the spin/boost parameter $\mathcal{B}$ between STT and QKT.

Final expressions

Knowing the spin-boost parameter $\mathcal{B}$ between STT and QKT we find that the values of Ψ_2 and Ψ_4 in QKT are given by

$$\begin{aligned}\Psi_2^{QKT} &= -\frac{I^{\frac{1}{2}}}{2\sqrt{3}}\left(\Theta + \Theta^{-1}\right), \\ \Psi_4^{QKT} &= \frac{\mathcal{B}^2 I^{\frac{1}{2}}}{2i}\left(\Theta - \Theta^{-1}\right).\end{aligned}$$

Moreover: the spin coefficient σ in QKT vanishes in the Kerr limit and is naturally related to

$$\sigma^{QKT} = \frac{\partial h_+^{TT}}{\partial t} + i\frac{\partial h_\times^{TT}}{\partial t}.$$

No need for numerical integration!

The Ricci identities

It is known that the Ricci identities in STT simplify to

$$
\begin{aligned}
\nabla_a A^a &= A_a A^a - B_a B^a - \frac{2I^{\frac{1}{2}}}{\sqrt{3}}\left(\Theta + \Theta^{-1}\right) \\
\nabla_a B^a &= -2B_a C^a + 2i\Psi_- \\
\nabla_a C^a &= A_a A^a - B_a B^a + 2A_a C^a - \frac{4I^{\frac{1}{2}}}{\sqrt{3}}\left(\Theta + \Theta^{-1}\right)
\end{aligned}
$$

Our result of obtaining A_a, B_a and C_a as functions of $\nabla_a I$, $\nabla_a \Theta$ and maybe $\nabla_a \Phi$ would then lead to equations of the type

$$
\begin{aligned}
\nabla_a \nabla^a I &= \ldots \\
\nabla_a \nabla^a \Theta &= \ldots \\
\nabla_a \nabla^a \Phi &= \ldots
\end{aligned}
$$

A set of three non-linear wave-like equations for I, Θ and Φ.

Conclusions

- Transverse tetrads ($\Psi_1 = \Psi_3 = 0$) are an elegant form of fixing the gauge in the Newman-Penrose formalism for wave extraction.
- Using STT as a starting point, we only need to calculate the spin-boost parameter $\mathcal{B}$ to obtain the scalars in QKT.
- Bianchi identities and the divergence of D^*_{abcd} allow to find the expression for the spin coefficients in STT, and consequently $\mathcal{B}$.
- Work in progress to determine whether the additional degree of freedom ($\mathcal{S}_a$) can in general be expressed as gradient of a third potential (it can in the Petrov type D limit).
- This procedure allows to study alternative quantities to Ψ_4, like σ, related to the first time derivative of the strain.