Coupled Currents in Cosmic Strings

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Outline

- 1. How do cosmic strings arise?
- 2. Some numbers on cosmic strings.
- 3. How do cosmic strings evolve?
- 4. Cosmic strings in field theory.
- 5. Currents in cosmic strings.
- 6. Cosmic strings in the worldsheet approach.
- 7. Coupled currents in cosmic strings.
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How do cosmic strings arise? (Kibble, '81)

Universe expands from a hot thermal local equilibrium state

It is assumed that at high energy, interactions are unified (enhanced symmetry)

With the cooling, spontaneous symmetry breaking occurs

This is the Higgs mechanism

For a field with a local $U(1)$ gauge symmetry Φ, the effective potential at $T = 0$ is

$$V(\Phi) = \frac{\lambda}{8}\left(|\Phi|^2 - \eta^2\right)^2$$

On distance larger than the horizon scale (causally disconnected), the phase (χ) choice is uncorrelated

The phase is **single-valued** and a **continuous** function:

On a closed path, $\Delta\chi = 2\pi n$

$\Rightarrow$ if $n \neq 0$, there exists a **singular point** within the closed path: $\Phi = 0$

$\Rightarrow$ strings are either infinite in length or loops

Some facts on cosmic strings

Correlation length: $\xi \sim m_h^{-1}$,

String loop radius: m_h^{-1}

Energy per unit length: $U \sim \eta^2$

For grand unification scales: 10^{15} tons/cm

Number density: $n_b \sim \xi^{-3}$

80% of strings at formation are of length (radius) > horizon
$\Rightarrow$ At first they grow!

Probability for intercommutation is of order 1

Generates a population of loops

Gravitational energy loss

Shrink under the effect of string tension

(note: stabilizing mechanisms exist, c. f. , later)

Cosmological Considerations

The spectrum of primordial fluctuations for cosmic strings disagrees with CMB data (Durrer *et al.*, '96, Bouchet *et al.*, '01)

SUSY D-term inflation models ⇒ cosmic "D-term" strings (Jeannerot *et al.* '03)

Tension between particle physics model building and cosmological data
⇒ Constraints

Renewed interest:

Brane inflationary model building

D-term and F strings produced at the end of brane inflation

Conjecture: ⇔ D-strings (Dvali *et al.*, '04)

⇒ Distinction: topological vs. string theory strings

Nambu-Goto Strings I

Abelian Higgs model (Nielsen & Olesen, '73):

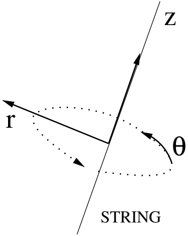
$$\mathcal{L}_{\rm H} = -D_\mu \phi^* D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda_\phi}{2} \left(|\phi|^2 - \eta^2\right)^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu = \partial_\mu + ieA_\mu$$

In a straight and static string with cylindrical symmetry:

$$\phi = \varphi(r) e^{i\chi} \text{ (with } \chi \text{ some angle)}$$

$$A_\mu = \frac{1}{e^2} \frac{j_\mu}{|\phi|^2} - \frac{1}{e} \partial_\mu \chi$$



Electromagnetic flux (⇒ stability against relaxation to $|\eta|$):

$$\Phi = \int F_{\mu\nu} \mathrm{d}\sigma^{\mu\nu} = \oint_\ell A_\mu \mathrm{d}x^\mu = -\oint_\ell \frac{\partial_\mu \chi}{e} \mathrm{d}x^\mu = -\frac{2\pi n}{e}$$

Nambu-Goto Strings II

We assume that

$$\chi = n\theta \Rightarrow \phi = \varphi(r)e^{in\theta}$$

We obtain a flux tube:

$$A_\mu = A_\theta(r)\delta_{\mu\theta} \quad \Rightarrow \quad F_{\mu\nu} = \partial_r A_\theta(r)\delta_{\mu r}\delta_{\nu\theta}$$

In the worldsheet description:

Energy-momentum tensor in flat spacetime in the thin string limit (Peter, 1994)

$$\overline{T}^{\mu\nu} = \int T^{\mu\nu} \mathrm{d}^2 x^\perp \quad \Rightarrow \quad \overline{T}_{tt}, \ \overline{T}_{zz}$$

Nambu-Goto action:

$$\mathcal{S} = \alpha \int \mathrm{d}^2\xi\sqrt{-\gamma}, \quad \mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \gamma_{ab}\mathrm{d}\xi^a \mathrm{d}\xi^b \quad (a = 0, 1)$$

Straight and static configuration, $\xi^0 = t$ and $\xi^1 = z$:

$$\boxed{\overline{T}_{tt} = \overline{T}_{zz} \Rightarrow U = T}$$

Superconducting Strings (Witten, '85)

Couple the string to a field Σ with $U(1)^{\mathrm{local}}$:

$$\mathcal{L} = \mathcal{L}_{\mathrm{H}} - D_\mu \Sigma^* D^\mu \Sigma - \frac{1}{4} H_{\mu\nu} H^{\mu\nu} - V(\Sigma, \phi)$$

$$V(\Sigma, \phi) = \beta_\sigma \left(|\phi|^2 - \eta^2 \right) |\Sigma|^2 + m^2 \Sigma^2 + \frac{\lambda}{2} |\Sigma|^4$$

$$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad D_\mu = \partial_\mu + iqB_\mu$$

Consider a of string with **local** cylindrical symmetry:

$$\Sigma = \sigma(r) e^{i\psi(t,z)}, \quad j_\mu = \sigma^2 (\nabla_\mu \psi + qB_\mu), \quad \psi(t, z) = \omega t - kz$$

$$\boxed{\oint \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} \mathrm{d}\zeta = 2\pi m \quad \Rightarrow \quad \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = \frac{m}{R}}$$

If $m \neq 0$, then the string is superconducting...

Also note:

$$\boxed{U \neq T}$$

Neutral strings (Peter, '92)

Key effect of currents on the string is mechanical (Davis & Shellard, '88)

We can consider two frames (Carter, '90):

- a rotating frame ($\overline{T}_{\mu\nu}$ diagonal)
- a local stationary frame in which we have $J = 2\pi R^2 \mathcal{T}^{01}$

$\Rightarrow$ Equilibrium configurations for $R^2\Omega^2 = T/U$ (Ω: ang. velocity)

Are they **really** stable?

Witten string model is not stricly local:

- the gauge field B_μ diverges logarithmically with r,
- leads to divergent integrals,
- necessitates the introduction of characteristic and cut-off scales.

Neutral limit:

- coupling constant $q \rightarrow 0$,
- mechanical properties of the string well reproduced!

A choice of field theory fixes all parameters expect k and ω
Can define

$$\overline{\mathcal{L}}(w) = \int \mathcal{L}\,\mathrm{d}^2 x^{\perp}, \qquad w = k^2 - \omega^2 = g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi$$

where w is the **only** possible Lorentz scalar.

Introduce the first fundamental tensor:

$$\boxed{\eta^{\mu\nu} = h^{ab}\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}}$$

If $w > 0$ (spacelike), $\sqrt{w}v^\mu = \eta^{\mu\nu}\nabla_\nu\psi$ (magnetic string), $v^\mu v_\mu = 1$

If $w < 0$ (timelike), $\sqrt{-w}u^\mu = \eta^{\mu\nu}\nabla_\nu\psi$ (electric string), $u^\mu u_\mu = -1$

Define preferred frame:

$$\boxed{\eta^{\mu\nu} = -u^\mu u^\nu + v^\mu v^\nu}$$

$\Rightarrow \overline{T}_{\mu\nu}$ diagonal

Properties of the worldsheet & of the currents (Carter, '89)

Definitions

Orthogonal projector $\perp^\mu_\nu = g^\mu_\nu - \eta^\mu_\nu$ Worldsheet derivative $\overline{\nabla}_\mu = \eta^\nu_\mu \nabla_\nu$

Antisymmetric tensor $\varepsilon^{\mu\rho}\varepsilon^\nu_\rho = \eta^{\mu\nu}$ Extrinsic curvature $K^\rho_{\mu\nu} = \eta^\sigma_\mu \overline{\nabla}_\nu \eta^\rho_\sigma$

Conservation equations:

Integrability of the worldsheet: $K^\rho_{[\mu\nu]} = 0$

Conservation of $\overline{T}^{\mu\nu}$: $\overline{\nabla}_\mu \overline{T}^{\mu\nu} = 0$

Current conservation: $\overline{\nabla}_\mu c^\mu = 0$

Irrot. : $c^\mu \propto \overline{\nabla}^\mu \psi \Leftrightarrow \varepsilon^{\mu\nu}\overline{\nabla}_\mu(\overline{\nabla}_\nu \psi) = 0$

Note: The geometry of the string worldsheet decouples from its internal dynamics for small perturbations

$\Rightarrow$ split conservation equations in $\perp$ and $\parallel$ components

Stability

The idea is to understand if the straight and static string is stable under small perturbations to the currents

Perturbed current:

$$\delta\left(c_\mu\right) = \delta c_\mu e^{i\chi}, \quad \overline{\nabla}_\mu \chi = k_\mu, \quad \omega = k_\mu u^\mu, \quad k = -k_\mu v^\mu$$

Perturbations do not grow if $\chi \in \mathbb{R}$

Transverse perturbations ($\perp_\mu^\nu$)

$$c_{\mathrm{T}}^2 = \frac{\omega_{\mathrm{T}}^2}{k_{\mathrm{T}}^2} = \frac{T}{U} \Rightarrow T > 0$$

Longitudinal perturbations (η_μ^ν)

$$c_{\mathrm{L}}^2 = \frac{\omega_{\mathrm{L}}^2}{k_{\mathrm{L}}^2} = -\frac{dT}{dU} \Rightarrow \frac{\mathrm{d}T}{\mathrm{d}U} < 0$$

Electric/magnetic Duality (Carter, '89)

Define: $$\boxed{\mathcal{K} = -2\frac{\mathrm{d}\overline{\mathcal{L}}}{\mathrm{d}w} = -\int \sigma(r)^2 \mathrm{d}x^{\perp}}$$

Then:

$$\overline{T}^{tt} = \omega^2 \int \sigma(r)^2 \mathrm{d}^{\perp}x - \overline{\mathcal{L}}, \qquad \overline{T}^{zz} = k^2 \int \sigma(r)^2 \mathrm{d}^{\perp}x + \overline{\mathcal{L}}$$

If $w < 0$ (electric) $\Rightarrow$ $w = -\omega^2$ and $k = 0$:

$$\overline{T}^{tt} = -w\mathcal{K} - \overline{\mathcal{L}} = -\tilde{\mathcal{L}} = U, \qquad \overline{T}^{zz} = \overline{\mathcal{L}} = -T$$

If $w > 0$ (magnetic) $\Rightarrow$ $w = k^2$ and $\omega = 0$:

$$\overline{T}^{zz} = w\mathcal{K} + \overline{\mathcal{L}} = \tilde{\mathcal{L}} = -T, \qquad \overline{T}^{tt} = -\overline{\mathcal{L}} = U$$

In short:

$$\boxed{T = -\overline{\mathcal{L}}, \quad U = -\tilde{\mathcal{L}} \quad (w < 0)} \qquad \boxed{T = -\tilde{\mathcal{L}}, \quad U = -\overline{\mathcal{L}} \quad (w > 0)}$$

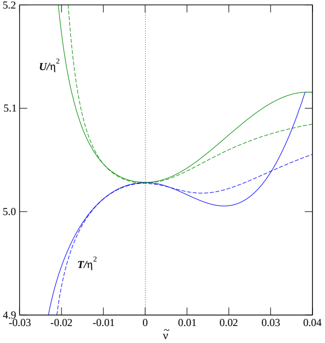
$$\Rightarrow \quad U - T = |w|\mathcal{K}$$

$\Rightarrow$ Simple expressions for c_T and c_L

Electric ($w < 0$)

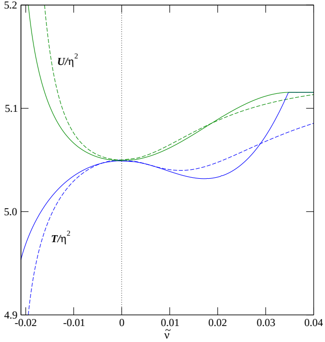
FT indicates 1st order pôle in $\mathcal{K}$ (Phase frequency threshold, $w = -m^2$)

$$\overline{\mathcal{L}}(w) = -{m_h}^2 - \frac{m^2}{2} \ln\left(1 + \frac{w}{m^2}\right)$$



Magnetic ($w > 0$)

$$\overline{\mathcal{L}}(w) = -{m_h}^2 - \frac{w}{2}\left(1 + \frac{w}{m^2}\right)^{-1}$$



Can we extend the formalism to more than one condensate?

What are the similarities and differences with the one condensate case?

Does it lead to new instabilities? What is the effect of the

coupling between the N additional fields?

Can the numerical results be reproduced with an analytic model?

Microscopic physics for N=2 currents II

Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\left(D_\mu H\right)^\dagger\left(D_\mu H\right) - \frac{1}{4}C_{\mu\nu}C^{\mu\nu} - \frac{1}{2}\partial_\mu\Phi^\star\partial^\mu\Phi - \frac{1}{2}\partial_\mu\Sigma^\star\partial^\mu\Sigma - V$$

$$D_\mu H = (\partial_\mu + iqC_\mu)H \qquad \text{and} \qquad C_{\mu\nu} \equiv \partial_\mu C_\nu - \partial_\nu C_\mu$$

Potential:

$$\begin{aligned} V \quad = \quad & \frac{\lambda}{8}(|H|^2 - \eta^2)^2 + \frac{1}{2}m_\phi^2|\Phi|^2 + \frac{1}{2}m_\sigma^2|\Sigma|^2 + \\ & \frac{1}{2}\left(|H|^2 - \eta^2\right)\left(f_\phi|\Phi|^2 + f_\sigma|\Sigma|^2\right) + \\ & \frac{1}{4}\lambda_\phi|\Phi|^4 + \frac{1}{4}\lambda_\sigma|\Sigma|^4 + \frac{g}{2}|\Phi|^2|\Sigma|^2 \end{aligned}$$

Ansatze, $\overline{\mathcal{L}}$, $\overline{T}^{ab}$, U, T and state parameters

Ansatze:

$$H(x^\alpha) = h(r)\mathrm{e}^{in\theta}, \qquad C_\mu(x^\alpha) = \frac{Q(r) - n}{q}\delta_\mu^\theta$$

$$\Phi(x^\alpha) = \phi(r)\mathrm{e}^{i\psi_\phi}, \qquad \Sigma(x^\alpha) = \sigma(r)\mathrm{e}^{i\psi_\sigma}$$

$\overline{\mathcal{L}}$ and $\mathcal{K}$:

$$\mathcal{L} \to \overline{\mathcal{L}} = \overline{\mathcal{L}}(w_\phi, w_\sigma)$$

$$\mathcal{K}_i = 2\frac{\mathrm{d}\overline{\mathcal{L}}}{\mathrm{d}w_i}$$

3 L-I state parameters:

$$w_\sigma = k_\sigma^2 - \omega_\sigma^2$$
$$w_\sigma = k_\phi^2 - \omega_\phi^2$$
$$\boxed{x = k_\sigma k_\phi - \omega_\sigma\omega_\phi}$$

Diagonalize $\overline{T}^{\mu\nu}$:

$$U = A + B$$
$$T = A - B$$

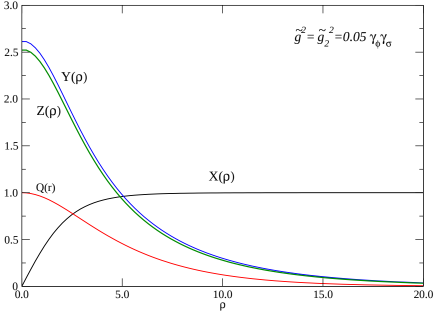
$$A: \ A\left(w_\phi, w_\sigma, x, \mathcal{K}_\phi, \mathcal{K}_\sigma\right)$$
$$B: \ B\left(w_\phi, w_\sigma, x, \mathcal{K}_\phi, \mathcal{K}_\sigma\right)$$

Field theory side: field profiles

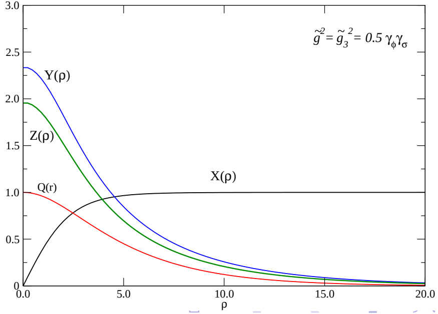
Results of numerical relaxation method for $w_\sigma = w_\phi = 0$

- **FT physics independent of** x
- fix microscopic parameters
- choose w_i's
- switch to dimensionless variables X, Y, Z, Q

weak coupling



strong coupling

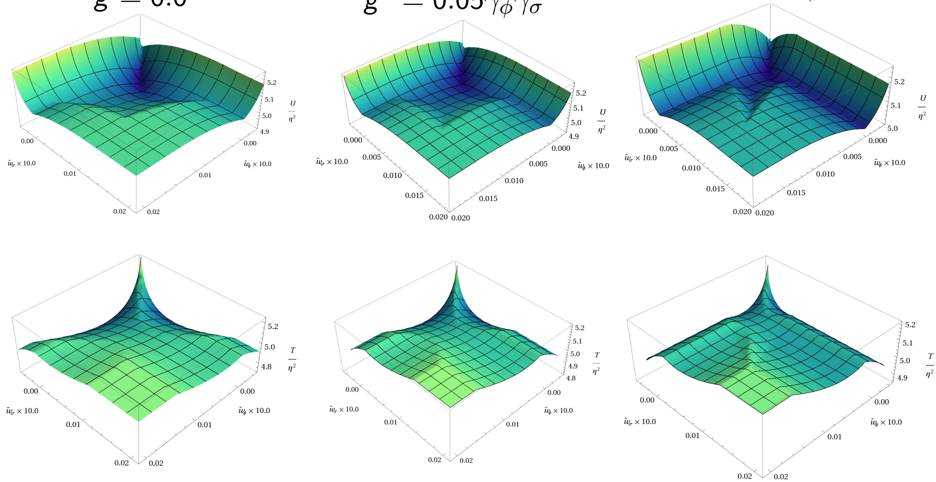


Energy and tension

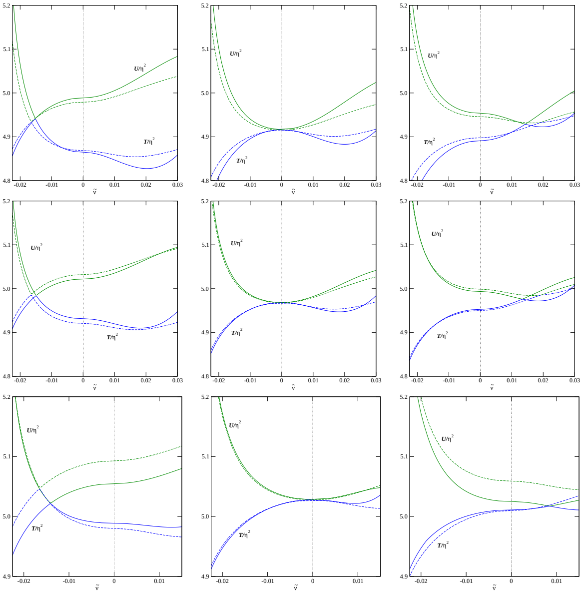
no coupling
$\tilde{g} = 0.0$

weak coupling
$\tilde{g}^2 = 0.05\gamma_\phi\gamma_\sigma$

strong coupling
$\tilde{g}^2 = 0.5\gamma_\phi\gamma_\sigma$



Comparison with simplified model



Worldsheet formalism for N condensates

There are $2N+4$ degrees of freedom

Conservation of $\overline{T}^{\mu\nu}$: 4 equations

Integrability (transverse): 2 equations

Current conservation/irrotationality: $2N-2$ equations

$\parallel$ equations

Current Conservation (N-1):

$$\overline{\nabla}_\mu \left(\frac{\delta \overline{\mathcal{L}}}{\delta(\overline{\nabla}_\mu \varphi_i)} \right) = 0$$

Irrotationality (N-1):

$$\epsilon^{\mu\nu} \overline{\nabla}_\mu \left(\overline{\nabla}_\nu \psi_i \right) = 0$$

$\parallel$ $\overline{T}^{\mu\nu}$ (2):

$$\eta^\rho_\nu \overline{\nabla}_\mu \overline{T}^{\mu\nu} = 0$$

$\perp$ equations (Identical to $N=1$)

Integrability condition (2):

$$K^\rho_{[\mu\nu]} = 0 \text{ for } (u_\mu, v_\mu)$$

$\perp$ $\overline{T}^{\mu\nu}$ (2):

$$\perp^\rho_\nu \overline{\nabla}_\mu \overline{T}^{\mu\nu} = 0$$

Reference frame

No preferred reference frame in the sense of the single current case

If one current is made colinear w/ u^μ or v^μ, the other is **not**!

$\Rightarrow$ Conditions for stability are *a priori* different for $N > 1$

Project the currents onto the lightlike directions:

- Places currents on an equal footing,
- Is a convenient choice.

Introduce

$$e^+_{i\mu} = \frac{1}{2}\left[d_{i\mu} - (-1)^i \varepsilon^\nu_\mu d_{i\nu}\right]$$

$$e^-_{i\mu} = \frac{1}{2w_i}\left[d_{i\mu} + (-1)^i \varepsilon^\nu_\mu d_{i\nu}\right]$$

with $d_{i\mu} = \varepsilon^{\mu\nu}\overline{\nabla}_\nu \psi_i$, the most meaningful quantity

Note: $4e^\pm_i$ for 2 lightlike directions:

$\Rightarrow$ provides a convenient way to relate the currents

Stability for $N = 2$ currents

Reminder: the perturbation equation for transverse perturbations decouples from perturbations in the currents

$\perp$ perturbations

$$c_{\mathrm{T}}^2 = \frac{\omega_{\mathrm{T}}^2}{k_{\mathrm{T}}^2} = \frac{T}{U} \Rightarrow T > 0$$

$\perp$ microscopic condition!

$$\boxed{x^2 \leq \frac{1}{\mathcal{K}_\phi \mathcal{K}_\sigma}\left[A^2 - \left(\frac{1}{2} w_\phi \mathcal{K}_\phi - \frac{1}{2} w_\sigma \mathcal{K}_\sigma\right)^2\right]}$$

$\parallel$ perturbations

$$\overline{\nabla}_\mu \left(\mathcal{K}^{ij} \overline{\nabla}^\mu \psi_j\right) = 0$$
$$\epsilon^{\mu\nu} \overline{\nabla}_\mu \left(\overline{\nabla}_\nu \psi_i\right) = 0$$

$\Rightarrow$ Set of 4 coupled equations in the currents and the perturbations

$\parallel$ stability

Determinant:

$$c_0 + c_1 \xi + c_2 \xi^2 + c_3 \xi^3 + c_4 \xi^4 = 0$$

$$\xi = \frac{-\zeta e_{1t}^- + e_{1z}^-}{-\zeta e_{2t}^- + e_{2z}^-}$$

$$\boxed{\zeta^2 = \frac{\omega_t^2}{k_z^2} \neq -\frac{\mathrm{d}T}{\mathrm{d}U}}$$

Stability of simplified model

$\overline{\mathcal{L}}$ for decoupled fields:

$$\overline{\mathcal{L}} = \overline{\mathcal{L}}_1(\chi_1) + \overline{\mathcal{L}}_2(\chi_2) + m^2$$

$\parallel$ stability:

Decoupling of longitudinal modes $\Rightarrow -m_i^2 \leq w_i \leq m_i^2$

$\perp$ stability:

$$T > 0 \Rightarrow x^2 \leq x_{\mathrm{lim}}$$

$w_1 w_2 \geq 0$

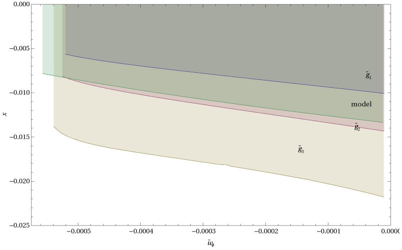
$$x_{\mathrm{lim}}^2 = \frac{w_1 w_2 (U_1 - m^2 + T_2)(U_2 - m^2 + T_1)}{(U_1 - T_1)(U_2 - T_2)}$$

$w_1 w_2 \leq 0$

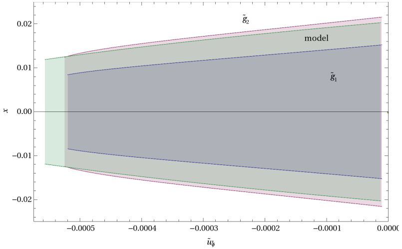
$$x_{\mathrm{lim}}^2 = \frac{|w_1 w_2| \left(T_1 + T_2 - m^2\right)\left(U_1 + U_2 - m^2\right)}{(U_1 - T_1)(U_2 - T_2)}$$

Stability: Simplified model vs. numerical result

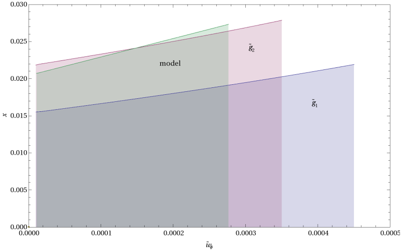
$w_1 < 0,\ w_2 < 0$



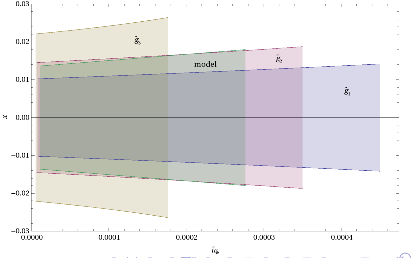
$w_1 < 0,\ w_2 > 0$



$w_1 > 0,\ w_2 > 0$



$w_1 > 0,\ w_2 < 0$



Conclusions

- Have investigated of the simplest extension of the Witten neutral current-carrying string
- Extented the microscopic description to the case of N=2 currents
- Three rather than two state parameters are necessary to describe the macroscopic dynamics
- Have determined the range of parameter space in which two simultaneous currents are present
- Extended Carter's macroscopic formalism to a string carrying N currents
- Have found no new type of instability at first order in perturbations
- Have shown that the perturbation equations at first order couple to the perturbed field to all other currents

Conclusions

- Have shown that the stability of transverse modes has a direct interpretation in terms of the scalar product of the phase gradients (*i.e.*, in terms of x)
- Have investigated the possibility to reproduce the results using a simplified model based on a log-fuction for $w < 0$ and a rational function for $w > 0$ for different values of the coupling g.
- The results are consistent with expectations
  1. U and T are best reproduced at weak coupling or when one current largely dominates
  2. Reproducing the stability in the 0 and weak coupling regime is okay
  3. Reproducing the stability in the strong coupling regime fails to capture much of the physics