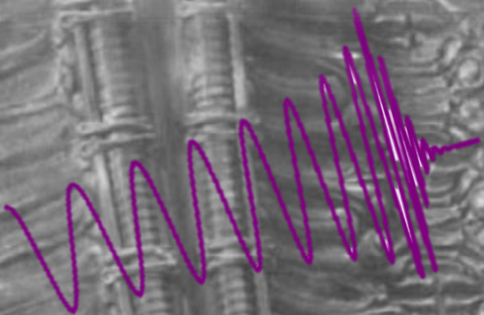
DOES THE GRAVITATIONAL WAVEFORM DEPEND ON THE SPIN SUPPLEMENTARY CONDITIONS?



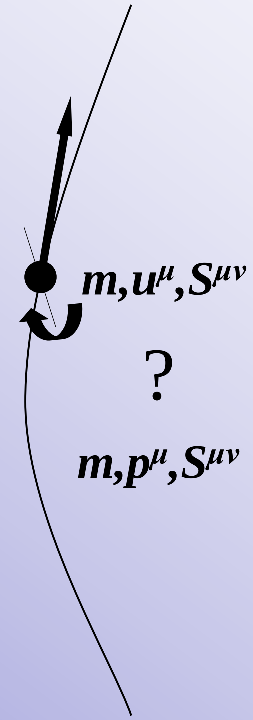
Balázs Mikóczi

HAS Wigner RCP

The Era of Gravitational-Wave Astronomy
Paris, 26-30 June, 2017

Outline

- The equations of motion of a spinning test particle i.e. **Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations**.
- Introducing of the different ***spin supplementary conditions*** (SSCs).
- **Acceleration-dependent Lagrangian of the compact binaries for spin-orbit (SO) interaction** (there are some examples where we can eliminate the acceleration terms).
- Construction of the **generalized Ostrogradski Hamiltonian mechanics** of the SO interaction.
- The orbital evolution of SO dynamics.
- Gravitational radiation corrections of SO contribution, i.e. **the energy, the angular momentum losses, and the waveform**.
- The main question is: **Do the conserved and the dissipative quantities by the gravitational radiation depend on SSC?**



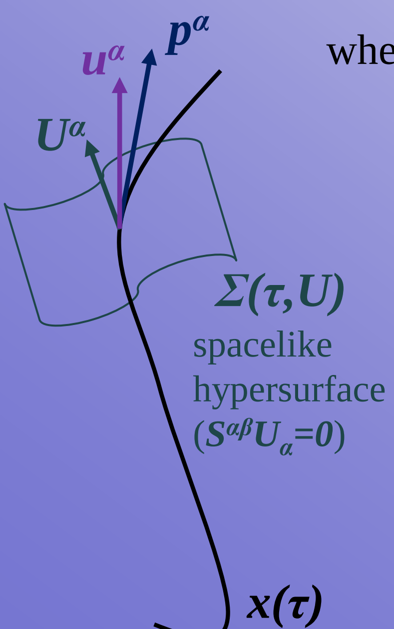
Motion of a spinning particle

MPTD-equations:

$$\dot{p}^{\alpha} = -\frac{1}{2} R^{\alpha}_{\gamma\delta\beta} S^{\delta\beta} u^{\gamma}$$

$$\dot{S}^{\alpha\beta} = p^{[\alpha} u^{\beta]}$$

(Mathisson 1937, Papapetrou 1951, Tulczyjew 1959 and Dixon 1964)



where "•" $\equiv D/D\tau = u^{\alpha}\nabla_{\alpha}$

$$u^{\alpha} = \frac{dX^{\alpha}}{d\tau}$$ the tangent vector of the wordline

$$u^{\alpha}u_{\alpha} = -1$$

$$S^{\alpha\beta} = \int_{\tau=konst} T^{[\alpha 0}\delta x^{\beta]} \sqrt{-g}\, d^3x$$

$$p^{\alpha} = m u^{\alpha} - u_{\beta}\dot{S}^{\alpha\beta}$$

- The MPTD equations can be derived from the properties of the energy-momentum tensor (symmetric and divergence-free).
- These equations beyond the point particle equations (EIH eqs.) are the pole-dipole eqs. in the multipole expansion.
- The MPTD eqs. are not closed! We have to impose one of the *spin supplementary conditions (SSCs).*
- There are generalized MPTD eqs. for the electromagnetic field (Dixon-Souria eqs.)
- There are two distinct masses:

$$\mu^2 = -p^{\alpha}p_{\alpha} \qquad m = -p^{\alpha}u_{\alpha}$$

- m=μ if the tangent vector u^{α} is parallel to momentum p^{α}: $u^{\alpha} = p^{\alpha}/\mu$
- We can introduce the magnitude of the spin: $$2S^2 = S^{\alpha\beta}S_{\alpha\beta}$$
- **Quantities S, m, μ are not conserved**! (We have to use one of the SSCs!)

Spin supplementary conditions (SSCs)

I. Frenkel-Mathisson-Pirani (SSC I):

$$S^{\alpha\beta} u_\alpha = 0$$

Frenkel J. Z. Phys. 37, 243 (1926).
Mathisson M. , Acta. Phys. Polon. 6, 167 (1937).
Pirani, F.A.E. Acta Phys. Polon. 15, 389 (1956).

II. Newton-Wigner-Pryce (SSC II):

$$2S^{0\beta} + u_\alpha S^{\alpha\beta} = 0$$

Pryce, M.H.L. Pme R. Soc. A 195, 62 (1948).
Newton, T.D. & Wigner, E.P. Rev. Mod. Phys. 21, 400 (1949).

III. Corinaldesi-Papapetrou (SSC III):

$$S^{\alpha 0} = 0$$

Corinaldesi, E. & Papapetrou, A.. Proc. Roy. Soc. A/09, 259 (1951).

IV. Tulczyjew-Dixon (SSC IV):

$$S^{\alpha\beta} p_\alpha = 0$$

Tulczyjew, W. M. Acta Phys. Polon. 18, 393. (1959).
Dixon, W. Nuovo Cim. 34, 317. (1964).

In short:

$$S^{a0} - kS^{ab} v_a = 0$$

where v^a is spacelike of the u^α

with *k* means the SSC dependence

k=1 SSC I or SSC IV
k=½ SSC II
k=0 SSC III

L. E. Kidder, PRD 52, 821 (1995)

***m* in SSC I, *μ* in SSC IV,**
and *S* in SSC I/SSC IV are conserved quantities!

$$m = -p^\alpha u_\alpha$$

$$\mu = \sqrt{-p^\alpha p_\alpha}$$

$$S = \sqrt{S^{\alpha\beta} S_{\alpha\beta}/2}$$

Compact binaries with leading-order spin-orbit interaction

The SO acceleration is not unique, it depends on SSC (Kidder 1995):

$$\mathbf{a} = -\frac{Gm}{r^3}\mathbf{r} + \frac{G}{c^2 r^3}\left[\frac{3}{r^2}\mathbf{r}\left[(\mathbf{r}\times\mathbf{v})\cdot\left(2\mathbf{S} + \boxed{(k+1)\boldsymbol{\sigma}}\right)\right] - \mathbf{v}\times(4\mathbf{S}+3\boldsymbol{\sigma}) + \frac{3\dot{r}}{r}\mathbf{r}\times\left(2\mathbf{S} + \boxed{(2-k)\boldsymbol{\sigma}}\right)\right]$$

SSC-dependent

where

$$m = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$

$$\boldsymbol{\sigma} = \frac{m_2}{m_1}\mathbf{S}_1 + \frac{m_1}{m_2}\mathbf{S}_2$$

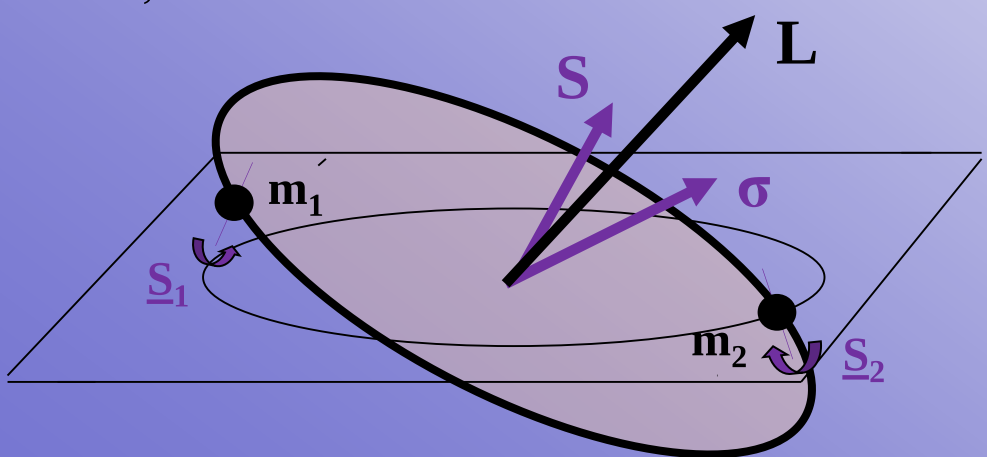
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,$$

$$\mathbf{v} = \dot{\mathbf{r}}, \mathbf{a} = \ddot{\mathbf{r}}$$

k=1	Pirani-Tulczyjew-Dixon	SSC I
k=½	Newton-Wigner-Pryce	SSC II
k=0	Corinaldesi-Papapetrou	SSC III

The transformation between the SSCs is

$$\mathbf{r}^{(\tilde{k})} = \mathbf{r}^{(k)} + \frac{k-\tilde{k}}{mc^2}(\mathbf{v}\times\boldsymbol{\sigma})$$



For the leading-order SO corrections:

$\mathcal{O}(S^2)\approx 0$ és $\mathcal{O}(1/c^4)\approx 0$

The spin-orbit dynamics of the compact binary

The Lagrangian of the SO interaction is

$$\mathcal{L} = \frac{\mu}{2}\mathbf{v}^2 + \frac{G\mu m}{r} + \frac{G\mu}{c^2 r^3}\mathbf{v}\cdot[\mathbf{r}\times(2\mathbf{S}+(k+1)\boldsymbol{\sigma})] + \frac{(2k-1)\mu}{2c^2 m}\mathbf{v}\cdot(\mathbf{a}\times\boldsymbol{\sigma})$$

Newtonian terms **SO terms** **acceleration-dependent term (SSC-dependent)**

Conserved quantities $E, L, A, S_1, S_2, \mathbf{J} = \mathbf{L} + \mathbf{S}$

where A is the magnitude of the *Newtonian* Laplace-Runge-Lenz vector (LRL)

$$\mathbf{A} = \frac{\mathbf{p}}{\mu}\times\mathbf{L} - \frac{Gm\mu}{r}\mathbf{r}$$

The SO motion is described by E and L, which **depend on SSC**:

$$E = \frac{\mu}{2}\mathbf{v}^2 - \frac{G\mu m}{r} + \frac{G\mu(1-2k)}{c^2 r^3}\mathbf{v}\cdot(\mathbf{r}\times\boldsymbol{\sigma}),$$

$$\mathbf{L} = \mu\mathbf{r}\times\mathbf{v} + \frac{G\mu}{c^2 r^3}\mathbf{r}\times[\mathbf{r}\times(2\mathbf{S}+(2-k)\boldsymbol{\sigma})] - \frac{(2k-1)\mu}{2c^2 m}\mathbf{v}\times(\mathbf{v}\times\boldsymbol{\sigma}).$$

REM: But, the magnitude of the LRL vector (A) is only conserved for the Newtonian case:

$$\mu A^2 = G^2 m^2 \mu^3 + 2EL^2$$

It can be proved that the evolutions of the relative angles between the spinvectors and the angular momentum are at least $\mathcal{O}(1/c^2)$ (first-order corr.):

$$(\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}_1), (\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}_2), (\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_1)$$

These angles can be considered the conserved quantities in linear order terms.

Example I.: Elimination of the acceleration from the Lagrangian

With help of the **constrained dynamics** (Riewe 1972, Ellis 1975).

$$\mathcal{L}(\mathbf{r},\mathbf{v},\mathbf{a}) \rightarrow \mathcal{L}^*(\mathbf{r},\mathbf{v},\boldsymbol{\lambda},\dot{\boldsymbol{\lambda}},\boldsymbol{\delta})$$

⇨ **Degenerate Lagrangian**

with $\boldsymbol{\lambda} = \dot{\mathbf{r}}$ and $\boldsymbol{\delta}$ is the Lagrange multiplier

$$\det \frac{\partial^2 \mathcal{L}^*}{\partial \dot{\mathbf{q}}^i \partial \dot{\mathbf{q}}^j} = 0$$

The new Lagrangian is

$$\mathcal{L}^* = \frac{\mu}{2}\mathbf{v}^2 + \frac{Gm\mu}{r} + \frac{G\mu}{c^2 r^3}\mathbf{v}\cdot[\mathbf{r}\times(2\mathbf{S}+(1+k)\boldsymbol{\sigma})] + \frac{(2k-1)\mu}{2c^2 m}\mathbf{v}\cdot(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\sigma}) + \boldsymbol{\delta}\cdot(\mathbf{v}-\boldsymbol{\lambda})$$

The E-L equations are

$$\mu\mathbf{a} = -\frac{Gm\mu}{r^3}\mathbf{r} - \frac{2G\mu}{c^2 r^3}\mathbf{v}\times(2\mathbf{S}+(1+k)\boldsymbol{\sigma}) + \frac{3G\mu}{c^2 r^5}\mathbf{r}[(\mathbf{r}\times\mathbf{v})\cdot(2\mathbf{S}+(1+k)\boldsymbol{\sigma})]$$

$$+\frac{3G\mu\dot{r}}{c^2 r^4}\mathbf{r}\times(2\mathbf{S}+(1+k)\boldsymbol{\sigma}) - \frac{(2k-1)\mu}{2c^2 m}(\ddot{\boldsymbol{\lambda}}\times\boldsymbol{\sigma}) - \dot{\boldsymbol{\delta}},$$

$$0 = \boldsymbol{\delta} - \frac{(2k-1)\mu}{2c^2 m}(\mathbf{a}\times\boldsymbol{\sigma}),$$

$$0 = \mathbf{v} - \boldsymbol{\lambda}.$$

Example I.. Hamiltonian dynamics

We have **three conjugate momenta**: $\mathbf{p}_\mathbf{r} = \frac{\partial \mathcal{L}^*}{\partial \dot{\mathbf{r}}}, \qquad \mathbf{p}_\lambda = \frac{\partial \mathcal{L}^*}{\partial \dot{\boldsymbol{\lambda}}}, \qquad \mathbf{p}_\delta = \frac{\partial \mathcal{L}^*}{\partial \dot{\boldsymbol{\delta}}}$

Thus

$$\mathbf{p}_\mathbf{r} = \mu \mathbf{v} + \frac{G\mu}{c^2 r^3} \mathbf{r} \times (2\mathbf{S} + (1+k)\boldsymbol{\sigma}) + \frac{(2k-1)\mu}{2c^2 m} \dot{\boldsymbol{\lambda}} \times \boldsymbol{\sigma} + \boldsymbol{\delta},$$

$$\mathbf{p}_\lambda = -\frac{(2k-1)\mu}{2c^2 m} \mathbf{v} \times \boldsymbol{\sigma},$$

$$\mathbf{p}_\delta \approx 0$$

the symbol ≈ denotes the weak equality

The final **Hamiltonian** is

$$\mathcal{H} = \frac{\mathbf{p}_\mathbf{r}^2}{2\mu} - \frac{Gm\mu}{r} + \frac{G(2k-1)}{2c^2 r^3} \mathbf{p}_\mathbf{r} \cdot (\mathbf{r} \times \boldsymbol{\sigma}) - \frac{G}{c^2 r^3} \mathbf{p}_\mathbf{r} \cdot [\mathbf{r} \times (2\mathbf{S} + (1+k)\boldsymbol{\sigma})] + \frac{G(2k-1)(\mathbf{p}_\mathbf{r} \cdot \mathbf{r})}{2c^2 \mu r^5} \mathbf{p}_\delta \cdot (\mathbf{r} \times \boldsymbol{\sigma}) - \frac{G(2k-1)}{2c^2 r^3} \mathbf{p}_\delta \cdot (\mathbf{v} \times \boldsymbol{\sigma}) - \boldsymbol{\delta} \cdot \left(\frac{\mathbf{p}_\mathbf{r}}{\mu} - \boldsymbol{\lambda} \right) - \frac{Gm}{r^3} \mathbf{p}_\lambda \cdot \mathbf{r} - \frac{G}{c^2 \mu r^3} \mathbf{p}_\lambda \cdot [\mathbf{p}_\mathbf{r} \times (4\mathbf{S} + 3\boldsymbol{\sigma}) - \mathbf{r}[(\mathbf{r} \times \mathbf{p}_\mathbf{r}) \cdot (2\mathbf{S} + (1+k)\boldsymbol{\sigma})] - \frac{3(\mathbf{r} \cdot \mathbf{p}_\mathbf{r})}{r^2} \mathbf{r} \times (2\mathbf{S} + (2-k)\boldsymbol{\sigma})].$$

The Hamilton's equations are

$$\dot{\mathbf{p}}_\mathbf{r} = -\frac{\partial \mathcal{H}}{\partial \mathbf{r}}, \qquad \dot{\mathbf{p}}_\lambda = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}, \qquad \dot{\mathbf{p}}_\delta = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\delta}},$$

$$\dot{\mathbf{r}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}_\mathbf{r}}, \qquad \dot{\boldsymbol{\lambda}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}_\lambda}, \qquad \dot{\boldsymbol{\delta}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}_\delta}.$$

Example II.: elimination of the acceleration from the Lagrangian

With the help of the ***Double zero formalism*** (Barker & O'Connell 1980).

We can add some terms to the original Lagrangian in which the zeroth-order (Newtonian) equations of motions can be used

$$\mathcal{L}' = \mathcal{L} + Z_1 Z_2$$

$$\Rightarrow$$

$$0 = \frac{d\mathcal{L}}{d\mathbf{r}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v}} + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\mathbf{a}} + \\ + Z_2\frac{dZ_1}{d\mathbf{r}} - Z_2\frac{d}{dt}\frac{\partial Z_1}{\partial\mathbf{v}} + Z_2\frac{d^2}{dt^2}\frac{\partial Z_1}{\partial\mathbf{a}} \\ + Z_1\frac{dZ_2}{d\mathbf{r}} - Z_1\frac{d}{dt}\frac{\partial Z_2}{\partial\mathbf{v}} + Z_1\frac{d^2}{dt^2}\frac{\partial Z_2}{\partial\mathbf{a}}$$

Thus the leading order Euler-Lagrange equations are unchanged.

Disadvantage: there are zeroth-order conserved quantities in the Lagrangian:

$$\mathcal{L}'' = \frac{\mu}{2}\mathbf{v}^2 + \frac{Gm\mu}{r} + \frac{G\mu}{2c^2r^3}\mathbf{v}\cdot[\mathbf{r}\times(4\mathbf{S}_0 + 3\boldsymbol{\sigma}_0)] \\ - \frac{(2k-1)}{2c^2m}\left[\left(\frac{\mathbf{v}^2}{r^2} - \frac{Gm}{r^3} - \frac{2(\mathbf{v}\cdot\mathbf{r})^2}{r^4}\right)\boldsymbol{\sigma}_0\cdot(\mathbf{L}-2\mathbf{L}_0) \right. \\ \left. - \left(\frac{Gm}{r^5}(\mathbf{r}\cdot\mathbf{L}_0) + \frac{2(\mathbf{v}\cdot\mathbf{r})}{r^4}(\mathbf{v}\cdot\mathbf{L}_0)\right)(\boldsymbol{\sigma}_0\cdot\mathbf{r}) + \frac{(\mathbf{v}\cdot\mathbf{L}_0)}{r^2}(\boldsymbol{\sigma}_0\cdot\mathbf{v})\right].$$

where $\mathbf{L}_0$, $\mathbf{S}_0$ and $\boldsymbol{\sigma}_0$ are zeroth-order conserved quantities.

Generalized canonical dynamics

The generalized Legendre transformation is $\mathcal{H} = \mathbf{p}\cdot\mathbf{v} + \mathbf{q}\cdot\mathbf{a} - \mathcal{L}$, where **(p,r)** and **(v,q)** are canonical pairs (**p** and **q** are the canonical momenta)

$$\mathbf{p} = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}} - \dot{\mathbf{q}}, \qquad \mathbf{q} = \frac{\partial\mathcal{L}}{\partial\mathbf{a}}$$

The Hamiltonian is nontrivial because we do not know which canonical moment to use in the Legendre transformation. Solution ⇨

The Hamiltonian has to satisfy the generalized Hamilton's equations and it has to be consistent with the Newtonian limit of the Hamiltonian.

$$\mathcal{H}_N = \frac{\mathbf{p}^2}{2\mu} - \frac{Gm\mu}{r}$$

Thus the **generalized (Ostrogradski) Hamiltonian** is

$$\mathcal{H} = \frac{\mathbf{p}^2}{2\mu} - \frac{G\mu m}{r} + \frac{G}{2c^2r^3}\mathbf{r}\cdot[2\mathbf{p}\times[2\mathbf{S}+(2-k)\boldsymbol{\sigma}]+(2k-1)\mu\mathbf{v}\times\boldsymbol{\sigma}] - \frac{Gm}{r^3}\mathbf{q}\cdot\mathbf{r}$$
$$+\frac{G}{c^2\mu r^3}\mathbf{q}\cdot\left[\frac{3}{r^2}\mathbf{r}[(\mathbf{r}\times\mathbf{p})\cdot(2\mathbf{S}+(k+1)\boldsymbol{\sigma})] - \mathbf{p}\times(4\mathbf{S}+3\boldsymbol{\sigma}) + \frac{3(\mathbf{r}\cdot\mathbf{p})}{r^2}\mathbf{r}\times(2\mathbf{S}+(2-k)\boldsymbol{\sigma})\right],$$

Generalized Hamilton's equations:

$$\dot{\mathbf{p}} = -\frac{\partial\mathcal{H}}{\partial\mathbf{r}}, \qquad \dot{\mathbf{r}} = \frac{\partial\mathcal{H}}{\partial\mathbf{p}},$$
$$\dot{\mathbf{q}} = -\frac{\partial\mathcal{H}}{\partial\mathbf{v}}, \qquad \dot{\mathbf{v}} = \frac{\partial\mathcal{H}}{\partial\mathbf{q}},$$

Canonical structure

We introduce the generalized Poisson brackets with spin terms (Yang & Hirschfelder 1980)

$$\{f,g\} = \sum_{j,i=1}^{3,2}\left[\frac{\partial f}{\partial \mathbf{Q}_i^j}\frac{\partial g}{\partial \mathbf{P}_i^j} - \frac{\partial f}{\partial \mathbf{P}_i^j}\frac{\partial g}{\partial \mathbf{Q}_i^j} - \frac{\partial f}{\partial \mathbf{S}_i^j}\left(\mathbf{S}_i^j \times \frac{\partial g}{\partial \mathbf{S}_i^j}\right)\right]$$

where ***f*** and ***g*** depend on **canonical variables** (***P, Q***) and **spin variables** ($\boldsymbol{S_1},\boldsymbol{S_2}$).
and we introduced the useful vector notation: $\boldsymbol{Q_1}=\boldsymbol{r}$, $\boldsymbol{Q_2}=\boldsymbol{v}$ and $\boldsymbol{P_1}=\boldsymbol{p}$, $\boldsymbol{P_2}=\boldsymbol{q}$.
(the superscripts are the components of the vectors)
Thus the non-vanishing **fundamental Poisson brackets** are

$$\left\{P_i^l, Q_j^k\right\} = \delta_{ij}\delta_{lk},$$
$$\left\{S_i^l, S_j^k\right\} = \delta_{ij}\varepsilon_{lkm}S_i^m.$$

Thus the **time evolutions** are satisfied:

$$\dot{\mathbf{J}} = 0 \Leftrightarrow \begin{cases} \dot{\mathbf{S}}_i = \{\mathbf{S}_i, \mathcal{H}\} = \dfrac{G(4+3\nu_i)}{2c^2r^3}\mathbf{L}\times\mathbf{S}_i, \\ \dot{\mathbf{L}} = \{\mathbf{L}, \mathcal{H}\} = \dfrac{G}{2c^2r^3}(4\mathbf{S}+3\boldsymbol{\sigma})\times\mathbf{L}, \end{cases}$$

Corollary: The spin precession equations do not depend on SSC!

$$\begin{aligned}\dot{\mathbf{A}} = \{\mathbf{A}, \mathcal{H}\} &= \frac{G}{c^2r^3}[2\mathbf{S}+(2-k)\boldsymbol{\sigma}]\times\mathbf{A} \\ &+ \frac{3G}{c^2\mu r^5}(\mathbf{r}\times\mathbf{L})[2\mathbf{L}\cdot\mathbf{S}+(1+k)\mathbf{L}\cdot\boldsymbol{\sigma}] \\ &+ \frac{G(2k-1)}{c^2r^3}\left[\frac{\mu v^2}{2}\boldsymbol{\sigma}\times\mathbf{r}+(\mathbf{L}\cdot\boldsymbol{\sigma})\mathbf{v}\right].\end{aligned}$$

The evolution of the Newtonian LRL vector depends on SSC.

SO angular equation(s)

In general we have three Euler angles: φ, Θ, and Υ.

There are two different definitions of the ***inclination (Θ):*** $\Theta_N = \hat{\mathbf{J}} \cdot \hat{\mathbf{L}}_\mathbf{N}$ or $\Theta = \hat{\mathbf{J}} \cdot \hat{\mathbf{L}}$

SSC-dependent **SSC-independent**

$$\dot{\mathbf{L}}_\mathbf{N} = -\frac{\mu G}{c^2 r^3}\mathbf{r} \times [\mathbf{v} \times (4\mathbf{S} + 3\boldsymbol{\sigma})] + \frac{3\mu G \dot{r}}{c^2 r^4}\mathbf{r} \times [\mathbf{r} \times (2\mathbf{S} + (2-k)\boldsymbol{\sigma})]$$

SSC-dependent

$$\dot{\mathbf{L}} = \frac{G\mu}{2c^2 r^3}(4\mathbf{S} + 3\boldsymbol{\sigma}) \times \mathbf{L}$$

SSC-independent
(pure precession)

e.g.: $\dot{\Theta} = \frac{3G\mu\left(\nu^{-1} - \nu\right)}{2c^2 r^3 J}\mathbf{L} \cdot (\mathbf{S}_1 \times \mathbf{S}_2)$ where $\nu = m_2/m_1$

The evolution of the inclination ($\dot{\Theta}$ or $\dot{\Theta}_N$) is quadratic in spin $\boldsymbol{\sigma}(S^2)$

Then we only consider the linear corrections in spin $\boldsymbol{\sigma}(S^2)\approx 0$

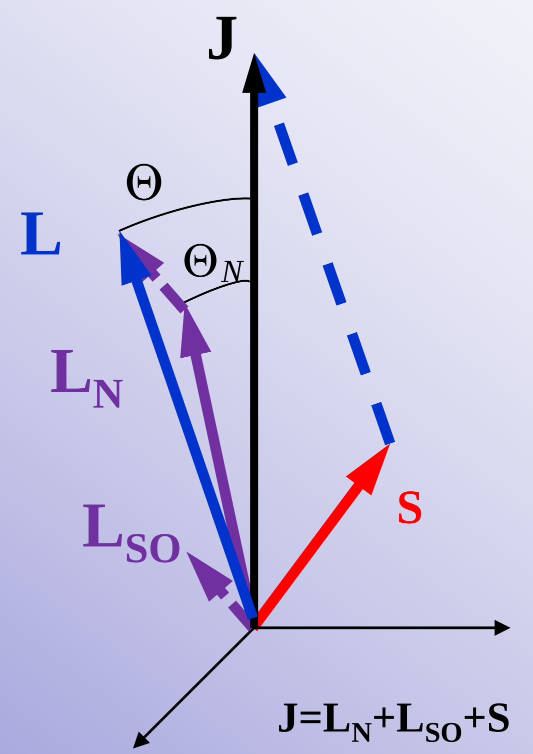
Thus we only have one angular equation

with

$$\dot{\varphi} = \frac{L}{\mu r^2}\left(1 - \frac{\lambda_{SO}}{L^2}\right) - \cos\Theta_N\,\dot{\Upsilon},$$

$\lambda_{SO} = \mathbf{L}_\mathbf{N} \cdot \mathbf{L}_\mathbf{SO}$

$$\dot{\Upsilon} = -\frac{\tan\varphi}{\sin^2\Theta_N}(\cos\Theta_N)^{\cdot} \quad]$$



Z. Keresztes, B Mikóczi, L. Á. Gergely and Vasúth J.Phys.Conf.Ser 2010

The orbital evolution of SO contribution

We neglected the precession of the orbital plane because the evolution of the angle is squared in magnitude of spin $o(S^2)\approx 0$. Thus we only have 2 equations i.e. the radial and the angular equations (φ is a azimuthal polar angle in orbital plane).

$$\dot r^2 = \dot r_N^2 + \frac{2G}{c^2\mu r^3}(2\mathbf{L}\cdot\mathbf{S} + (2-k)\mathbf{L}\cdot\boldsymbol{\sigma}) - \frac{2(2k-1)E}{c^2 m\mu^2 r^2}(\mathbf{L}\cdot\boldsymbol{\sigma})$$

$$\dot\varphi = \dot\varphi_N + \frac{G}{c^2 L r^3}(2\mathbf{L}\cdot\mathbf{S} + 3(1-k)\mathbf{L}\cdot\boldsymbol{\sigma}) - \frac{(2k-1)E}{c^2 m\mu L r^2}(\mathbf{L}\cdot\boldsymbol{\sigma})$$

where the Newtonian terms are:

$$\dot r_N^2 = \frac{2E}{\mu} + \frac{2Gm}{r} - \frac{L^2}{\mu^2 r^2}$$

$$\dot\varphi_N = \frac{L}{\mu r^2}$$

The radial equation can be integrated up to o(1/c²) order with the help of the generalized eccentric anomaly ***u*** parametrization.

$$n(t-t_0) = u - e_t \sin u$$

$$r = a_r(1 - e_r\cos u)$$

where the radial orbital parameters are

$$a_r = \frac{Gm\mu}{-2E} + \frac{G\mu}{c^2 L}[2\mathcal{S} + (2-k)\Sigma],$$

$$e_r^2 = 1 + \frac{2EL^2}{G^2m^2\mu^3} + \frac{4E}{c^2 mL}\left\{4\left[1 + \frac{EL^2}{G^2m^2\mu^3}\right]\mathcal{S} + \left[4 - 2k + \frac{(5-4k)EL^2}{G^2m^2\mu^3}\right]\Sigma\right\},$$

$$e_t^2 = 1 + \frac{2EL^2}{G^2m^2\mu^3} + \frac{4E}{c^2 mL}\left\{2\mathcal{S} + \left[2 - k + \frac{(1-2k)EL^2}{G^2m^2\mu^3}\right]\Sigma\right\}$$

$$n = \frac{1}{Gm}\left(\frac{-2E}{\mu}\right)^{3/2},$$

$$\mathcal{S} = \hat{\mathbf{L}}\cdot\mathbf{S} \text{ and } \Sigma = \hat{\mathbf{L}}\cdot\sigma$$

Corollary: The radial orbital parameters (a_r, e_r and e_t) depend on SSC.

The angular motion of SO contribution

The angular equation is $\dot{\varphi} = \frac{L}{\mu r^2} + \frac{G}{c^2 L r^3}(2\mathcal{S} + 3(1-k)\Sigma) - \frac{(2k-1)E}{c^2 m \mu L r^2}\Sigma$

with $\mathcal{S} = \hat{\mathbf{L}} \cdot \mathbf{S}$ and $\Sigma = \hat{\mathbf{L}} \cdot \sigma$

Solutions

I. Damour-Deruelle parametrization (v)

$$\varphi - \varphi_0 = (1+\tilde{k})v$$

with $\tan\frac{v}{2} = \sqrt{\frac{1+e_\theta}{1-e_\theta}}\tan\frac{u}{2}$

$$\tilde{k} = -\frac{G^2 m \mu^3 (4\mathcal{S} + 3\Sigma)}{c^2 L^3},$$

$$e_\theta^2 = 1 + \frac{2EL^2}{G^2 m^2 \mu^3} + \left(1 + \frac{EL^2}{G^2 m^2 \mu^3}\right)\frac{4E(4\mathcal{S}+3\Sigma)}{c^2 m L}.$$

The angular parameters do not depend on parameter *k*.

II. Generalized true anomaly parametrization (χ)

$$\varphi - \varphi_0 = K\chi - Q\sin\chi$$

with $\tan\frac{\chi}{2} = \sqrt{\frac{1+e_r}{1-e_r}}\tan\frac{u}{2}$

$$K = 1 - \frac{G^2 m \mu^3 (4\mathcal{S} + 3\Sigma)}{c^2 L^3},$$

$$Q = \frac{G\mu^2(2k-1)A\Sigma}{c^2 L^3}.$$

The parameter Q depends on parameter *k*.

The relationships between quantities for the angular motion are

$$K = 1 + \tilde{k},$$

$$Q = \frac{Gm\mu^2 A}{2EL^2}\left(1 - \frac{e_r}{e_\theta}\right)$$

The energy and the orbital angular momentum losses

We have to calculate the mass and current quadrupole moments:

$$\mathcal{I}_{ij} = \mu(r_i r_j)^{STF} + \frac{2\mu}{3c^2 m}(\varepsilon_{ipq}[(1+3k)x_j v_p - 2v_j x_p]\boldsymbol{\sigma}_q)^{STF},$$

SSC-dependent

$$\mathcal{J}_{ij} = -\mu\frac{\delta m}{m}(\varepsilon_{ipq}x_j x_p v_q)^{STF} + \frac{3\mu}{2\delta m}(x_i[S_j - \boldsymbol{\sigma}_j])^{STF},$$

SSC-independent

STF- part of symmetric trace-free

The losses are up to SO-order:

$$\frac{dE}{dt} = -\frac{G}{5c^5}\left(\dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}_{ij} + \frac{16}{9c^2}\dddot{\mathcal{J}}_{ij}\dddot{\mathcal{J}}_{ij}\right),$$

$$\frac{dL}{dt} = -\frac{2G}{5c^5}\varepsilon_{ipq}\left(\ddot{\mathcal{J}}_{pj}\dddot{\mathcal{I}}_{qj} + \frac{16}{9c^2}\ddot{\mathcal{J}}_{pj}\dddot{\mathcal{J}}_{qj}\right)\hat{L}_i,$$

Newtonian terms

$$\frac{dE}{dt} = \frac{8G^3m^2\mu^2}{15c^5r^4}(11\dot{r}^2 - 12v^2) - \frac{8G^3m\mu L}{15c^7r^6}\left[27\dot{r}^2 - 37v^2 - 12\frac{Gm}{r}\right]\mathcal{S} - \frac{8G^3m\mu L}{15c^7r^6}\left[3(22k-5)\dot{r}^2 - (48k-5)v^2 + 4(6k-5)\frac{Gm}{r}\right]\Sigma,$$

$$\frac{dL}{dt} = \frac{8G^2m\mu L}{5c^5r^5}\left(3\dot{r}^2 - 2v^2 - \frac{2Gm}{r}\right) + \frac{12G^2\mu^2}{45c^7r^7}\left[6(3\dot{r}^2v^2 - 4\dot{r}^4 + v^4) - 26\frac{Gm}{r}(\dot{r}^2 - v^2) - 6\frac{G^2m^2}{r^2}\right]\mathcal{S} + \frac{12G^2\mu^2}{45c^7r^7}\left[6(16-21k)\dot{r}^2v^2 - (78-90k)\dot{r}^4 - (17-36k)v^4 + \frac{Gm}{r}[(7-24k)\dot{r}^2 - 8(1-3k)v^2] - 5\frac{G^2m^2}{r^2}\right]\Sigma,$$

with

$$\mathcal{S} = \hat{\mathbf{L}}\cdot\mathbf{S} \text{ and } \Sigma = \hat{\mathbf{L}}\cdot\sigma$$

The averaged energy and angular momentum losses

It can be proved that, the averaged energy and angular momentum losses for one Newtonian orbital period, the explicit k-dependence disappears, but *E* and *L* depend on SSC.

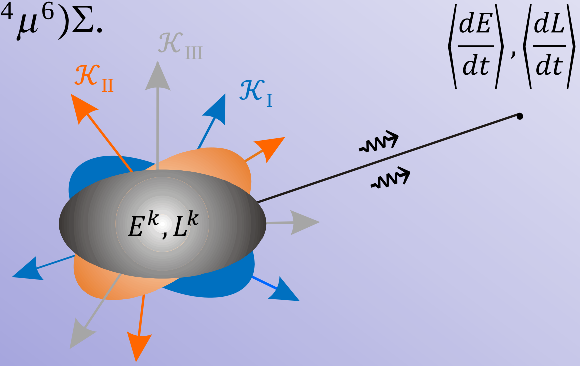
$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{G^2 m(-2E\mu)^{3/2}}{15c^5L^7}(148E^2L^4 + 732G^2m^2\mu^3EL + 425G^4m^4\mu^6)$$
$$+\frac{G^2(-2E\mu)^{3/2}}{10c^7L^9}[(520E^3L^6 + 10740G^2m^2\mu^3E^2L^4 + 24990G^4m^4\mu^6EL^2 + 12579G^6m^6\mu^9)]\mathcal{S}$$
$$+\frac{G^2(-2E\mu)^{3/2}}{10c^7L^9}(256E^3L^6 + 6660G^2m^2\mu^3E^2L^4 + 16660G^4m^4\mu^6EL^2 + 8673G^6m^6\mu^9)\Sigma,$$

$$\left\langle \frac{dL}{dt} \right\rangle = -\frac{4G^2 m(-2E\mu)^{3/2}}{5c^5L^4}(14EL^2 + 15G^2m^2\mu^3)$$
$$+\frac{G^2(-2E\mu)^{3/2}}{15c^7L^6}(1188E^2L^4 + 6756G^2m^2\mu^3EL^2 + 5345G^4m^4\mu^6)\mathcal{S}$$
$$+\frac{G^2(-2E\mu)^{3/2}}{15c^7L^6}(772E^2L^4 + 4476G^2m^2\mu^3EL^2 + 3665G^4m^4\mu^6)\Sigma.$$

It agrees with the results of

R. Rieth and G. Schäfer, CQG 14,2357 (1997). for the **SSC II**,
L. Á. Gergely, Z. Perjés, and M. Vasúth, PRD 58, 124001 (1998). for the **SSC I**.

$$\langle X \rangle = \frac{1}{T}\int_0^T X dt$$



The gravitational waveform

The gravitational waveform is $h_{ij} = \frac{2G}{c^4 D}\left[\ddot{I}_{ij} + \frac{4}{3c^2}\varepsilon_{kl(i}\ddot{\mathcal{J}}_{j)k}N_l + \frac{1}{2c^2}\varepsilon_{kl(i}\dddot{\mathcal{J}}_{j)km}N_l N_m\right]_{TT}$

$$\mathcal{J}_{ijk} = \mu(1-3\eta)(r_i r_j \varepsilon_{kpq} r_p v_q)^{STF} + 2\eta(r_i r_j \sigma_k)^{STF}$$

where „TT" means the part of „transverse traceless"

The mass octupole moment doesn't depend on SSC.

Our result is $h_{ij} = \frac{2G\mu}{c^4 D}\left[Q_{ij} + PQ_{ij}^{SO} + P^{1.5}Q_{ij}^{SO}\right]_{TT}$ (We neglected the PN corrections)

where $Q_{ij} = 2\left(v_i v_j - \frac{Gm}{r^3} r_i r_j\right),$ Newtonian terms

$$PQ_{ij}^{SO} = \frac{2m}{r^3 \delta m}[(\boldsymbol{\sigma}-\mathbf{S})\times\mathbf{N}]_{(i} r_{j)},$$ Leading-order SO

SSC-independent

$$P^{1.5}Q_{ij}^{SO} = \frac{2}{r^3}\left\{\frac{3r_i r_j}{r^2}(\mathbf{r}\times\mathbf{v})\cdot[2\mathbf{S} + (1+k)\boldsymbol{\sigma}] - r_{(i}[\mathbf{v}\times(4\mathbf{S} + (3+2k)\boldsymbol{\sigma})]_{j)} - 2kv_{(i}(\mathbf{r}\times\boldsymbol{\sigma})_{j)} + \frac{6\dot{r}}{r} r_{(i}[\mathbf{r}\times(\mathbf{S}+\boldsymbol{\sigma})]_{j)} + \left[\left(\frac{3\dot{r}}{r}\mathbf{r} - 2\mathbf{v}\right)(\mathbf{N}\cdot\mathbf{r}) - 2\mathbf{r}(\mathbf{N}\cdot\mathbf{v})\right]_{(i}(\boldsymbol{\sigma}\times\mathbf{N})_{j)}\right\},$$

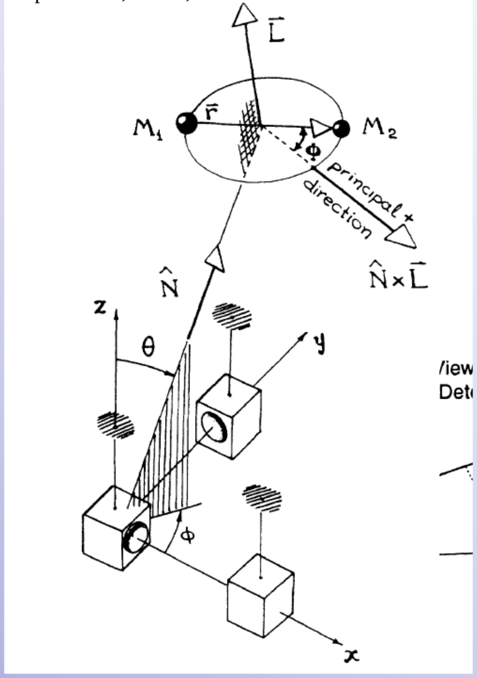
Next-leading-order SO

SSC-dependent!

with $\delta m = m_1 - m_2$

It agrees with the Kidder's result for the SSC I

Apostolatos, Cutler, Sussman and Thorne 1994



N is an unit vector which points from the source to the observer (*line of sight*)

Summary

- We presented the **spin supplementary conditions** for the leading order spin-orbit interaction of the compact binaries.
- The Lagrangian contains **acceleration-dependent** terms in some cases of SSC.
- We calculated the generalized (Ostrogradski) Hamiltonian for all cases and extended the canonical structure with the generalized Poisson brackets.
- The **radial and angular motion** of the compact binaries represent the SSC dependence of any orbital parameter for eccentric orbits.
- We calculated the energy and the orbital angular momentum losses due to gravitational radiation in each SSC, and we concluded that **the dependence on SSC apparently disappears since we used averaging over one orbital period**. However, these expressions are SSC-dependent because the E and the L depend on SSC.
- It has been proved **that the leading-order SO waveform does not depend on SSC, but the next-to-leading-order SO contribution does**.

$$h_{ij} = \frac{2G\mu}{c^4 D}\left[Q_{ij} + \boxed{PQ_{ij}^{SO}} + \boxed{P^{1.5}Q_{ij}^{SO}}\right]_{TT}$$

SSC-independent **SSC-dependent**

Thank you for your attention!