New Cosmological Solutions in Massive Gravity

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$c = \hbar = 1$

Contents

- **Introduction**

  History of Massive Gravity

  (Excellent review, Hinterbichler 1105.3735)

  Basics of Massive Gravity

  Absence of Flat Friedmann Universe
- **Painleve-Gullstrand metric**
- **Cosmological Solution**

  Friedmann Universe
- **Inhomogeneous spherical collapse of dust**
- **Summary**

Introduction

(Theoretical) motivation of Massive gravity

- **General relativity (GR)**

  The theory of an interacting massless helicity 2 particle

  ↓

- **Massive gravity**

  A theory of an interacting massive spin 2 particle

What is it ?

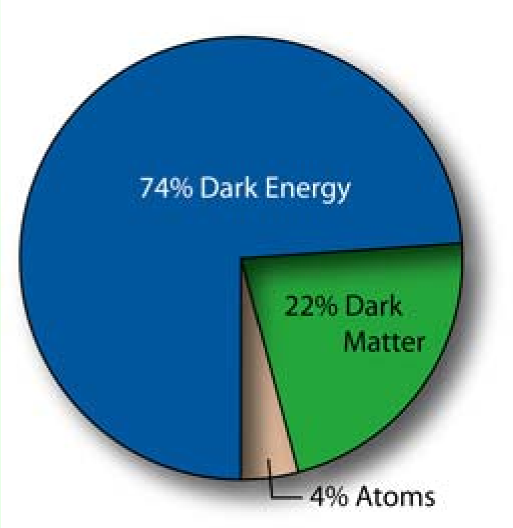
(Observational) motivation of Massive gravity

The Universe is now accelerating !!

- **Dark Energy is introduced**

  or

- **GR may be modified in the IR limit**



WMAP

One possibility ⇓ **Massive gravity**

If the graviton has a mass comparable to the present Hubble scale, gravity is suppressed beyond that scale.

➔ the present Universe looks accelerating.

(Long) History of Massive Gravity

Fierz and Pauli theory of Massive gravity

$$S=\frac{1}{2\kappa^2}\int d^4x\left[\underbrace{-\frac{1}{4}h_{\mu\nu,\lambda}h^{\mu\nu,\lambda}+\frac{1}{2}h_{\nu\lambda,\mu}h^{\mu\lambda,\nu}-\frac{1}{2}h_{,\nu}h^{\mu\nu,\mu}+\frac{1}{2}h_{,\lambda}h^{,\lambda}}_{\mathbf{R}}\underbrace{-\frac{1}{4}m^2\left(h_{\mu\nu}h^{\mu\nu}-h^2\right)}_{\text{graviton mass term}}\right].$$

$$\left(h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu},\ \ h=h^{\mu}_{\mu}\right).$$

- **Mass term :** $a_1h^2+a_2h_{\mu\nu}h^{\mu\nu}$

$\Rightarrow$ $\mathcal{L}\supset\frac{(a_1+a_2)^2}{3a_2^2}\left(h_{,\mu}h^{,\mu}+m_0^2h^2\right),\ \ m_0^2=\frac{2a_2(4a_1+a_2)}{a_1+a_2}.$

(spin 0 mode : h)

$a_1+a_2=0$ **(Fierz Pauli tuning)** $\Rightarrow$ **No ghost**

- **EOM :** $4\kappa^2\frac{\delta S}{\delta h^{\mu\nu}}=\Box h_{\mu\nu}-h^{\lambda}_{\nu,\mu\lambda}-h^{\lambda}_{\mu,\nu\lambda}+\eta_{\mu\nu}h^{\lambda\sigma}_{,\lambda\sigma}+h_{\mu\nu}-\eta_{\mu\nu}\Box h-m^2(h_{\mu\nu}-\eta_{\mu\nu}h)=0.$

$\Leftrightarrow$ $\left(\Box-m^2\right)h_{\mu\nu}=0,\ \ \partial^{\mu}h_{\mu\nu}=0,\ \ h=0.$

[10 – 4 – 1 = 5 (real space) d.o.f]

van Dam, Veltman, Zakharov (vDVZ) discontinuity I

- $S_{\mathrm{FP}} = \frac{1}{2\kappa^2}\int d^4x\, \frac{1}{4} h_{\mu\nu}\mathcal{O}^{\mu\nu,\alpha\beta}h_{\alpha\beta}$

$$\mathcal{O}^{\mu\nu}{}_{\alpha\beta} = \left(\eta^{(\mu}_{\alpha}\eta^{\nu)}_{\beta} - \eta^{\mu\nu}\eta_{\alpha\beta}\right)(\Box - m^2) - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}_{\beta)} + \partial^\mu\partial^\nu\eta_{\alpha\beta} + \partial_\alpha\partial_\beta\eta^{\mu\nu}.$$

$$\mathcal{O}^{\mu\nu,\alpha\beta}\mathcal{D}_{\alpha\beta,\sigma\lambda} = \frac{i}{2}(\delta^\mu_\sigma\delta^\nu_\lambda + \delta^\nu_\sigma\delta^\mu_\lambda).$$

$$\Rightarrow \quad \mathcal{D}_{\alpha\beta,\sigma\lambda} = \frac{-i}{p^2+m^2}\left[\frac{1}{2}\left(\eta_{\alpha\sigma}\eta_{\beta\lambda} + \eta_{\alpha\lambda}\eta_{\beta\sigma}\right) - \frac{1}{3}\eta_{\alpha\beta}\eta_{\sigma\lambda}\right] + \mathcal{O}(p^2)$$

Fourier space

(Massive graviton propagator)

- $S_{\mathrm{EH}} + S_{GF} = \frac{1}{2\kappa^2}\int d^4x \left[\frac{1}{4}h_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}h_{\alpha\beta} - \frac{1}{2}\left(h^{;\nu}_{\mu\nu} - \frac{1}{2}h_{,\mu}\right)^2\right] = \frac{1}{2\kappa^2}\int d^4x\, \frac{1}{4}h_{\mu\nu}\widetilde{\mathcal{O}}^{\mu\nu,\alpha\beta}h_{\alpha\beta}.$

$$\widetilde{\mathcal{O}}^{\mu\nu,\alpha\beta} = \Box\left[\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right) - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right].$$

$$\widetilde{\mathcal{O}}^{\mu\nu,\alpha\beta}\widetilde{\mathcal{D}}_{\alpha\beta,\sigma\lambda} = \frac{i}{2}(\delta^\mu_\sigma\delta^\nu_\lambda + \delta^\nu_\sigma\delta^\mu_\lambda).$$

$$\Rightarrow \quad \widetilde{\mathcal{D}}_{\alpha\beta,\sigma\lambda} = \frac{-i}{p^2}\left[\frac{1}{2}\left(\eta_{\alpha\sigma}\eta_{\beta\lambda} + \eta_{\alpha\lambda}\eta_{\beta\sigma}\right) - \frac{1}{2}\eta_{\alpha\beta}\eta_{\sigma\lambda}\right].$$

Fourier space

(Massless graviton propagator)

van Dam, Veltman, Zakharov (vDVZ) discontinuity II

Coupling of graviton with a conserved source: $\mathcal{L}_{\text{int}} = h_{\mu\nu} T^{\mu\nu}.$

Point particle with a mass M at rest at origin : $T^{\mu\nu}(x) = M\delta^{\mu}_{0}\delta^{\nu}_{0}\delta^{3}(\boldsymbol{x})$

- **Massive gravity**

$$\left\{\begin{array}{lcl} h_{00}(x) & = & \frac{8GM}{3}\frac{e^{-mr}}{r}, \\ h_{0i}(x) & = & 0, \\ h_{ij}(x) & = & \frac{4GM}{3}\frac{e^{-mr}}{r}\delta_{ij}. \end{array}\right.$$

(m => 0) $\Longrightarrow$

$$\left\{\begin{array}{lcl} \phi & = & -\frac{h_{00}}{2} = -\frac{4}{3}\frac{GM}{r} \\ \psi & = & -\frac{h^i_i}{6} = -\frac{2}{3}\frac{GM}{r}. \end{array}\right.$$

- **Massless gravity (GR)**

$$\left\{\begin{array}{lcl} h_{00}(x) & = & \frac{2GM}{r}, \\ h_{0i}(x) & = & 0, \\ h_{ij}(x) & = & \frac{2GM}{r}\delta_{ij}. \end{array}\right.$$

$\Longrightarrow$

$$\left\{\begin{array}{lcl} \phi & = & -\frac{h_{00}}{2} = -\frac{GM}{r} \\ \psi & = & -\frac{h^i_i}{6} = -\frac{GM}{r}. \end{array}\right.$$

GR is not recovered in the limit m → 0 of massive gravity.

(G → 3G/4 leads to 25% off of light bending.)

Vainshtein effects

Non-linear massive gravity with the flat absolute metric $\eta^{\mu\nu}$

SFP ➔ $$S = \frac{1}{2\kappa^2}\int d^4x \left[(\sqrt{-g}R) - \underbrace{\frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right)}_{\text{FP graviton mass}} \right].$$

Non-linear extension of kinetic term　　　**FP graviton mass**

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

Spherical symmetric solution : $g_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2.$

flat

Expand around the flat space solution

(mr << 1)

$$\begin{cases} B(r) - \overset{\text{flat}}{1} = -\frac{8}{3}\frac{GM}{r}\left(1 - \frac{1}{6}\frac{GM}{m^4 r^5} + \cdots\right), \\ C(r) - 1 = -\frac{8}{3}\frac{GM}{m^2 r^3}\left(1 - 14\frac{GM}{m^4 r^5} + \cdots\right), \\ A(r) - 1 = \frac{4}{3}\frac{GM}{m^2 r^3}\left(1 - 4\frac{GM}{m^4 r^5} + \cdots\right). \end{cases}$$

The non-linear effects becomes dominant for $r < r_V = \left(\frac{GM}{m^4}\right)^{\frac{1}{5}} \to \infty.$ **(m->0)**

➡ **vDVZ discontinuity may be an artifact of linear theory.**

Boulware Deser (BD) ghost I

$$S = \frac{1}{2\kappa^2}\int d^4x \left[(\sqrt{-g}R) - \cancel{\frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right)}\right].$$

ADM decomposition : $g_{00} = -N^2 + g^{ij}N_iN_j, \;\; g_{0i} = N_i, \;\; g_{ij} = g_{ij}.$

$$\frac{1}{2\kappa^2}\int d^4x \sqrt{-g}R. \Rightarrow \frac{1}{2\kappa^2}\int d^4x \sqrt{{}^{(3)}g}N\left[{}^{(3)}R - K^2 + K^{ij}K_{ij}\right].$$

$$\Rightarrow \quad p^{ij} = \frac{\delta L}{\delta \dot{g}_{ij}} = \frac{1}{2\kappa^2}\sqrt{{}^{(3)}g}\left(K^{ij} - Kg^{ij}\right), \;\; K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - {}^{(3)}\nabla_i N_j - {}^{(3)}\nabla_j N_i\right).$$

$$\Rightarrow \quad H = \left(\int_{\Sigma_t} d^dx \; p^{ij}\dot{g}_{ij}\right) - L = \frac{1}{2\kappa^2}\int_{\Sigma_t} d^3\boldsymbol{x}\left(N\mathcal{C} + N_i\mathcal{C}^i\right).$$

$$\mathcal{C} = -\sqrt{{}^{(3)}g}\left[{}^{(3)}R + K^2 - K^{ij}K_{ij}\right], \;\; \mathcal{C}^i = 2\sqrt{{}^{(3)}g}\,{}^{(3)}\nabla_j\left(K^{ij} - Kh^{ij}\right).$$

As is well known, the lapse N and the shift Ni serve as Lagrange multipliers.

$\Rightarrow$ { **Hamiltonian constraint C=0,**
Momentum constraints Ci =0 } **(both are first class constraints)**

$(g_{ij} \;\&\; p_{ij})$ **12 – 4 x 2 = 4 phase space d.o.f = 2 real space d.o.f**

Boulware Deser (BD) ghost II

$$S=\frac{1}{2\kappa^2}\int d^4x\left[(\sqrt{-g}R)-\frac{1}{4}m^2\eta^{\mu\alpha}\eta^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta}-h_{\mu\alpha}h_{\nu\beta}\right)\right].$$

ADM decomposition : $g_{00}=-N^2+g^{ij}N_iN_j,\ g_{0i}=N_i,\ g_{ij}=g_{ij}.$

$$\Rightarrow\ S=\frac{1}{2\kappa^2}\int d^4x\left\{2\kappa^2p^{ij}\dot{g}_{ij}-\left(N\mathcal{C}+N_i\mathcal{C}^i\right)-\frac{m^2}{4}\left[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl}-h_{ik}h_{jl}\right)+2\delta^{ij}h_{ij}-2N^2\delta^{ij}h_{ij}+2N_i\left(g^{ij}-\delta^{ij}\right)N_j\right\}\right].$$

For m ≠ 0, the lapse N and the shift Ni serve as auxiliary fields rather than Lagrange multipliers because they are quadratic in the action.

$$\Rightarrow\ N=\frac{\mathcal{C}}{m^2\delta^{ij}h_{ij}},\quad N_i=-\frac{1}{m^2}\left(g^{ij}-\delta^{ij}\right)^{-1}\mathcal{C}^j$$

$$\Rightarrow\ H=\frac{1}{2\kappa^2}\int d^3x\left\{\frac{1}{2m^2}\frac{\mathcal{C}^2}{\delta^{ij}h_{ij}}-\frac{1}{2m^2}\mathcal{C}^i\left(g^{ij}-\delta^{ij}\right)^{-1}\mathcal{C}^j+\frac{m^2}{4}\left[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl}-h_{ik}h_{jl}\right)+2\delta^{ij}h_{ij}\right]\right\}\neq 0.$$

$(g_{ij}\ \&\ p_{ij})$ **12 – 0 = 12 phase space d.o.f = 6 real space d.o.f**

1 extra d.o.f. ⇔ BD ghost (Hamiltonian unbounded)

Stuckelberg trick

Non-linear massive gravity with the flat absolute metric $\eta^{\mu\nu}$

$$S = \frac{1}{2\kappa^2}\int d^4x \left[(\sqrt{-g}R) - \frac{1}{4}m^2\eta^{\mu\tau}\eta^{\nu\sigma}\left(h_{\mu\nu}h_{\tau\sigma} - h_{\mu\tau}h_{\nu\sigma}\right)\right].$$

The presence of the (fixed) absolute metric seems to break general covariance.

Stuckelberg fields

$$\eta_{\mu\nu} \Rightarrow \eta_{ab}(\phi(x))\,\partial_\mu\phi^a\partial_\nu\phi^b \equiv \Sigma_{\mu\nu}.$$

scalar(fixed) **scalars** **tensor**

(**Unitary gauge:** $\phi^a = \delta^a_\mu x^\mu = x^a \Rightarrow \Sigma_{\mu\nu} = \eta_{\mu\nu}$)

In order to recover general covariance,

$$\left\{\begin{array}{l} h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \\ S = \frac{1}{2\kappa^2}\int d^4x \left[(\sqrt{-g}R) + m^2 V(g,h)\right]. \end{array}\right. \Rightarrow \left\{\begin{array}{l} H_{\mu\nu} = g_{\mu\nu} - \Sigma_{\mu\nu} \\ S = \frac{1}{2\kappa^2}\int d^4x \left[(\sqrt{-g}R) + m^2 V(g,H)\right]. \end{array}\right.$$

Stuckelberg trick II

Expand the Stuckelberg field : $\phi^a = x^a - A^a$

$$H_{\mu\nu} = g_{\mu\nu} - \Sigma_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu - \partial_\mu A^a \partial_\nu A_a$$

$$\left(h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu},\ \ \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b = \Sigma_{\mu\nu}\right)$$

In order to extract the helicity 0 mode, $A_\mu \to A_\mu + \partial_\mu \pi$

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\pi - \partial_\mu A^\alpha \partial_\nu A_\alpha - \partial_\mu A^\alpha \partial_\nu\partial_\alpha\pi - \partial_\mu\partial^\alpha\pi\partial_\nu A_\alpha - \partial_\mu\partial^\alpha\pi\partial_\nu\partial_\alpha\pi.$$

helicity2 **helicity1** **helicity0**

d.o.f 2 2 2

π **always appears in second derivatives.**

BD ghost

Nonlinear extention of mass term

$$S = \frac{1}{2\kappa^2}\int d^4x\left[(\sqrt{-g}R) - m^2 V(g, H)\right].$$

$$(H_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\pi - \partial_\mu A^\alpha \partial_\nu A_\alpha - \partial_\mu A^\alpha \partial_\nu\partial_\alpha\pi - \partial_\mu\partial^\alpha\pi\partial_\nu A_\alpha - \partial_\mu\partial^\alpha\pi\partial_\nu\partial_\alpha\pi.)$$

In order to avoid BD ghost, the (self)-interaction terms of the Stuckelberg scalar π should not appear, namely, they should reduce to the total derivatives.

$$\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi.$$

$$\left\{\begin{array}{lcl}
\mathcal{L}_1^{\mathrm{TD}}(\Pi) & = & [\Pi], \\
\mathcal{L}_2^{\mathrm{TD}}(\Pi) & = & [\Pi]^2 - [\Pi^2], \\
\mathcal{L}_3^{\mathrm{TD}}(\Pi) & = & [\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3], \\
\mathcal{L}_4^{\mathrm{TD}}(\Pi) & = & [\Pi]^4 - 6[\Pi^2][\Pi]^2 + 8[\Pi^3][\Pi] + 3[\Pi^2]^2 - 6[\Pi^4].
\end{array}\right.$$

Brackets represent taking a trace.

(In order to take a trace, $g^{-1}\Pi$ is relevant.)

Nonlinear extention of mass term II

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\pi - \partial_\mu A^\alpha \partial_\nu A_\alpha - \partial_\mu A^\alpha \partial_\nu\partial_\alpha\pi - \partial_\mu\partial^\alpha\pi\partial_\nu A_\alpha - \partial_\mu\partial^\alpha\pi\partial_\nu\partial_\alpha\pi.$$

$h_{\mu\nu} = 0, A_\mu = 0$ $\quad (2x - x^2)$

$$\Rightarrow \quad H_{\mu\nu} = g_{\mu\nu} - \Sigma_{\mu\nu} = 2\partial_\mu\partial_\nu\pi - \partial_\mu\partial^\alpha\pi\partial_\nu\partial_\alpha\pi.$$

$$\Rightarrow \quad g^{\mu\tau}\Pi_{\tau\nu} = \delta^\mu_\nu - \left(\sqrt{g^{-1}(g-H)}\right)^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}\Sigma}\right)^\mu_\nu.$$

$(\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi := x)$ $\quad \left(1 - \sqrt{1-(2x-x^2)} = x\right)$

$$\left(\sqrt{C}\right)^\mu_\tau \left(\sqrt{C}\right)^\tau_\nu = C^\mu_\nu.$$

Recovering h & A, $\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \left(\sqrt{g^{-1}\Sigma}\right)^\mu_\nu$

$$\left\{\begin{array}{rcl} \cancel{\mathcal{L}_1^{\mathrm{TD}}(\mathcal{K})} & \cancel{=} & \cancel{[\mathcal{K}],} \\ \mathcal{L}_2^{\mathrm{TD}}(\mathcal{K}) & = & [\mathcal{K}]^2 - [\mathcal{K}^2], \\ \mathcal{L}_3^{\mathrm{TD}}(\mathcal{K}) & = & [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{L}_4^{\mathrm{TD}}(\mathcal{K}) & = & [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]. \end{array}\right.$$

There are still couplings with tensors. $\xrightarrow{\textbf{unmix}}$ **Galileon terms remain.**

Nonlinear (ghost-free) Massive Gravity

Non-linear massive gravity

de Rham & Gabadadze 2010
de Rham, Gabadadze, Tolley 2011

$$S = \frac{M_{\rm pl}^2}{2}\int \mathrm{d}^4x\sqrt{-g}\left(R + m^2\mathcal{U}\right) + S_{\rm m}.$$

Potential for the graviton: $\mathcal{U} := \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4.$

$$\begin{cases} \mathcal{U}_2 := [\mathcal{K}]^2 - [\mathcal{K}^2], \\ \mathcal{U}_3 := [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3], \\ \mathcal{U}_4 := [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4]. \end{cases}$$

$[\mathcal{K}] = \mathrm{Tr}\mathcal{K}...$

$$\begin{cases} \mathcal{K}_\mu^{\ \nu} := \delta_\mu^{\ \nu} - \left(\sqrt{g^{-1}\Sigma}\right)_\mu^{\ \nu}, \\ \Sigma_{\mu\nu} := \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b. \end{cases}$$

$$\left(\begin{array}{c} \mathcal{K}_\nu^\mu = \partial^\mu\partial_\nu\pi \\ \Uparrow \\ \phi^a = \delta_\mu^a x^\mu - \eta^{a\mu}\partial_\mu\pi. \end{array} \right.$$

ϕ^a : Stuckelberg fields

Unitary gauge: $\phi^a = \delta_\mu^a x^\mu$

$$\Rightarrow \quad M_{\rm pl}^2\left(G_{\mu\nu} + m^2 X_{\mu\nu}\right) = T_{\mu\nu}.$$

Note on non-linear massive gravity

de Rham & Gabadadze 2010
de Rham, Gabadadze, Tolley 2011

- **Absence of (BD) ghost was proven by Hassan & Rosen.**

- **A cut-off scale as an effective field theory is raised as**

$$\Lambda_5 = \left(M_{\rm pl} m^4\right)^{\frac{1}{5}} \Longrightarrow \Lambda_3 = \left(M_{\rm pl} m^2\right)^{\frac{1}{3}} .$$

- **Vainstein radius becomes smaller.**

$$r_{V,5} \sim \left(\frac{M}{M_{\rm pl}}\right)^{\frac{1}{5}} \frac{1}{\Lambda_5} \sim \left(\frac{GM}{m^4}\right)^{\frac{1}{5}} \Longrightarrow r_{V,3} \sim \left(\frac{M}{M_{\rm pl}}\right)^{\frac{1}{3}} \frac{1}{\Lambda_3} \sim \left(\frac{GM}{m^2}\right)^{\frac{1}{3}} .$$

M~M☼~10^{33}g

$$r_{V,3} \sim 10^{16}\text{km} \sim 1\text{kpc}.$$

Cosmology of Massive Gravity

Absence of Flat Friedmann Universe

D'Amico et al. 2011

Homogeneous & isotropic solution:

- **real metric:** $ds^2 = -dt^2 + a^2(t)d\boldsymbol{x}^2$
- **imposes the same symmetry on Stuckelberg fields**

$$\phi^0 = f(t), \quad \phi^i = x^i.$$

$$\Rightarrow \quad \Sigma_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} = \mathrm{diag}\left(-\dot{f}^2, 1, 1, 1\right)$$

$$\Rightarrow \quad \left(g^{-1}\Sigma\right)^\mu_\nu = \mathrm{diag}\left(\dot{f}^2, a^{-2}, a^{-2}, a^{-2}\right)$$

$$\Rightarrow \quad \mathcal{K}^\nu_\mu = \delta^\nu_\mu - (\sqrt{g^{-1}\Sigma})^\nu_\mu = \mathrm{diag}\left(1 - |\dot{f}|, 1 - a^{-1}, 1 - a^{-1}, 1 - a^{-1}\right)$$

Absence of Flat Friedmann Universe II

$$\begin{cases} \mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2] = -6(1-a^{-1})\left(a^{-1} - 2 + |\dot{f}|\right), \\ \mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3] = -6(1-a^{-1})^2\left(a^{-1} - 4 + 3|\dot{f}|\right), \\ \mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4] \\ \quad = -24(1-a^{-1})^3\left(-1 + |\dot{f}|\right). \end{cases}$$

dot{f} appears only linearly !!

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2} a^3 \left[6(\dot{H} + 2H^2) + m^2 \mathcal{U}\right] = \frac{M_{\rm pl}^2}{2} a^3 \left[-6H^2 + m^2 \mathcal{U}\right].$$

Integration by part

$$\frac{\delta \mathcal{L}}{\delta f} = 3M_{\rm pl}^2 m^2 \frac{\mathrm{d}}{\mathrm{d}t}\left[a^3 - a^2 + 3\alpha_3 a(a-1)^2 + 4\alpha_4 (a-1)^3\right] = 0.$$

$$\dot{a} = 0$$

The scale factor cannot evolve !!

There is no homogeneous and isotropic flat Universe.

What can we do ?

- Consider open Friedmann Universe instead of flat one.

  ⇒ Gumrukcuoglu, Lin and Mukohyama 2011.

- Abandon imposing the same symmetry on the Stuckelberg field.

  ⇒ Our work by use of Painleve-Gullstrand meric

Note that the same idea was done in D'Amico et al. 2011 & Gratia, Hu, and Wyman 2012 as well.

Painleve-Gullstrand meric

Spherically symmetric vacuum solution in (massless) GR

Schwarzschild metric:

$$ds^2 = -f(r)dt_s^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2.$$

$$f(r) = 1 - 2M/r.$$

This metric has a coordinate singularity at the horizon r = 2M.

In GR, this is not a real singularity and can be removed by coordinate transformation.

Danger of coordinate singularity in Massive Gravity

Gruzinov & Mirbabayi 2011
Berezhiani et al. 2012

New invariant in Massive Gravity:

$$I^{ab} = g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b.$$

This quantity is invariant under coordinate transformation, namely, a scalar quantity and should have the same position as $R, R_{\mu\nu}R^{\mu\nu}, \cdots.$

In the unitary gauge: $\phi^a = \delta^a_\mu x^\mu$

$$\Longrightarrow \quad I^{ab} = g^{\mu\nu}\delta^a_\mu\delta^b_\nu.$$

Any inverse metric with divergence leads to singularity in this invariant.

(Though such a singularity does not affect the geodesic motion, it would cause a problem for perturbations around classical solutions because inverse metric could change its sign across the singularity.)

Danger of coordinate singularity in Massive Gravity II

Gruzinov & Mirbabayi 2011
Berezhiani et al. 2012

The Schwarzschild like metric in Massive Gravity
(Schwarzschild, Schwarzschild-De Sitter, Reissner-Nordstrom…)
can be dangerous.

Needs the metric without coordinate singularity

Painleve-Gullstrand meric !!

- **BH solutions in PG metric:**

  Berezhiani, Chkareuli, de Rham, Gabadadze, Tolley 2011

- **Cosmological solutions in PG metric:**

  our work

Painleve-Gullstrand metric

Painleve 1922, Gullstrand 1922,
Kanai, Siino, Hosoya 2011

Merit of PG metric:

- includes an off-diagonal and spatially flat elements, which leads to no coordinate singularity (except real singularity at the origin)
- can cover both inside and outside the horizon by a single coordinate patch.
- Time coordinate as measured by an observer who is at rest at infinity and freely falls into the BH.
- The space described by the PG metric can be regarded as a river whose speed of current is the Newtonian escape velocity at each point.
- Generalized PG metric can also describe the FRLW universe.

Derivation of Painleve-Gullstrand metric

Painleve 1922, Gullstrand 1922,
Kanai, Siino, Hosoya 2011

Schwarzschild metric: $ds^2 = -f(r)dt_s^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2.$

$$f(r) = 1 - 2M/r.$$

Four velocity of an observer : $u_s^\mu = dx_s^\mu / d\tau = \dot{x}_s^\mu.$

normalization condition : $-1 = g_{\mu\nu} u_s^\mu u_s^\nu = -f \dot{t}_s^2 + f^{-1} \dot{r}^2.$

(conserved) energy per rest mass : $\epsilon = -g_{\mu\nu} \xi^\mu u_s^\nu = f \dot{t}_s.$

$\xi^\mu = (\partial / \partial t_s)^\mu$: timelike Killing vector

$$\begin{cases} u_s^\mu = \left(\dot{t}_s, \dot{r}, \dot{\theta}, \dot{\phi}\right) = \left(\frac{\epsilon}{f}, -\sqrt{\epsilon^2 - f}, 0, 0\right), \\ u_{s\,\mu} = g_{\mu\nu} u_s^\nu = \left(-\epsilon, -\frac{\sqrt{\epsilon^2 - f}}{f}, 0, 0\right). \end{cases}$$

Derivation of Painleve-Gullstrand metric II

t_P : the proper time of the free-falling observer

⟺ **The geodesic is orthogonal to the surface t_P = const.**

⟺ **The geodesic tangent vector $u_{s\mu}$ is equal to the gradient of t_P.**

$$u_{s\mu} = -\frac{\partial}{\partial x_s^\mu} t_P(x_s). \qquad u_{s\mu} = \left(-\epsilon, -\sqrt{\epsilon^2 - f}/f, 0, 0\right).$$

⟺ $$dt_p = \epsilon dt_s + \frac{\sqrt{\epsilon^2 - f}}{f} dr.$$

$$\begin{aligned} ds^2 &= -f(r)dt_s^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \\ &= -dt_p^2 + \frac{1}{\epsilon^2}(dr + v(r)dt_p)^2 + r^2 d\Omega^2. \end{aligned}$$

$v(r) = \sqrt{\epsilon^2 - f(r)}$: radially free-falling velocity

At the horizon f(r)=0 ⇔ r =2M, the metric is non-singular.

Derivation of Painleve-Gullstrand metric III

$$ds^2 = -dt_p^2 + \frac{1}{\epsilon^2}(dr + v(r)dt_p)^2 + r^2 d\Omega^2.$$

$$\Rightarrow \begin{cases} u_p^\mu &= \left(\dot{t}_p, \dot{r}, \dot{\theta}, \dot{\phi}\right) = (1, -v, 0, 0)\,, \\ u_{p\,\mu} &= g_{\mu\nu} u_p^\nu = (-1, 0, 0, 0)\,. \end{cases}$$

$$\frac{dr}{dt_p} = \frac{\dot{r}}{\dot{t}_p} = -v(r) = -\sqrt{\epsilon^2 - f(r)}. \quad \Rightarrow \quad E = \frac{1}{2}\left(\frac{dr}{dt_p}\right)^2 + \Phi(r).$$

$$\begin{cases} E = (\epsilon^2 - 1)/2 & \text{: conserved energy} \\ \Phi(r) = -M/r & \text{: gravitational potential} \end{cases}$$

For a particle freely falling from infinity at rest (ε =1 ⇔ E=0),

$$ds^2 = -dt_p^2 + \left(dr + \sqrt{2M/r}\,dt_p\right)^2 + r^2 d\Omega^2.$$

Standard form given by PG.

This is a vacuum solution, so we want a solution including matter.

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$$\Rightarrow \quad ds^2 = -dt_p^2 + \frac{1}{1 + 2E(t_p, r)}(dr + v(t_p, r)dt_p)^2 + r^2 d\Omega^2, \;\; v(t_p, r) = \sqrt{2E(t_p, r) + 2m(t_p, r)/r}.$$

Spherical gravitational collapse – from infinity

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Spherical gravitational collapse of matter with E = 0 :

$$ds^2 = -dt^2 + \left(dr + \sqrt{\frac{2m(r,t)}{r}}dt\right)^2 + r^2 d\Omega^2.$$

Einstein Eq.

$(8\pi T^\mu_\nu = R^\mu_\nu - \delta^\mu_\nu R/2)$

$$\left\{\begin{aligned} 8\pi T^t_t &= -\frac{2m'}{r^2}, \\ 8\pi T^r_t &= \frac{2\dot{m}}{r^2}, \\ 8\pi T^r_r &= -\frac{2m'}{r^2} + \frac{2\dot{m}}{r^2}\left(\frac{2m}{r}\right)^{-1/2}, \\ 8\pi T^\theta_\theta(T^\phi_\phi) &= -\frac{m''}{r} + \left(\frac{\dot{m}}{2r^2} + \frac{\dot{m}'}{r}\right)\left(\frac{2m}{r}\right)^{-1/2} - \frac{\dot{m}m'}{r^2}\left(\frac{2m}{r}\right)^{-3/2}. \end{aligned}\right.$$

$$\dot{} = \partial/\partial t, \quad ' = \partial/\partial r.$$

Only three are independent. ← $T^r_r = T^t_t + T^r_t (2m/r)^{-1/2}.$

Perfect fluid : $T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}$

$u^\mu = (1, -v(t,r), 0, 0)$ ⇒ $T^t_t = -\rho,\ T^r_r = P,\ T^r_t = (\rho + P)v.$

$v(t,r) = \sqrt{\frac{2m(t,r)}{r}}$ **is equal to the escape velocity.**

Spherical gravitational collapse – from infinity II

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- $8\pi T^t_t = -8\pi\rho = -2m'/r^2$:

$$\Rightarrow \quad m(t,r) = 4\pi \int_0^r \rho(t,r) r^2 dr \left(= \frac{2r^3}{9\gamma^2 t^2} \right)$$

- $8\pi T^r_t = 8\pi(\rho + P) = \frac{2\dot{m}}{r^2}, \quad P = (\gamma - 1)\rho, \quad \rho(t,r) = f(r)h(t)$:

$$\Rightarrow \quad \rho(t,r) = \frac{1}{6\pi\gamma^2 t^2}, \quad P(t,r) = \frac{\gamma - 1}{6\pi\gamma^2 t^2}.$$

B.C. $m|_{r=0} = 0, \quad \rho|_{t=0} = \infty.$ **(t : $-\infty \rightarrow 0$)**

Matter density (of the star) is uniform.

Though, in case of gravitational collapse, we need to match this inner solution with the outer solution given before, we are now interested in only the inner solution because…

Relation between this solution and Friedmann Universe

$$ds^2 = -dt^2 + (dr + v_-(t,r)dt)^2 + r^2 d\Omega^2.$$

$$v_-(t,r) = \sqrt{\frac{2m(t,r)}{r}} = \frac{2r}{3\gamma(-t)}.$$ **(t : -∞ ➔ 0)**

- $P = (\gamma - 1)\rho$ $\Rightarrow$ (**FLRW**) $H = \frac{2}{3\gamma t}$ $\Rightarrow$ $v_-(t,r) = -\frac{\dot{a}(t)}{a(t)} r.$

- **In fact,** $r = a(t)\tilde{r}$ $\Rightarrow$ $dr = \dot{a}(t)\frac{r}{a(t)}dt + a(t)d\tilde{r}$

$$\Rightarrow \quad \begin{aligned} ds^2 &= -dt^2 + (dr + v_-(t,r)dt)^2 + r^2 d\Omega^2 \\ &= -dt^2 + a^2(t)\left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right). \end{aligned}$$

The generalized Painleve-Gullstrand metric includes flat Friedmann Universe. (Expanding phase: v_- ➔ - v_-)

Spherical gravitational collapse – from a finite radius

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$$ds^2 = -dt^2 + \frac{1}{1+2E(t,r)}(dr + v_-(t,r)dt)^2 + r^2 d\Omega^2, \;\; v_-(t,r) = \sqrt{2E(t,r) + 2m(t,r)/r}.$$

The boundary surface r=a(t) that freely falls from a radius a_0 at rest :

Solving Einstein Eq. inside the boundary $\left(M = m(t,r)|_{r=a(t)} = \frac{4\pi}{3}a^3(t)\rho(t)\right)$

$$\Rightarrow \quad E(t,r) = -\frac{M}{a_0}\left(\frac{r}{a(t)}\right)^2 < 0, \quad v_-(t,r) = \sqrt{\frac{2M}{a(t)} - \frac{2M}{a_0}\frac{r}{a(t)}}.$$

- **In this case also,** $r = a(t)\tilde{r} \;\Rightarrow\; v_-(t,r) = -\frac{\dot{a}(t)}{a(t)}r.$

$$\Rightarrow \quad \begin{aligned} ds^2 &= -dt^2 + \frac{1}{1 - \frac{2M}{r_0}\left(\frac{r}{a(t)}\right)^2}(dr + v_-(t,r)dt)^2 + r^2 d\Omega^2 \\ &= -dt^2 + a^2(t)\left(\frac{d\tilde{r}^2}{1 - \frac{2M}{a_0}\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right). \end{aligned}$$

The generalized Painleve-Gullstrand metric includes closed (open ⇔ E>0) Friedmann Universe as well.

Cosmological solution

Generalized Painleve-Gullstrand metric as a cosmological solution

- **Our strategy is to find generalized PG metric in Massive Gravity instead of the standard FLRW metric.**

$$\mathrm{d}s^2 = -V^2(t,r)\mathrm{d}t^2 + U^2(t,r)\left(\mathrm{d}r + \epsilon\sqrt{f(t,r)}\mathrm{d}t\right)^2 + W^2(t,r)r^2\mathrm{d}\Omega^2.$$

$$(\epsilon = \pm 1)$$

- **Stuckelberg fields in the unitary gauge:**

$$\phi^0 = t, \qquad \phi^i = r\hat{n}^i. \qquad \hat{n} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$$

- **One parameter family:**

$$\alpha_3 = \frac{1}{3}(\alpha - 1), \quad \alpha_4 = \frac{1}{12}\left(\alpha^2 - \alpha + 1\right).$$

$$W(t,r) = \tilde{\alpha} := \frac{\alpha}{\alpha+1}. \quad \Longrightarrow \quad X_{\mu\nu} = \frac{1}{\alpha}g_{\mu\nu}.$$

Effective C.C.

$$M_{\mathrm{pl}}^2\left(G_{\mu\nu} + m^2 X_{\mu\nu}\right) = T_{\mu\nu}$$

Any PG-type metric in GR (with a cosmological constant) is also a solution to Massive Gravity.

Friedmann Universe in Massive Garvity

- **The FLRW metric can be rewritten in a general PG form :**

$$ds^2 = -\kappa^2 dt^2 + \frac{\tilde{\alpha}^2}{1 - K\tilde{\alpha}^2 r^2/a^2(t)}\left(dr - \frac{\dot{a}}{a} r dt\right)^2 + \tilde{\alpha}^2 r^2 d\Omega^2.$$

$$K = 0, \pm 1. \Longleftrightarrow$$ **All types of Friedmann Universe**

- **Perfect fluid :** $T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}$

$$\rho = \rho(t), \;\; P = P(t). \qquad u_\mu = (-\kappa, 0, 0, 0)$$

EOM $\Longrightarrow$

$$\begin{cases} \frac{3}{\kappa^2}\tilde{H}^2 = \frac{\rho}{M_{\rm pl}^2} + \frac{m^2}{\alpha} - \frac{3K}{a^2}, \\ -\frac{1}{\kappa^2}\left(3\tilde{H}^2 + 2\dot{\tilde{H}}\right) = \frac{p}{M_{\rm pl}^2} - \frac{m^2}{\alpha} + \frac{K}{a^2}. \end{cases} \qquad \left(\tilde{H}(t) := d\ln a/dt\right)$$

Rescaling time coordinate t ➔ $\tau = \kappa$ t with H:=d lna/d τ gives
the standard cosmological equation with effective C.C. $\Lambda_{\rm eff} = m^2/\alpha.$

Thus, our solution can accommodate spatially flat, open, and closed models.

More familiar form

$$ds^2 = -d\tau^2 + \frac{\tilde{\alpha}^2}{1 - K\tilde{\alpha}^2 r^2/a^2(\tau)}\left(dr - \frac{\dot{a}}{\kappa a} r d\tau\right)^2 + \tilde{\alpha}^2 r^2 d\Omega^2.$$

Coordinate transformation: $r \to \tilde{r} = \dfrac{\tilde{\alpha} r}{a(\tau)}.$

$$ds^2 = -d\tau^2 + a^2\left(\frac{d\tilde{r}^2}{1 - K\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right).$$

with Stuckelberg fields, which do not respect the same symmetry,

$$\phi^0 = \frac{\tau}{\kappa}, \qquad \phi^i = \frac{a(\tau)\tilde{r}}{\tilde{\alpha}}\hat{n}^i.$$

$$\Sigma_{\mu\nu} dx^\mu dx^\nu = -\left(\frac{1}{\kappa^2} - \frac{a^2 H^2 \tilde{r}^2}{\tilde{\alpha}^2}\right) d\tau^2 + 2\frac{a^2 H \tilde{r}}{\tilde{\alpha}^2} d\tau d\tilde{r} + \frac{a^2}{\tilde{\alpha}^2}\left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right).$$

Inhomogeneous spherical collapse of dust

Inhomogeneous spherical collapse of dust

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Spherically symmetric spacetime in general PG form:

$$ds^2 = -N(t,r)^2 dt^2 + \frac{\tilde{\alpha}^2}{1+2E(t,r)}(N_r(t,r)dt + dr)^2 + \tilde{\alpha}^2 r^2 d\Omega^2.$$

(N(t,r)>0 : lapse, Nr(t,r) : radial component of shift, E(t,r) > -1.)

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}.$$

ρ & P are no longer homegeneous and bare Λ may be included.

EOM

$$E = \frac{1}{2}\left(\frac{\tilde{\alpha}N_r}{N}\right)^2 - \frac{M}{\tilde{\alpha}r}$$ **with** $$M(r,t) := 4\pi \int^r \left(\rho + M_{\rm pl}^2 \Lambda\right) r'^2 dr'.$$

To TLB coordinates: (t, r, θ, ϕ) ➔ (T, R, θ, ϕ)

t=t(T)=T, $r = r(T,R) = \bar{r}(T,R)/\tilde{\alpha}$ **with** $\left(\frac{\partial \bar{r}}{\partial T}\right) = -\tilde{\alpha}N_r = -N\sqrt{\frac{2M}{\bar{r}} + 2E}.$

$$\mathrm{d}s^2 = -N(T)^2 \mathrm{d}T^2 + \frac{1}{1+2E(R)}\left(\frac{\partial \bar{r}}{\partial R}\right)^2 \mathrm{d}R^2 + \bar{r}^2 \mathrm{d}\Omega^2.$$

Inhomogeneous spherical collapse of dust represented by LTB.

Generalized Painleve-Gullstrand metric as a cosmological solution

- **Our strategy is to find generalized PG metric in Massive Gravity instead of the standard FLRW metric.**

$$\mathrm{d}s^2 = -V^2(t,r)\mathrm{d}t^2 + U^2(t,r)\left(\mathrm{d}r + \epsilon\sqrt{f(t,r)}\mathrm{d}t\right)^2 + W^2(t,r)r^2\mathrm{d}\Omega^2.$$

$$(\epsilon = \pm 1)$$

- **Stuckelberg fields in the unitary gauge:**

$$\phi^0 = t, \qquad \phi^i = r\hat{n}^i. \qquad \hat{n} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$$

- **One parameter family:**

$$\alpha_3 = \frac{1}{3}(\alpha - 1), \quad \alpha_4 = \frac{1}{12}\left(\alpha^2 - \alpha + 1\right).$$

$$W(t,r) = \tilde{\alpha} := \frac{\alpha}{\alpha+1}. \Longrightarrow X_{\mu\nu} = \frac{1}{\alpha}g_{\mu\nu}.$$

Effective C.C.

$$M_{\rm pl}^2\left(G_{\mu\nu} + m^2X_{\mu\nu}\right) = T_{\mu\nu}$$

Any PG-type metric in GR (with a cosmological constant) is also a solution to Massive Gravity.

Summary and comments

- **We have presented a spatially flat, open, and closed Friedmann Universe in Massive Gravity, though the Stuckelberg fields are inhomogeneous.**
- **Our analysis is based on the observation that any PG metric with the Stuckelberg fields in the unitary gauge generates an effective cosmological constant for a choice of one parameter family.**
- **Our choice of parameter is special in that fluctuation modes become non-dynamical at quadratic order. However, recently, it is suggested that they may acquire kinetic term at cubic order, signaling the ghost instabilities, though they use a different fiducial metric. I am also not sure what happens if we take into account quantum corrections.**

Relation to the work of Gratia, Hu, and Wyman

arXiv:1205.4241

Try to find spatially isotropic solution:

$$\mathrm{d}s^2 = -b^2(t,r)\mathrm{d}t^2 + a^2(r,t)\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right).$$

with Stuckelberg fields, which respects the same symmetry,

$$\phi^0 = f(t,r), \qquad \phi^i = g(t,r)\frac{x^i}{r}.$$

(N.B. b(t,r) = 1 & a(t,r) = a(t) ➔ Flat FRW metric)

Potential: $-\mathcal{U} = P_0\left(\frac{g}{ar}\right) + \sqrt{X}P_1\left(\frac{g}{ar}\right) + WP_2\left(\frac{g}{ar}\right).$

$$\begin{cases} P_0(x) = -12 - 2x(x-6) - 12(x-1)(x-2)\alpha_3 - 24(x-1)^2\alpha_4, \\ P_1(x) = 2(3-2x) + 6(x-1)(x-3)\alpha_3 + 24(x-1)^2\alpha_4, \\ P_2(x) = -2 + 12(x-1)\alpha_3 - 24(x-1)^2\alpha_4. \end{cases}$$

$$X = \left(\frac{\dot{f}}{b} + \mu\frac{g'}{a}\right)^2 - \left(\frac{\dot{g}}{b} + \mu\frac{f'}{a}\right)^2, \quad W = \frac{\mu}{ab}\left(\dot{f}g' - \dot{g}f'\right), \quad \mu = \mathrm{sgn}\left(\dot{f}g' - \dot{g}f'\right), \quad \dot{} = \frac{\partial}{\partial t}, \quad {}' = \frac{\partial}{\partial r}.$$

Relation to the work of Gratia, Hu, and Wyman II

EOMs for f & g :

$$\begin{cases} \partial_t\Big[\frac{a^3r^2}{\sqrt{X}}\big(\frac{\dot f}{b}+\mu\frac{g'}{a}\big)P_1+\mu a^2r^2g'P_2\Big]-\partial_r\Big[\frac{a^2br^2}{\sqrt{X}}\big(\mu\frac{\dot g}{b}+\frac{f'}{a}\big)P_1+\mu a^2r^2\dot g P_2\Big]=0, \\ -\partial_t\Big[\frac{a^3r^2}{\sqrt{X}}\big(\frac{\dot g}{b}+\mu\frac{f'}{a}\big)P_1+\mu a^2r^2f'P_2\Big]+\partial_r\Big[\frac{a^2br^2}{\sqrt{X}}\big(\mu\frac{\dot f}{b}+\frac{g'}{a}\big)P_1+\mu a^2r^2\dot f P_2\Big]=a^2br\left[P_0'+\sqrt{X}P_1'+WP_2'\right]. \end{cases}$$

The solution to the first EOM is given by $P_1(x_0)$ = 0 & g(t,r)= x_0 a r.

$$x_0=\frac{1+6\alpha_3+12\alpha_4\pm\sqrt{1+3\alpha_3+9\alpha_3^2-12\alpha_4}}{3(\alpha_3+4\alpha_4)}.$$

The second EOM reduces to

$$\sqrt{X}P_1'(x_0)\;=\Big[\frac{2P_2(x_0)}{x_0}-P_2'(x_0)\Big]W-P_0'(x_0).$$

Our parameter choice with $\alpha_3=\frac{1}{3}(\alpha-1),\;\alpha_4=\frac{1}{12}\left(\alpha^2-\alpha+1\right)$ **automatically satisfies this equation.**

$$\left(P_0'(x_0)=P_1'(x_0)=0,\quad P_2'(x_0)=\frac{2P_2(x_0)}{x_0}.\right)$$