Applications of the characteristic formalism in relativity

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Outline

- Background: Bondi-Sachs formalism
- Characteristic extraction
- Linearized solutions and applications
- Cosmology
- Conclusion

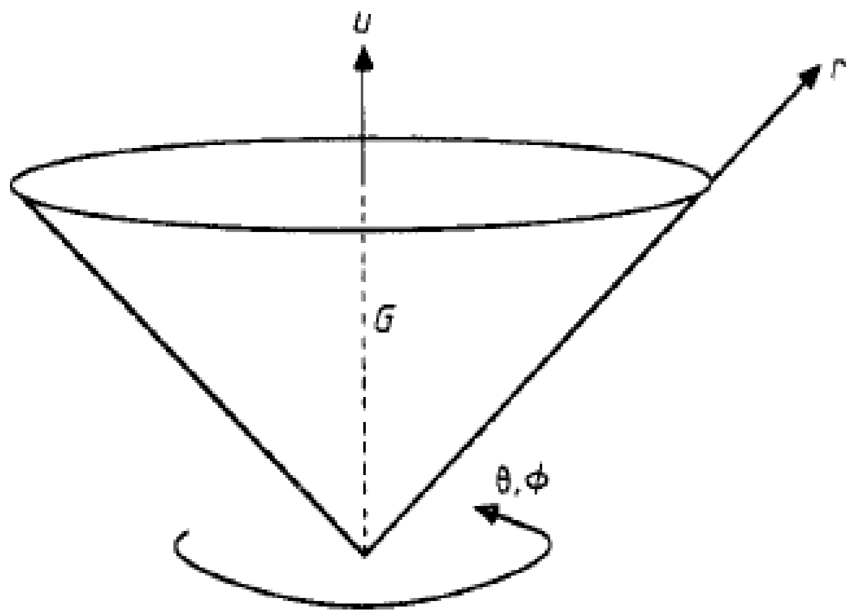


Figure 1. Null coordinates.

Background: Bondi-Sachs Formalism*

The Bondi-Sachs metric is

$$ds^2 = -\left(e^{2\beta}(1 + W_c r) - r^2 h_{AB} U^A U^B\right) du^2$$
$$-2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B,$$

where r is an area coordinate so that $det(h_{AB}) = det(q_{AB})$ with q_{AB} a unit sphere metric. We introduce a complex dyad q_A (e.g. in spherical polars, $q_A = (1, i \sin\theta)$). Then h_{AB} can be represented by

$$J = h_{AB} q^A q^B / 2.$$

We use the spin-weighted field $U = U^A q_A$ as well as the (complex differential "angular gradient") eth operators $\eth$ and $\bar{\eth}$. Einstein's equations $R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T)$ can be categorized as

*N.T. Bishop *et al.*: Phys. Rev. D **56** 6298 (1997)

- Hypersurface equations, $R_{11}, q^A R_{1A}, h^{AB} R_{AB}$,

$$\begin{aligned} \beta_{,r} &= f_1(J) \\ U_{,rr} &= f_2(J,\beta) \\ W_{c,r} &= f_3(J,\beta,U) \end{aligned}$$

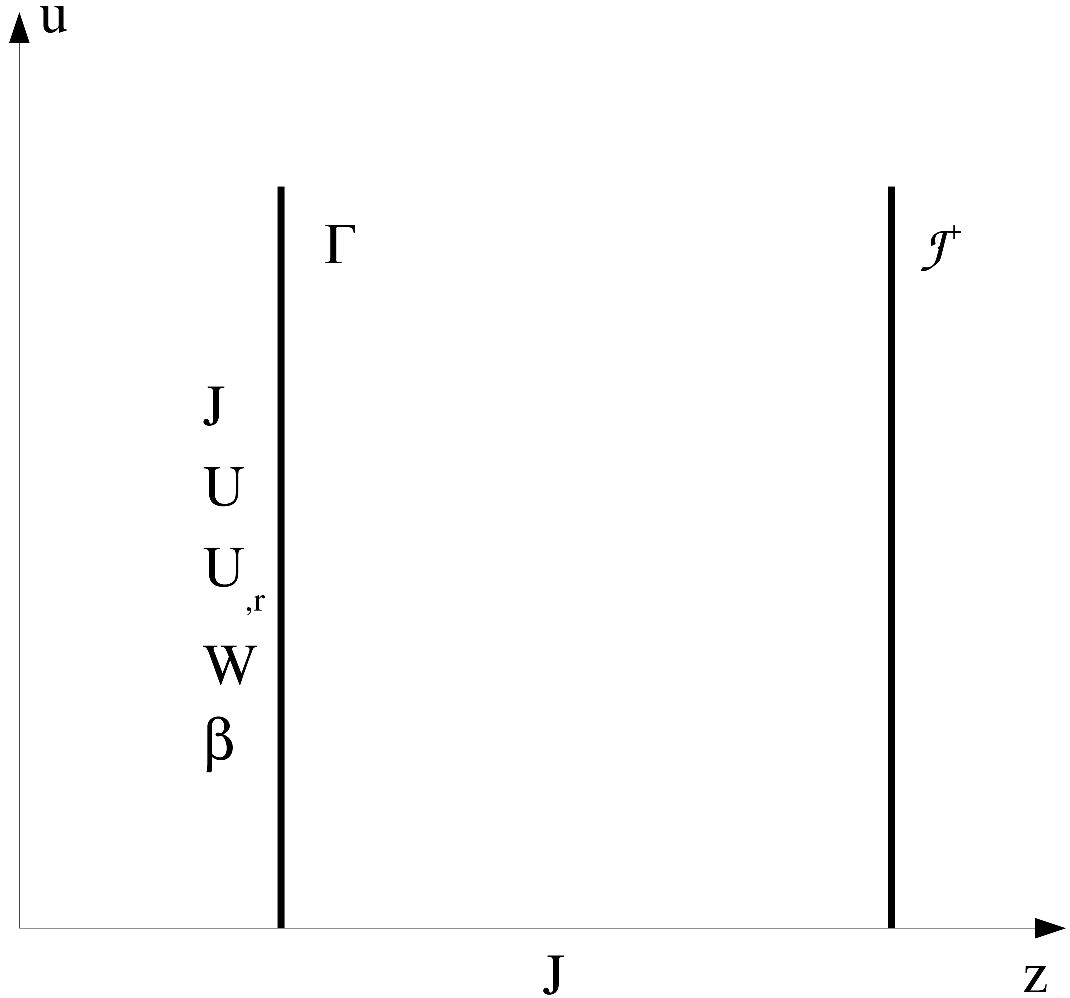
- Evolution equation $q^A q^B R_{AB}$

$$J_{,ur} = f_4(J,\beta,U,W_c)$$

where the f_i include hypersurface derivatives (∂_r, ∂_A) of the variables

- Constraints $R_{0\alpha}$.

Compactify: $r \to x$ with $r = \infty \to x = 1$


u
Γ
$\mathcal{J}^+$
J
U
$U_{,r}$
W
β
J
z

Inclusion of matter to the characteristic formalism[†]

The matter is characterized by

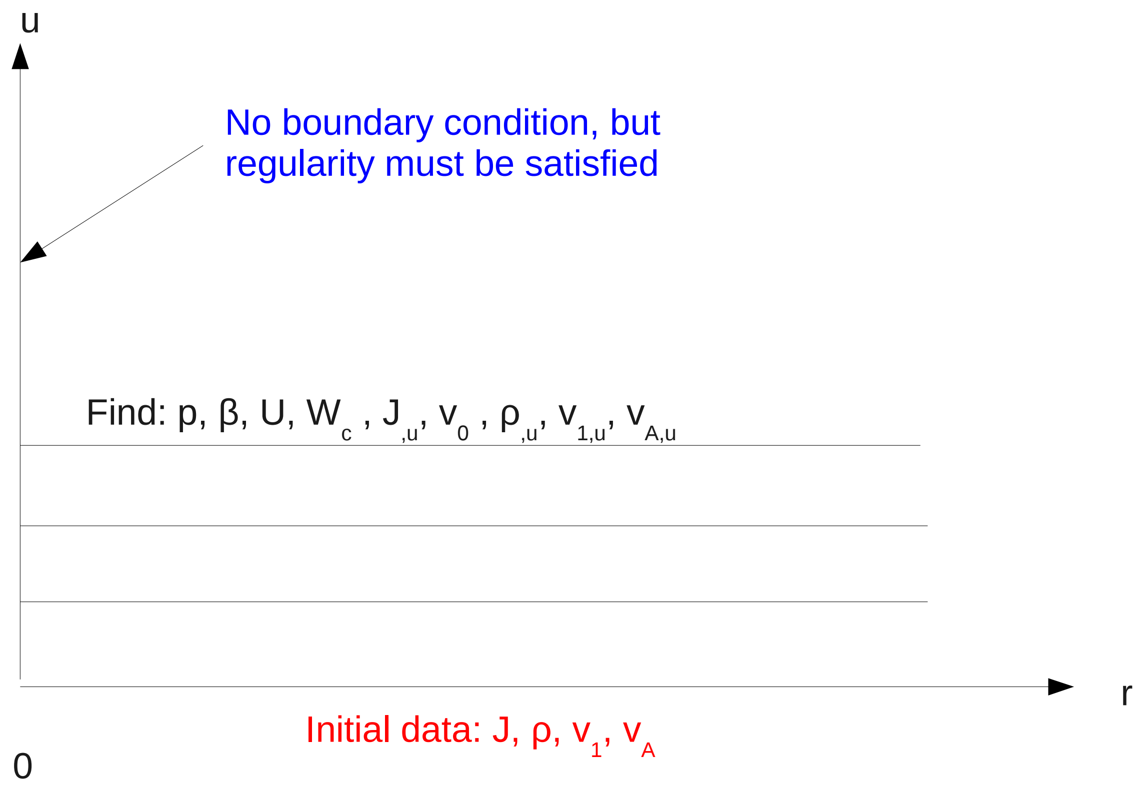
Density	ρ
Pressure	p
Covariant 4-velocity	$v_\alpha = (v_0, v_1, v_A)$
Equation of state	$p = p(\rho)$

Define $V = v_A q^A$, apply the Einstein equations, as well as the fluid conservation equations $T^a_{b;a} = 0$, to get

[†]N.T. Bishop *et al.*, Phys. Rev. D **60**, 024005 (1999)

$$
\begin{aligned}
p &= p(\rho) \\
\beta_{,r} &= f_1 + 2\pi r(\rho + p)(v_1)^2 \\
U_{,rr} &= f_2 + (\rho + p) v_1 V F_2(r, \beta, J) \\
W_{c,r} &= f_3 + F_3(\rho, p, v_1, V, r, \beta, J) \\
J_{,ur} &= f_4 + F_4(\rho, p, V, r, \beta, J) \\
v_0 &= F_5(v_1, V, r, \beta, U, J) \\
\rho_{,u} &= F_6(\rho, p, , v_1, V, v_0, r, \beta, J, U, W_c) \\
v_{1,u} &= F_7(\rho, p, v_1, V, r, \beta, J, U, W_c) \\
V_{,u} &= F_8(\rho, p, v_1, V, v_0, r, \beta, J, U, W_c).
\end{aligned}
$$

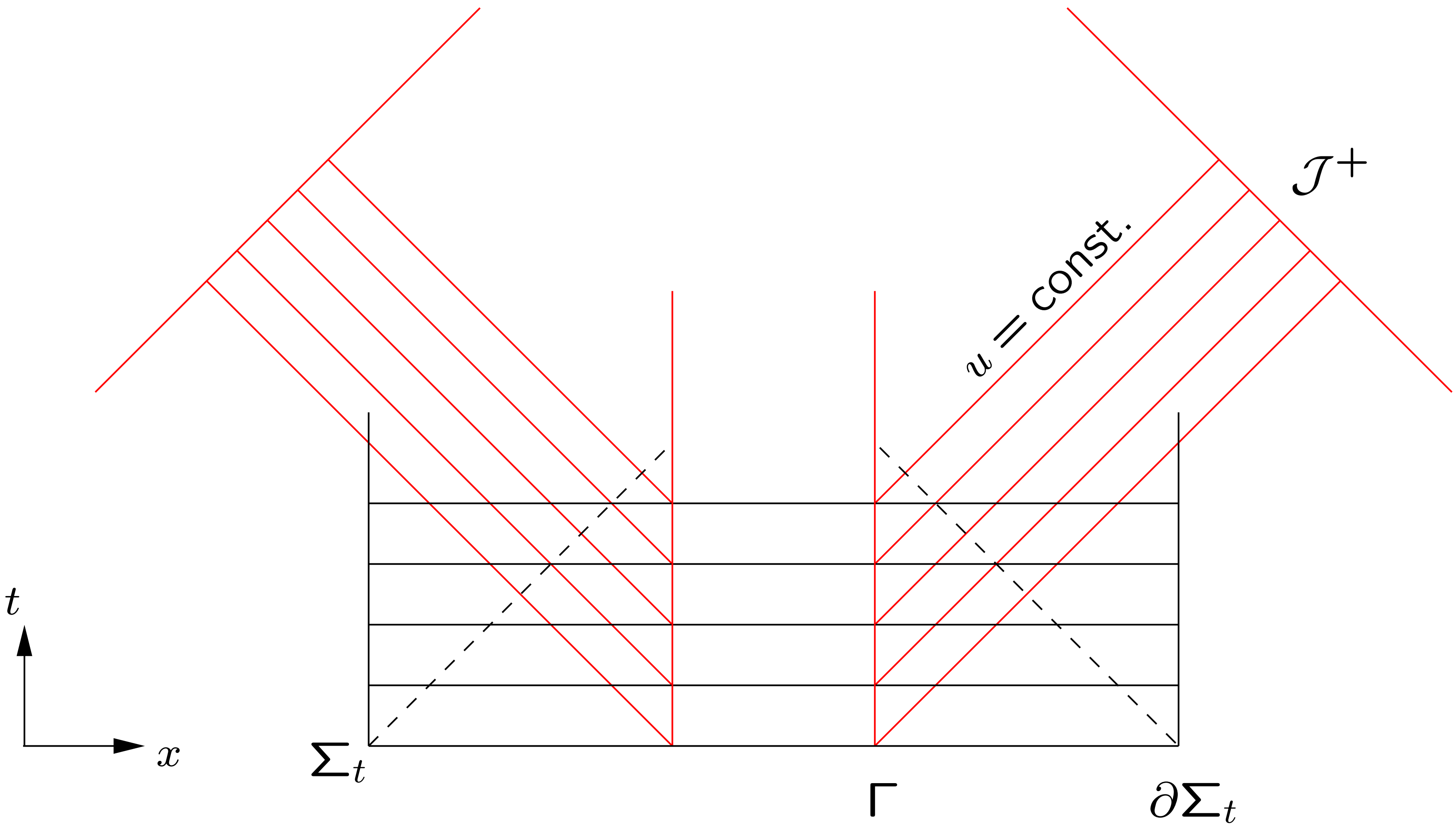
Evolution variables	J, ρ, v_1, V
Auxiliary variables	p, β, U, W_c, v_0


u
No boundary condition, but
regularity must be satisfied
Find: p, β, U, W_c , $J_{,u}$, v_0 , $\rho_{,u}$, $v_{1,u}$, $v_{A,u}$
r
Initial data: J, ρ, v_1, v_A
0

Characteristic extraction‡

- Gravitational radiation is defined at future null infinity ($\mathcal{J}^+$)
- But ... It is extracted by perturbative matching, or from $r\psi_4$, using data at a finite distance from the source
- Characteristic numerical relativity has many positive features – stability, convergence, inclusion of null infinity
- Idea: use data on a finite worldtube as input to a characteristic code, and thereby calculate the radiation at $\mathcal{J}^+$

‡C. Reisswig, N. T. Bishop, D. Pollney, and B. Szilagyi: Phys. Rev. Lett. **103**, 221101 (2009); Class. Quantum Grav. **27**, 075014 (2010)



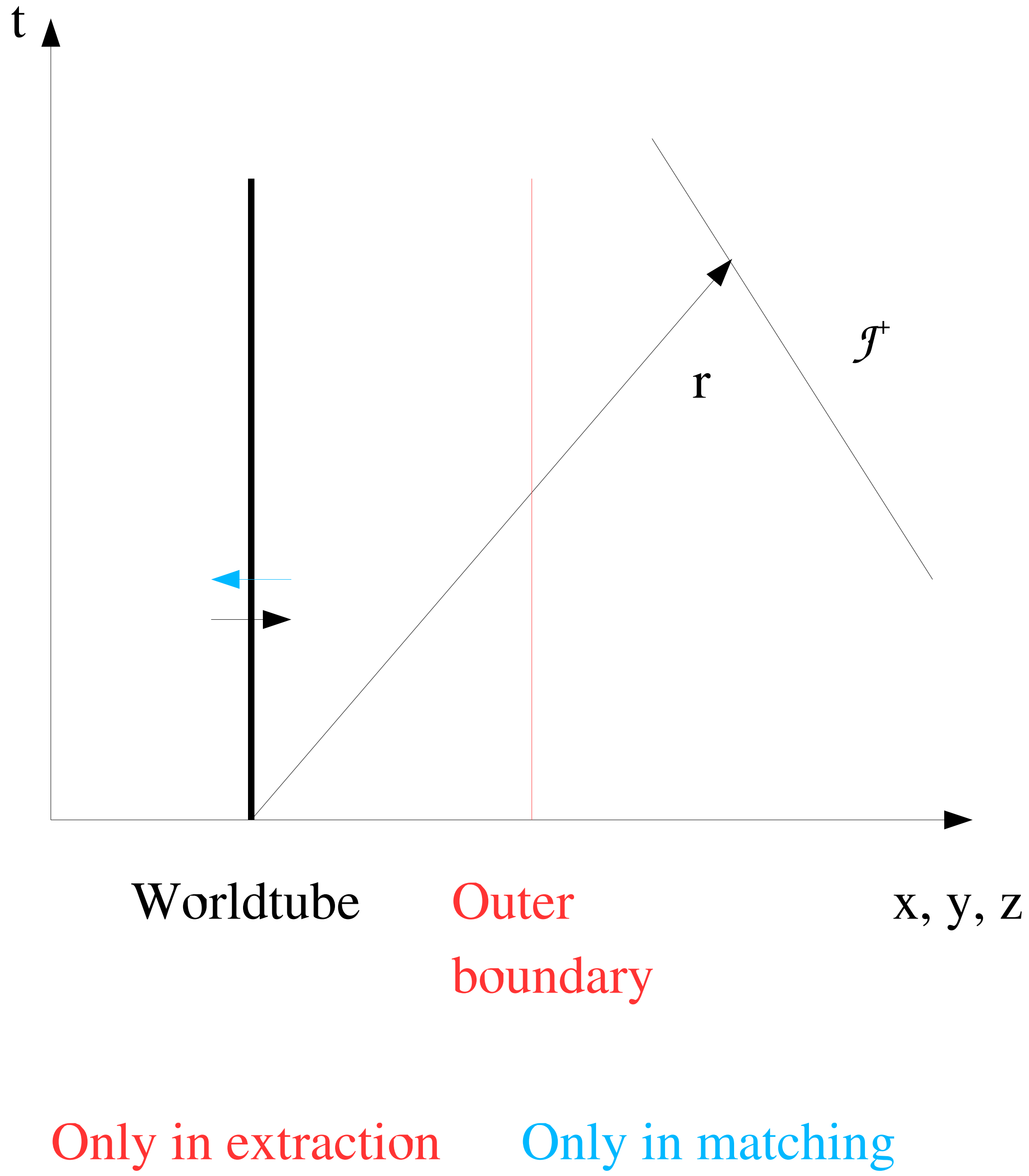
$\mathcal{J}^+$
$u = $ const.
t
x
Σ_t
Γ
$\partial\Sigma_t$

Characteristic extraction and matching

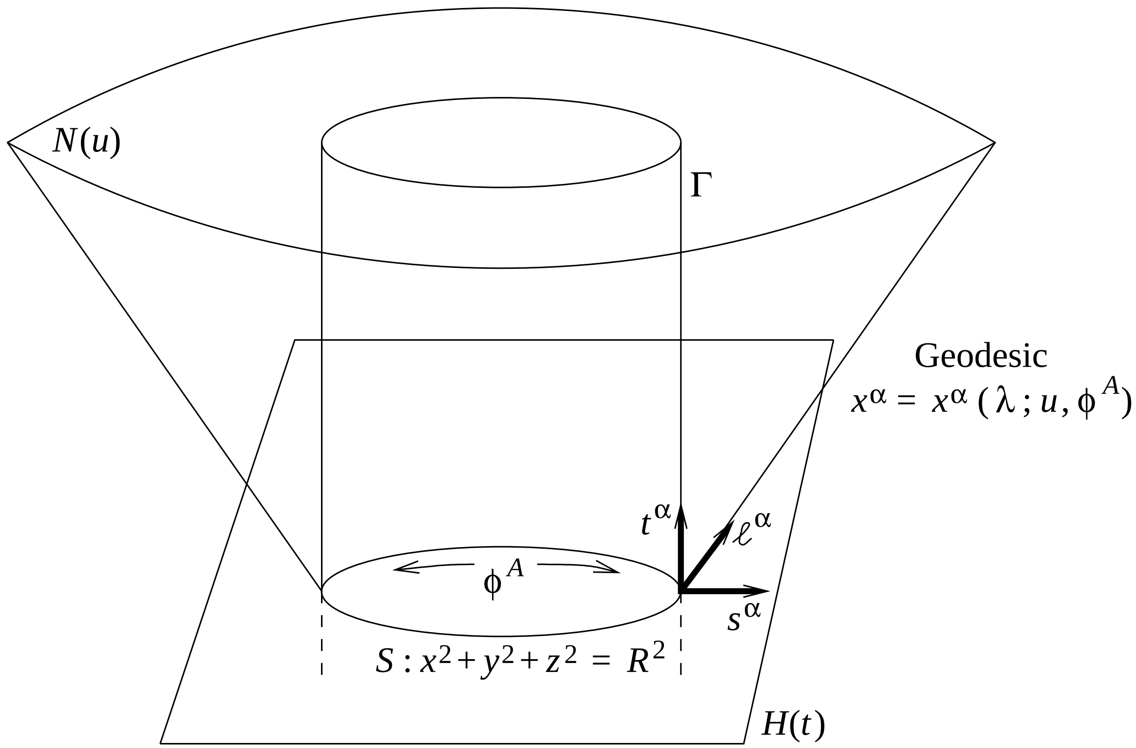
- Define a worldtube Γ, and use the Cauchy metric data to generate characteristic metric data on Γ; then use the characteristic Einstein equations to find metric data between Γ and $\mathcal{J}^+$, and compute the radiation at $\mathcal{J}^+$

- **In extraction:** Impose a standard outer boundary condition for the Cauchy evolution at some surface well outside Γ

- (In matching): Use characteristic metric data to provide an outer boundary condition for the Cauchy evolution


t
r
$\mathcal{J}^+$
Worldtube
Outer boundary
x, y, z
Only in extraction
Only in matching

Extraction procedure – analytic§



§N.T. Bishop *et al.* in B. Iyer and B. Bhawal (Eds.) "Black Holes, Gravitational Radiation and the Universe", Kluwer, Dordrecht, The Netherlands (1999)

Binary black hole evolution

- Equal-mass binary black hole inspiral and merger in two cases (a) $(a_1, a_2) = (0, 0)$, and (b) $(a_1, a_2) = (0.8, -0.8)$

- Initial data parameters are determined from a post-Newtonian evolution to find the momenta for quasi-circular trajectories some 4 orbits before merger.

- BSSN evolution proceeds for $\approx 1350M$, followed by merger and ringdown $\approx 100M$.

- Outermost extraction sphere at $1000M$, outer boundary at $3600M$

Comparison with extrapolated Cauchy waveforms

- Evaluate ψ_4 in a radially oriented tetrad at six radii ($r = 280, 300, 400, 500, 600, 1000M$) and perform an extrapolation based on a 3rd order polynomial least-squares fit to the time-series data

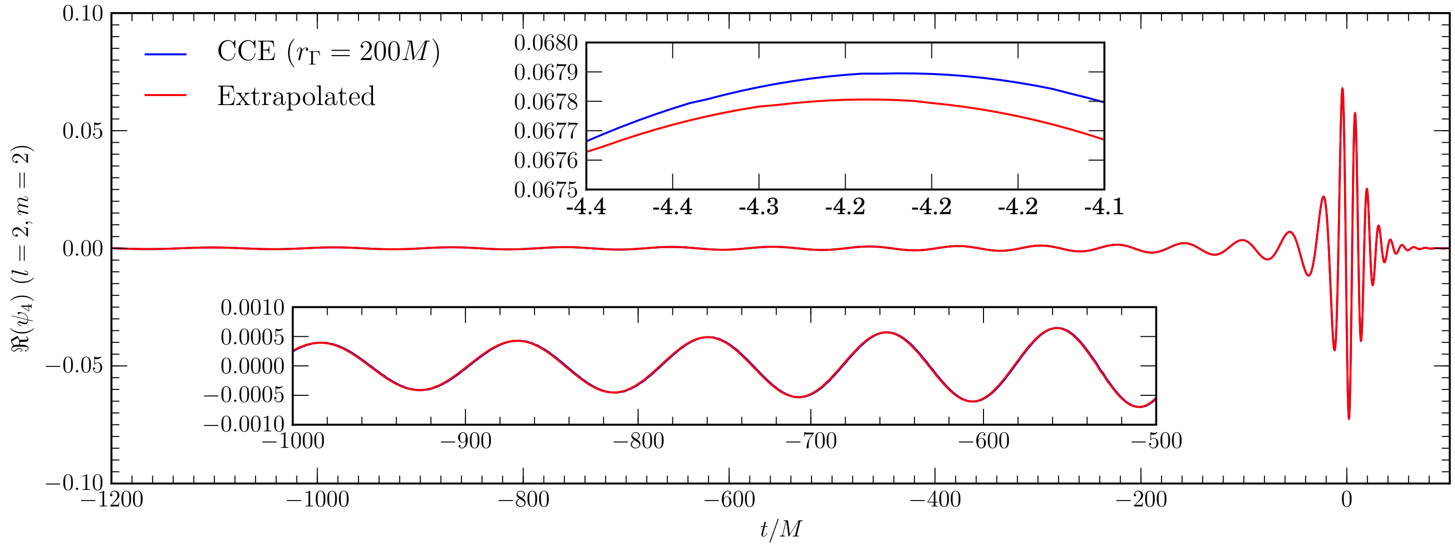
- Use the coordinate radius, r, as a radial coordinate for purposes of the extrapolation. The retarded time, t, is defined by $t_s - r^*$, where the Schwarzschild time t_s is approximated by the coordinate time and r^* is the tortoise coordinate radius

- The error of extrapolation is estimated to be about 0.001% in amplitude and 0.001 radians in phase[¶]

- For the highest resolution $h = 0.64M$, the differences between extrapolated and characteristic extraction waveforms are 1.09% (max) and 0.17% (mean) in amplitude with a dephasing of 0.019 radians (max) and 0.004 radians (mean). The correction is towards slightly larger amplitudes and frequencies

- Recall, the mean error of CCE is about 0.03% in amplitude with a dephasing of 0.002 radians

[¶]D. Pollney, C. Reisswig, N. Dorband, E. Schnetter, P. Diener, Phys.Rev.D **80**, 121502 (2009)


CCE ($r_\Gamma = 200M$)
Extrapolated
$\Re(\psi_4)$ $(l = 2, m = 2)$
t/M
0.10
0.05
0.00
−0.05
−0.10
−1200
−1000
−800
−600
−400
−200
0
0.0680
0.0679
0.0678
0.0677
0.0676
0.0675
-4.4
-4.4
-4.3
-4.2
-4.2
-4.2
-4.1
0.0010
0.0005
0.0000
−0.0005
−0.0010
−1000
−900
−800
−700
−600
−500

Observational significance

- The differences are not important for event detection

- Parameter estimation: We determine[‖] the minimum signal-to-noise ratio (SNR) at which an astrophysical interpretation of detector data would depend on whether the extrapolated or CCE waveform is used as template

- The Table shows the maximum distance for various merger events at which the difference between the two waveforms would be significant

[‖] L. Lindblom *el al.*, Phys. Rev. D **78** 124020 (2008)

Detector	Masses	Maximum distance
LIGO	$50M_{\odot} + 50M_{\odot}$	5Mpc
(e)LIGO	$50M_{\odot} + 50M_{\odot}$	8Mpc
Virgo	$50M_{\odot} + 50M_{\odot}$	14Mpc
AdLIGO	$50M_{\odot} + 50M_{\odot}$	197Mpc
AdVirgo	$50M_{\odot} + 50M_{\odot}$	177Mpc
LISA	$10^7 M_{\odot} + 10^7 M_{\odot}$	$> cH^{-1}$

Linearized solutions**

$$J = \Re(J_0(r)e^{i\nu u})\eth^2 Z_{\ell m}, \;\; U = \Re(U_0(r)e^{i\nu u})\eth Z_{\ell m},$$

$$\beta = \Re(\beta_0(r)e^{i\nu u}) Z_{\ell m}, \;\; W_c = -\frac{2M}{r^2} + \Re(w_0(r)e^{i\nu u}) Z_{\ell m},$$

where $Z_{\ell m}$ are the "real" $Y_{\ell m}$. Then the Einstein hypersurface equations reduce to a single master equation

$$-2(2x + 8Mx^2 + i\nu)J_2 + 2\left(2x^2 + i\nu x - 7x^3 M\right)\frac{dJ_2}{dx}$$

$$+x^3(1-2xM)\frac{d^2 J_2}{dx^2} = 0$$

where $J_2(x) \equiv d^2 J_0/dx^2$ and $x = 1/r$, and so we write

N.T. Bishop, Class. Quantum Grav. **22 2393 (2005)

$$
\begin{aligned}
J_0(r) &= \frac{C_1}{r} + C_2 f(r) + C_3 g(r) + C_4 \\
U_0(r) &= F_U(C_1, C_2, C_3 f(r), C_4 g(r), C_5) \\
w_0(r) &= F_W(C_1, C_2, C_3 f(r), C_4 g(r), C_5) + C_5 \\
\beta_0(r) &= C_6
\end{aligned}
$$

The constraints, $R_{00} = q^A R_{0A} = 0$, impose 2 conditions on $C_1, \cdots, C_6$, so the general solution involves 4 arbitrary constants.

In the case $M = 0$, $f(r), g(r)$ are simple, and for $\ell = 2$

$$\begin{aligned}
J_0 &= \frac{C_1}{4r} - \frac{C_2}{12r^3} + \frac{e^{2i\nu r} C_3}{4r^3}\left(r^2\nu^2 + 2ir\nu - 1\right) + \frac{iC_4}{\nu} \\
U_0 &= \frac{C_1}{2r^2} + \frac{C_2}{12r^4}(3 + 4i\nu r) + \frac{e^{2i\nu r} C_3}{4r^4}(3 - 2i\nu r) + C_4 + \frac{2C_6}{r} \\
w_0 &= \frac{C_2}{2r^4}(1 + 2i\nu r) + \frac{3e^{2i\nu r} C_3}{2r^4} + 6C_4\left(\frac{2i}{\nu r} - 1\right) + \frac{C_5}{r^2} - 10\frac{C_6}{r} \\
\beta_0 &= C_6
\end{aligned}$$

with

$$C_5 = -\nu^2 C_2,\ C_1 = \frac{\nu^2}{3} C_2 + \frac{12}{\nu^2} C_4 + \frac{8i}{\nu} C_6$$

Application 1: Code verification

- Use the solution on a Minkowski background, with no incoming radiation so that $C_3 = 0$, as an exact solution against which to test characteristic numerical realtivity code. Formulas for the gravitational news and ψ_4 are

$$\begin{aligned} \mathcal{N} &= \Re\left(i\nu^3 C_2 e^{i\nu u}\right)\, \eth^2 Z_{2m}, \\ \lim_{r\to\infty} r\psi_4 &= \Re\left(-2\nu^4 C_2 e^{i\nu u}\right)\, \bar{\eth}^2 Z_{2m} \end{aligned}$$

- These solutions have been a crucial tool in debugging characteristic codes.

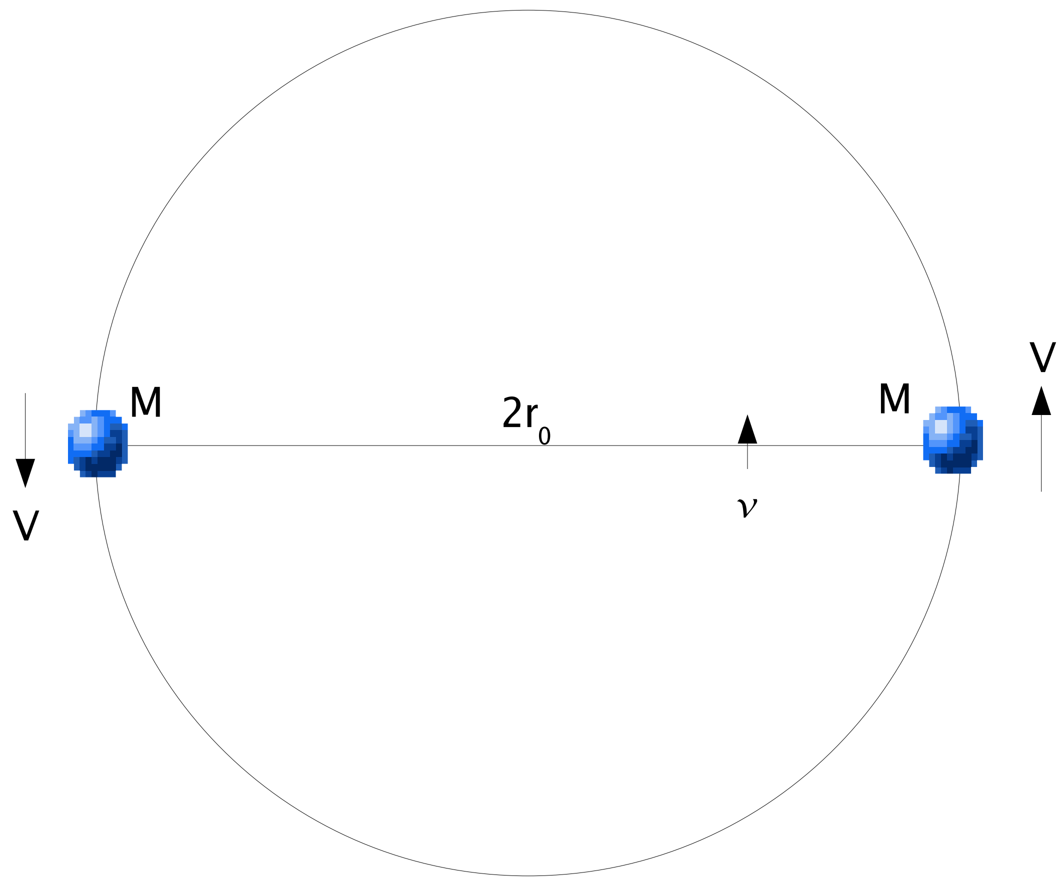
Application 2: Equal mass binary

- Two particles, each of mass M

- Circular orbit radius r_0, at $\theta = \pi/2$, about common centre of mass

- Orbital angular velocity ν, particle velocity V

$$\rho = \frac{M}{r_0^2}\delta(r - r_0)\delta(\theta - \frac{\pi}{2})[\delta(\phi - \nu u) + \delta(\phi - \nu u - \pi)]$$


M
M
2r0
V
V
ν

We express ρ in terms of spherical harmonics

$$\rho = \sum_{\ell,m} \Re\left(\rho_{\ell,m}\,\mathrm{exp}(|m|i\nu u)\right) Z_{\ell,m},$$

For $\ell \leq 2, m \neq 0$, the only nonzero coefficients are

$$\rho_{2,2} = \delta(r - r_0)\frac{M}{2r_0^2}\sqrt{\frac{15}{\pi}}, \quad \rho_{2,-2} = -i\delta(r - r_0)\frac{M}{2r_0^2}\sqrt{\frac{15}{\pi}}.$$

Now construct two separate linearized solutions, one valid in $r < r_0$ with constants $C_{1-}, \cdots, C_{6-}$, and the other valid in $r > r_0$ with constants $C_{1+}, \cdots, C_{6+}$. The 12 constants satisfy the conditions

Number of conditions	Description
2	Constraints in $r < r_0$
2	Constraints in $r > r_0$
1	No incoming radiation in $r > r_0$
3	As $r \to 0$, metric becomes Minkowskian
4	Boundary conditions at $r = r_0$

where the boundary conditions at $r = r_0$ are

$$\begin{aligned} J_{0+} &= J_{0-}, \;\; U_{0+} = U_{0-}, \\ \beta_{,r} &= 2\pi r_0 \rho (1 + V^2), \;\; w_{,r} = -4\pi\rho(1 + V^2) \end{aligned}$$

leading to

$$\mathcal{N} = M(1 + V^2) r_0^2 \nu^3 \sqrt{\frac{2\pi}{5}} 2^4 \left(\sin(2\nu u)\, {}_2Z_{22} - \cos(2\nu u)\, {}_2Z_{2-2} \right).$$

Integrating $N^2/(4\pi)$ over the sphere gives

$$\frac{dE}{du} = -\frac{M^2 (1 + V^2)^2 r_0^4 \nu^6 2^7}{5}$$

In Newtonian limit $V \ll 1$ and orbit is circular if $M = 2^2 r_0^3 \nu^2$

$$\frac{dE}{du} = -\frac{2M^5}{5 r_0^5}$$

which is the standard quadrupole formula.

Higher multipoles

The formalism can also be used to evaluate the gravitational radiation in higher modes, e.g for $\ell = 4$,

$$N = Mr_0^4\nu^5\Big(\quad 0.64\,(\sin(2\nu u)\,{}_2Z_{42} - \cos(2\nu u)\,{}_2Z_{4-2}) \\ + \quad 26.9\,(\sin(4\nu u)\,{}_2Z_{44} - \cos(4\nu u)\,{}_2Z_{4-4})\Big).$$

Application 3: Quasi-normal modes[††]

- Recall the master equation for a linearized solution

$$-2(2x+8Mx^2+i\nu)J_2+2\left(2x^2+i\nu x-7x^3M\right)\frac{dJ_2}{dx}$$
$$+x^3(1-2xM)\frac{d^2J_2}{dx^2}=0 \qquad (1)$$

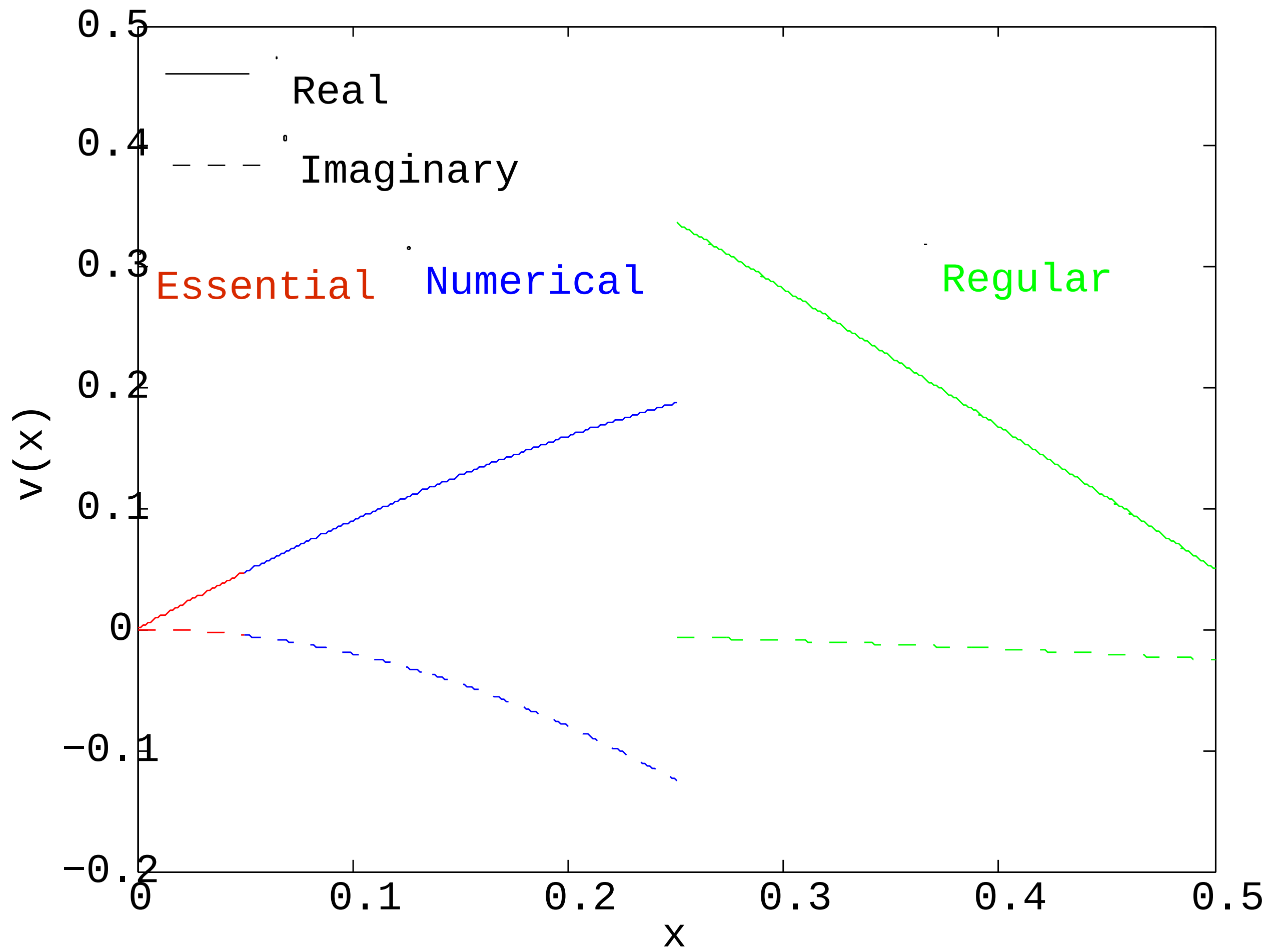
- Eq. (1) has singularities at $x = 0$ and $x = 0.5M$. The problem is to find values of ν for which there exists a solution to Eq. (1) that is regular everywhere in the interval [0, 0.5M]; these values of ν are the quasi-normal modes.

[††]N.T. Bishop and A.S. Kubeka, *Phys. Rev. D* **80**, 064011 (2009)

- While the differential equation is different, this is the same scenario as when finding the quasi-normal modes of a black hole. The first solution was obtained by using series solutions around the singular points, and a numerical solution in the interior of the interval. Nowadays, quasi-normal modes are usually found using the theory of 3-term recurrence relations, but for technical reasons that theory cannot be used here.

- We construct the asymptotic series about the essential singularity at $x = 0$, and use it to find a solution at a point $x_0 > 0$. We then use this solution as initial data for a numerical solution of Eq. (1) in the range (x_0, x_c) where $x_c < 0.5M$. Finally, we construct the regular series solution about $x = 0.5M$ and use it to find a solution at $x = x_c$. Then a value of ν is a quasi-normal mode if the difference at $x = x_c$ between the regular series solution and the numerical solution, vanishes.

- The difficult part is the essential singularity, because the asymptotic series is not convergent. However, we can calculate a rigorous bound on the error involved in approximating the solution by a given number of terms of the series, and ensure that this is less than machine precision.



Schematic plot of $v(x) = J_2(x)/J_2'(x)$ against x.

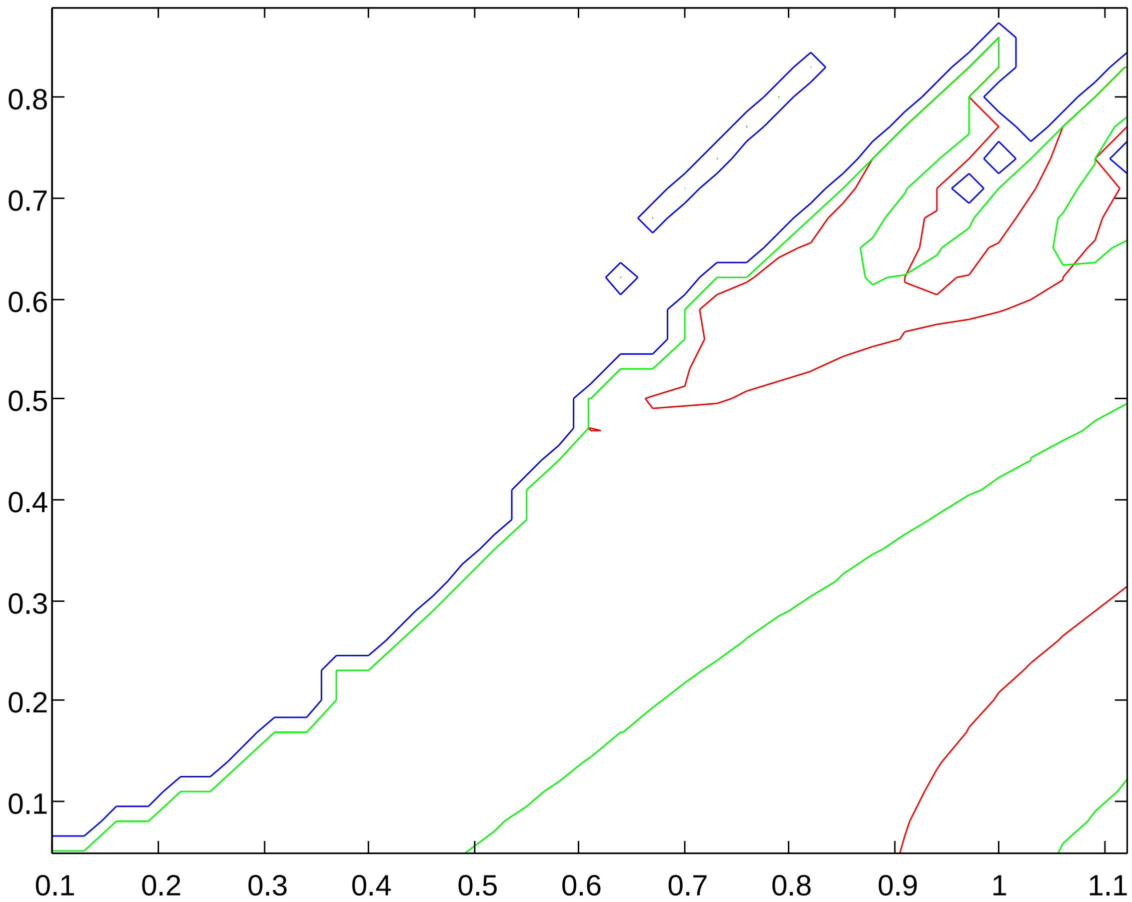
Results

- Defining

$$g_\nu = v_+ - v_-,$$

  the quasi-normal modes are those values of ν such that g_ν is indistinguishable from zero.

- We calculated g_ν for values of ν in the range $\nu = a + ib$, $0.1 \leq a \leq 1.07$, $0.05 \leq b \leq 0.89$, in increments of 0.03.

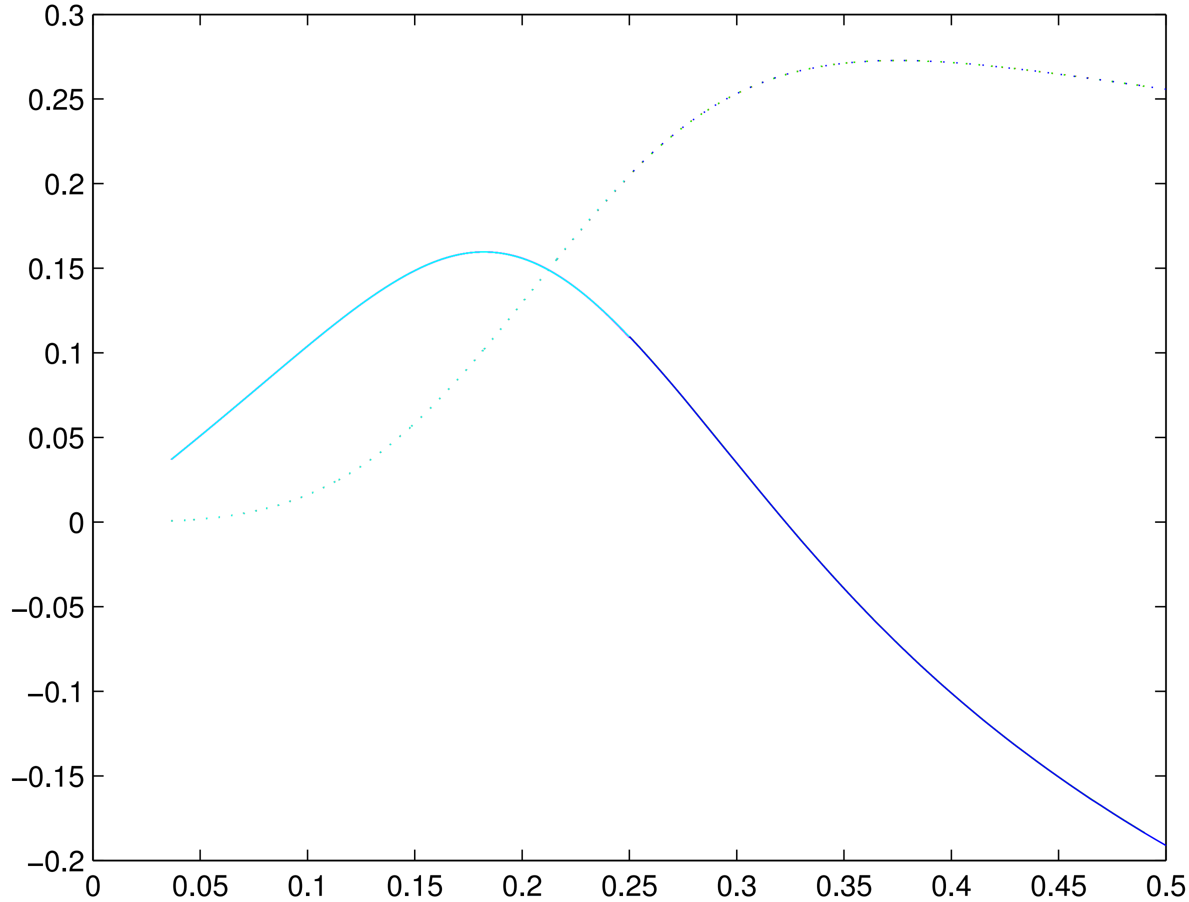


Contour plot in the complex plane: $\Im(g_\nu) = 0$, $\Re(g_\nu) = 0$, Boundary of reliable computation.

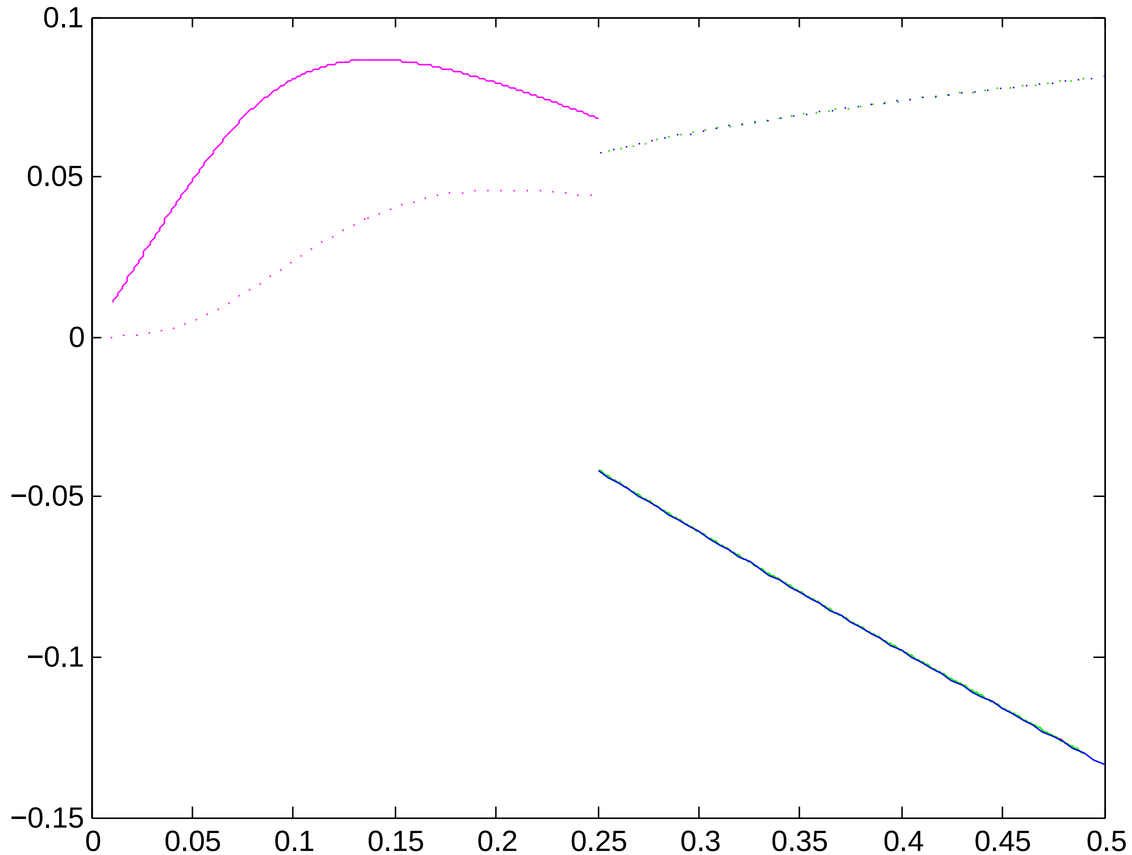
- We then applied a secant method, and calculated bounds on the possible error, obtaining the lowest quasi-normal mode

$$\nu = 0.883 + 0.614i + 0.003k$$

where k is a complex number satisfying $|k| \leq 1$.



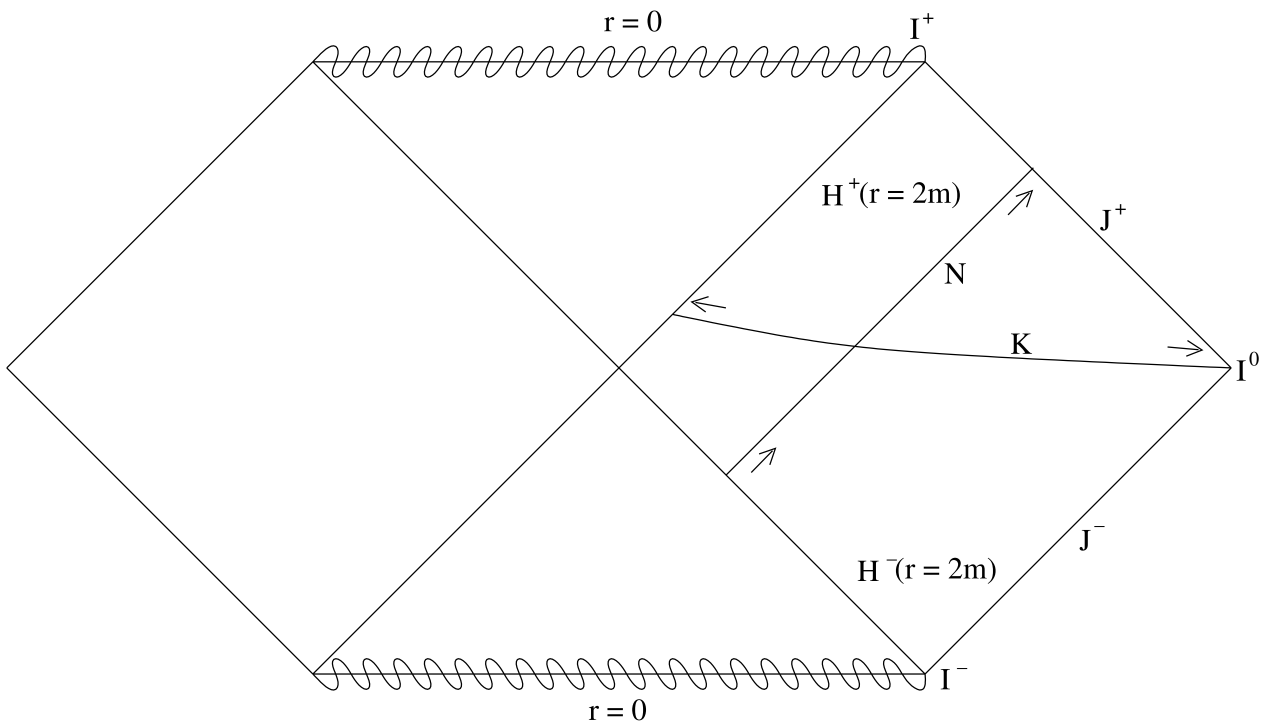
$v(x)$ for $\nu = 0.883 + 0.614i$.



$v(x)$ for $\nu = 0.37367 + 0.08896i$, which is the lowest quasi-normal mode of a Schwarzschild black hole.

Discussion

- How can the quasi-normal modes of Eq. (1) be different to the usual values of a Schwarzschild black hole?



- Geometrically, K is a typical hypersurface in the black hole case, and N is null. From the direction of wave propagation on N, the quasi-normal modes can be interpreted as perturbations of a white hole.

- Algebraically, Eq. (1) is a second order d.e., and there are two independent series solutions about the regular singularity at $x = 0.5$, i.e. at the horizon, say $as_1(x)+bs_2(x)$ with $s_2(x)$ unbounded near the horizon. For the calculations above we set $b = 0$, but if instead we set $a = 0$, the standard Schwarzschild QNM is found.

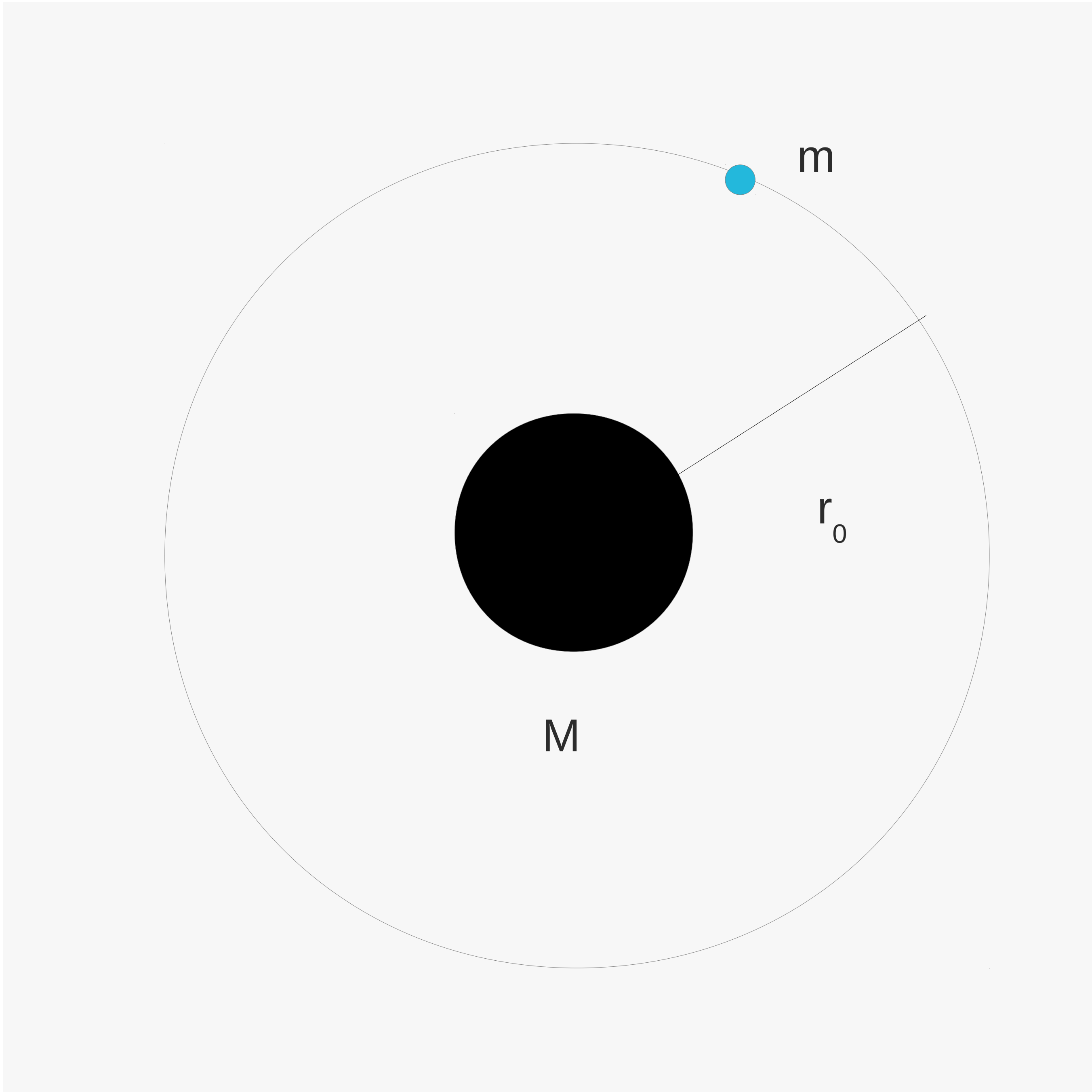
Application 4: Particle orbiting a Schwarzschild black hole

In principle, the calculation is similar to that on the Minkowski background, but is technically much more difficult because part of the solution is not known analytically.

- Black hole of mass M, particle of mass m_0 in circular orbit at $r_0 = 6M$, $\theta = \pi/2$

$$\frac{d\phi}{d\tau} = \frac{\sqrt{3}}{18M}, \quad \nu = \frac{d\phi}{du} = \frac{\sqrt{6}}{36M}$$

$$\rho = \frac{m_0}{r_0^2}\delta(r - r_0)\delta(\theta - \frac{\pi}{2})\delta(\phi - \nu u)$$


m
r0
M

We express ρ in terms of spherical harmonics

$$\rho = \sum_{\ell,m} \Re\left(\rho_{\ell,m}\,\mathrm{exp}(|m|i\nu u)\right) Z_{\ell,m},$$

For $\ell \leq 2, m \neq 0$, the only nonzero coefficients are

$$\rho_{2,2} = \delta(r-r_0)\tfrac{m_0}{r_0^2}\sqrt{\tfrac{15}{\pi}}, \quad \rho_{2,-2} = -i\delta(r-r_0)\tfrac{m_0}{r_0^2}\sqrt{\tfrac{15}{\pi}}.$$

Now construct two separate linearized solutions, one valid in $r < r_0$ with constants $C_{1-}, \cdots, C_{6-}$, and the other valid in $r > r_0$ with constants $C_{1+}, \cdots, C_{6+}$. The 12 constants satisfy the conditions

Number of conditions	Description
2	Constraints in $r < r_0$
2	Constraints in $r > r_0$
1	No incoming radiation in $r > r_0$
2	Bondi gauge conditions as $r \to \infty$
1	Exclude well-behaved solution at horizon
4	Boundary conditions at $r = r_0$

where the boundary conditions at $r = r_0$ are

$$\begin{aligned} J_{0+} &= J_{0-}, \;\; U_{0+} = U_{0-}, \\ \beta_{,r} &= 4\pi r_0 \rho, \;\; w_{,r} = -\frac{16}{3}\pi\rho \end{aligned}$$

leading to

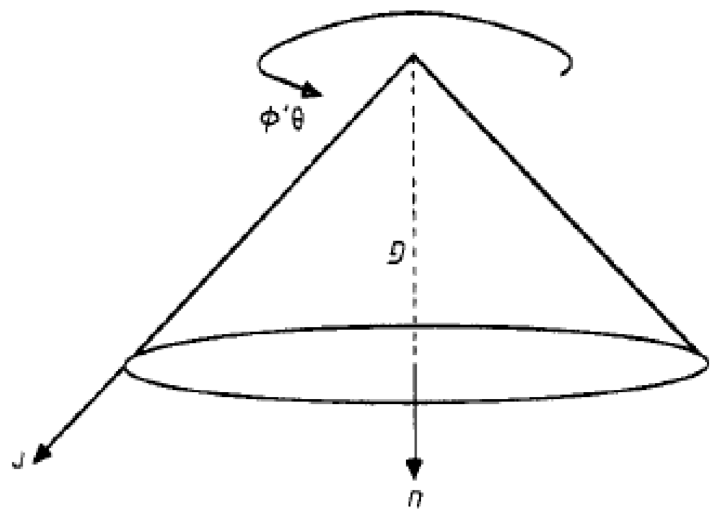
$$\begin{aligned}
\mathcal{N} &= \frac{m_0}{M}\Bigg(\Re\left[(0.014267+0.093467i)\exp\left(\frac{2i\nu u}{M}\right)\right]\,{}_2Z_{2,2} \\
&- \Re\left[(0.014267i-0.093467)\exp\left(\frac{2i\nu u}{M}\right)\right]\,{}_2Z_{2,-2}\Bigg) \\
dE/du &= -7.114\times 10^{-4}\frac{m_0^2}{M^2}
\end{aligned}$$

Compare value from quadrupole formula $-8.2305\times 10^{-4} m_0^2/M^2$.

Still work in progress – value for $\mathcal{N}$ should not be regarded as reliable.

COSMOLOGY

Figure 1. Null coordinates.



Observational cosmology[‡‡]

- Cosmological data is (almost all) from the past null cone, and it is natural to apply the characteristic approach.

- These ideas were developed by Ellis and others, using a different formalism.

- Here, we adapt the characteristic code so that it actually computes the past behaviour of the Universe. We take the cosmological fluid as dust, with $p = 0$, $T_{ab} = \rho v_a v_b$.

[‡‡] G.F.R. Ellis *et al.*, Phys. Rep., **124**, 315 (1985); P.J. van der Walt and N.T. Bishop, Phys. Rev. D: **82**, 084001 (2010); **85**, 044016 (2012)

Data on the past null cone

- Initial data required by the code: J, ρ, v_1, V, as functions of r, x^A.

- Observational data

x^A	Position on the sky
d_L	Luminosity distance: related to r by $d_L = (1+z)^2 r$
z	Red-shift : $v^0 = 1 + z, v_1 = -e^{2\beta}(1+z)$
$\frac{dx^A}{du}$	Angular velocity : $v^A = (1+z)\frac{dx^A}{du}, v_A = F_9(r, z, \beta, U, J, \frac{dx^A}{du})$
n	Observed number count : $N = \frac{ne^{-2\beta}}{1+z}$proper number count
J	Shear: from observed shape of spherical object

- A relationship for ρ needs to be assumed, say $\rho = F_{10}(N)$.

- The Einstein equation for R_{11} becomes

$$\beta_{,r} = f_1 + 2\pi r F_{10}\left(\frac{ne^{-2\beta}}{1+z}\right)\left(-e^{2\beta}(1+z)\right)^2$$

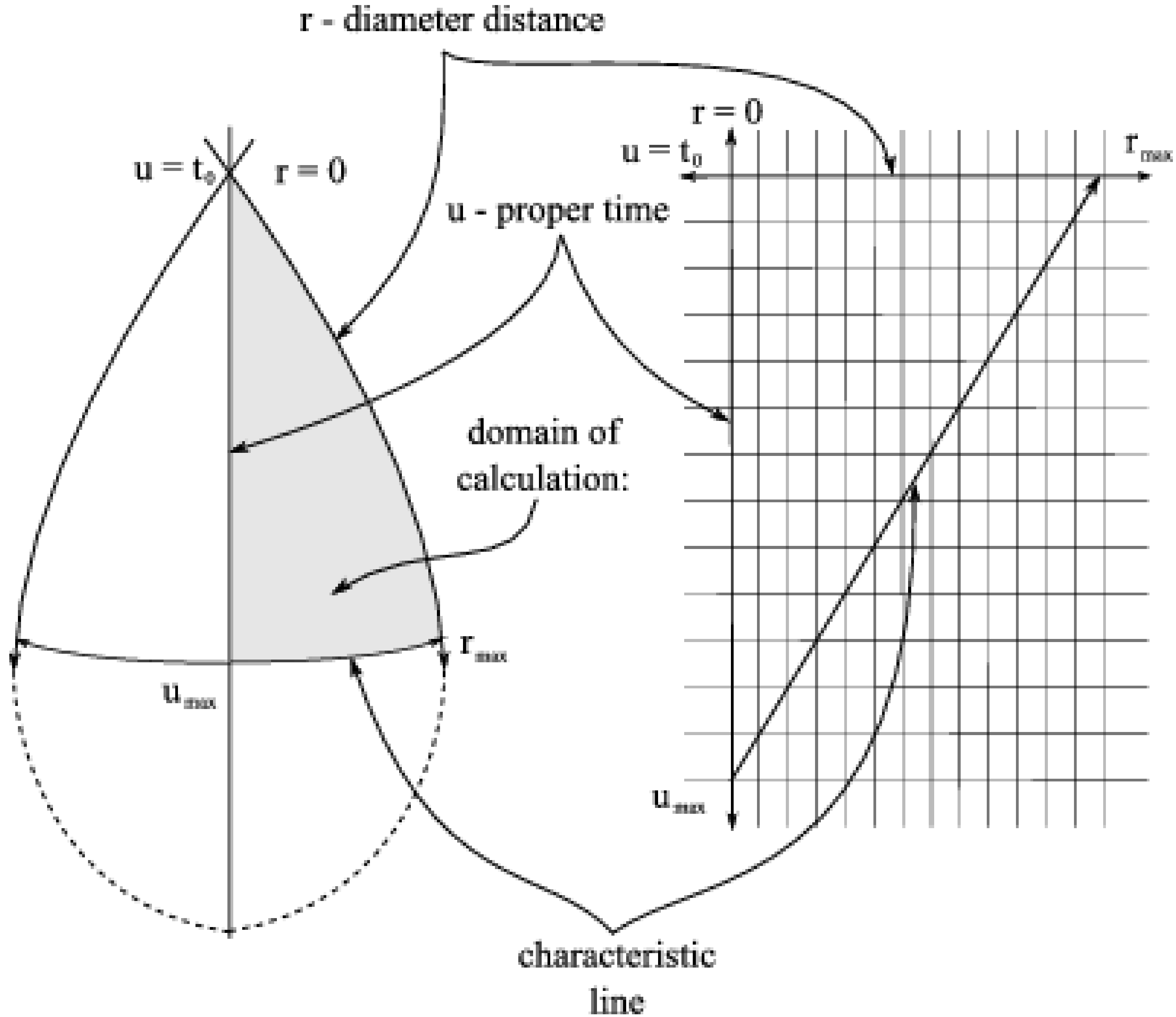
which remains an o.d.e. for β. Once solved, v_1 is also found.

- Similarly, the R_{1A} equations remain o.d.es. for U, and once solved, v_A and hence V are also found.

Spherical symmetry

- Cosmology code implemented and tested for spherical symmetry.

- Difficult numerical issues
  - Evolution near the origin
  - Outer boundary – incoming null geodesic.



r - diameter distance
u = t0
r = 0
u - proper time
domain of calculation:
r = 0
u = t0
rmax
rmax
umax
umax
characteristic line

Code testing

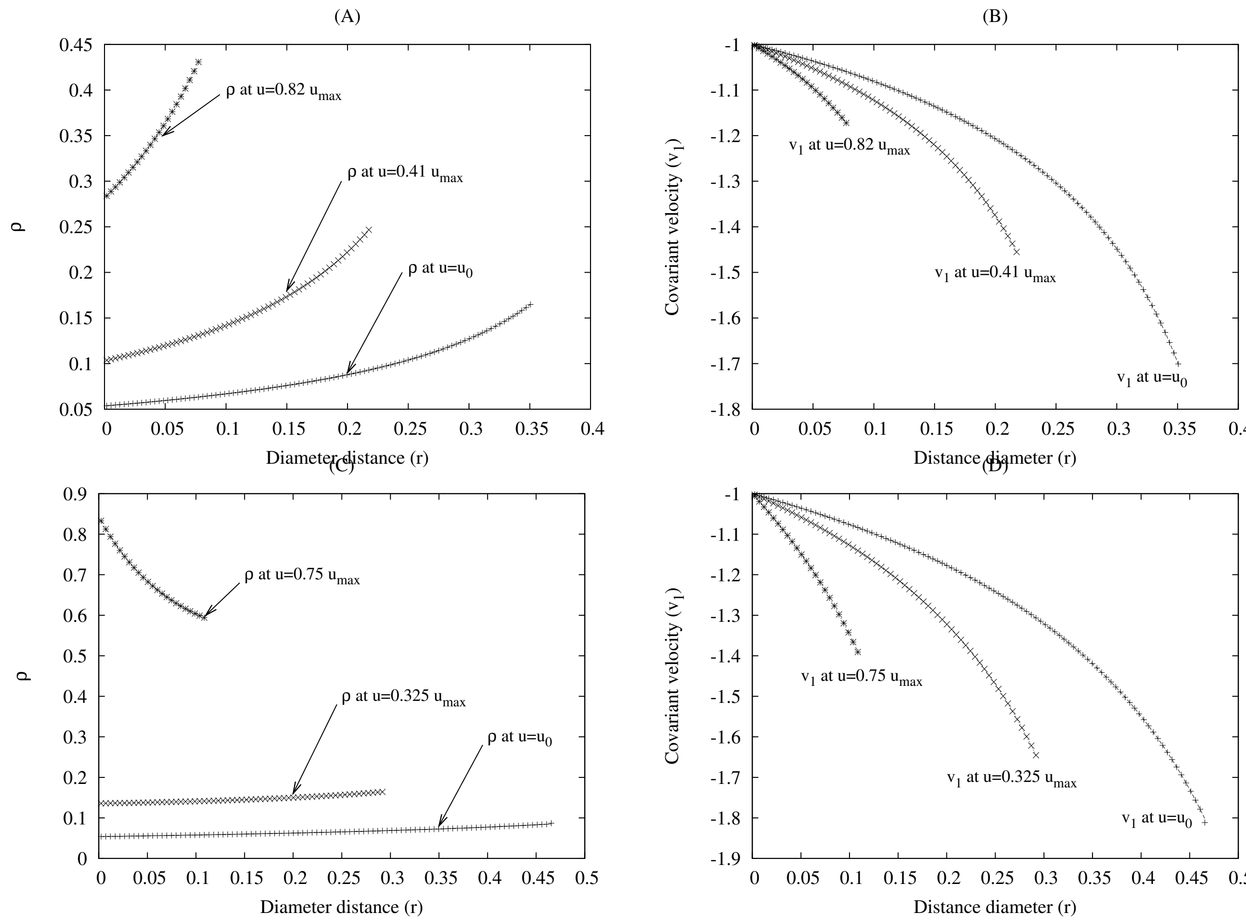
- Use Lemaître-Tolman-Bondi (LTB) model as exact solution

$$ds^2 = -dt^2 + [R_{,r}(t,r)]^2 dr^2 + [R(t,r)]^2 (d\theta^2 + sin^2\theta d\phi^2).$$

- Spherically symmetric inhomogeneous model, with FRW recovered when $R(t,r) = ra(t)$.

- We use $R(t,r) = r(t - br)^{2/3}$, so $b = 0$ is Einstein-de Sitter.

- Construct (numerically) coordinate transformation to characteristic coordinates.

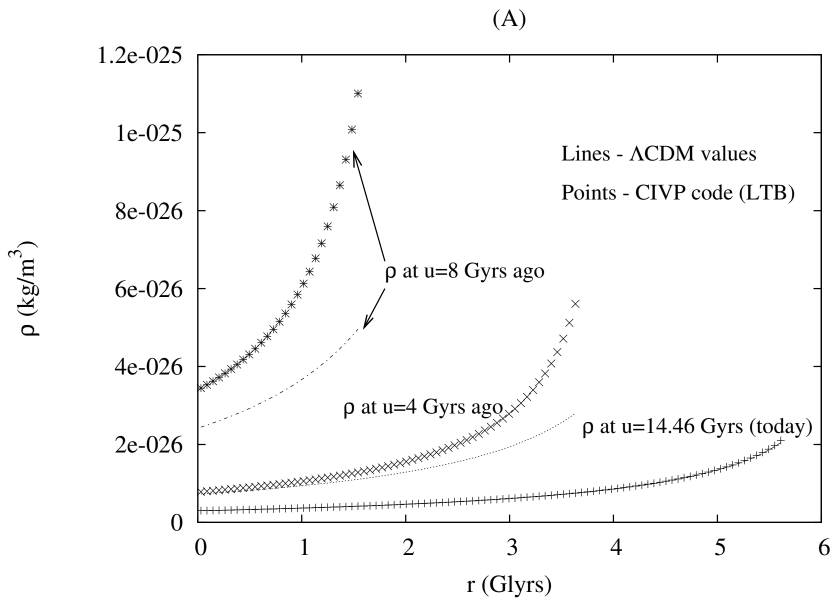


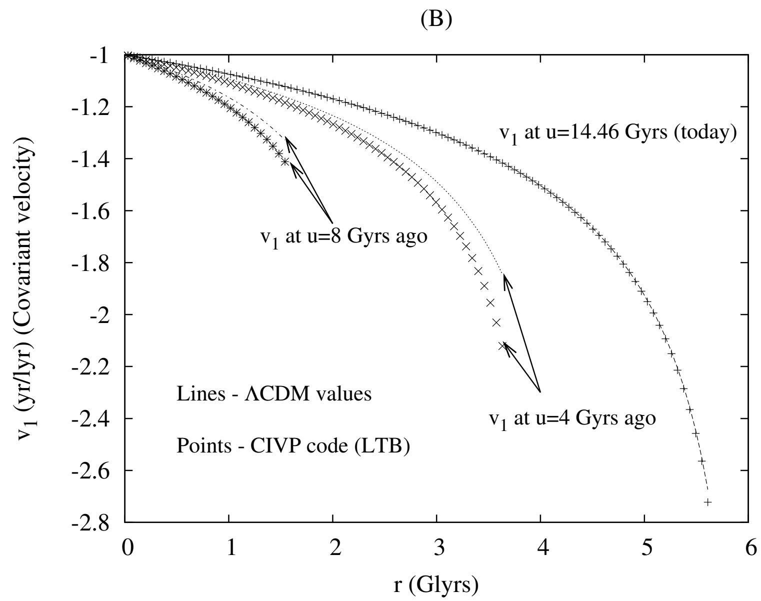
(A), (B): EdS; (C), (D): LTB with $b = -0.5$

LTB vs ΛCDM

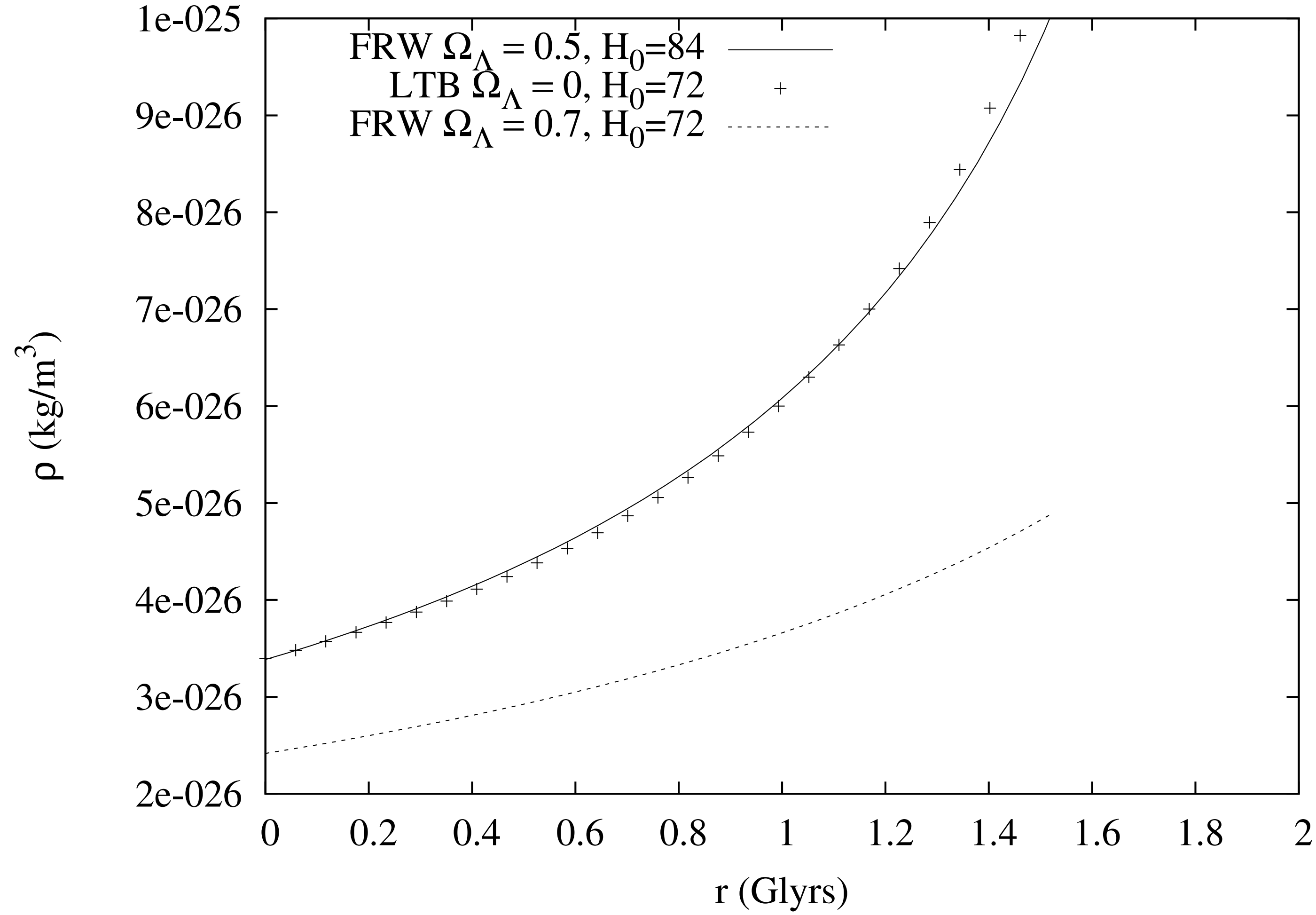
- SnIa data is usually regarded as caused by dark energy, but could also be explained by large scale inhomogeneities, i.e. an LTB model.

- However, the past behaviour of LTB and ΛCDM models is different.

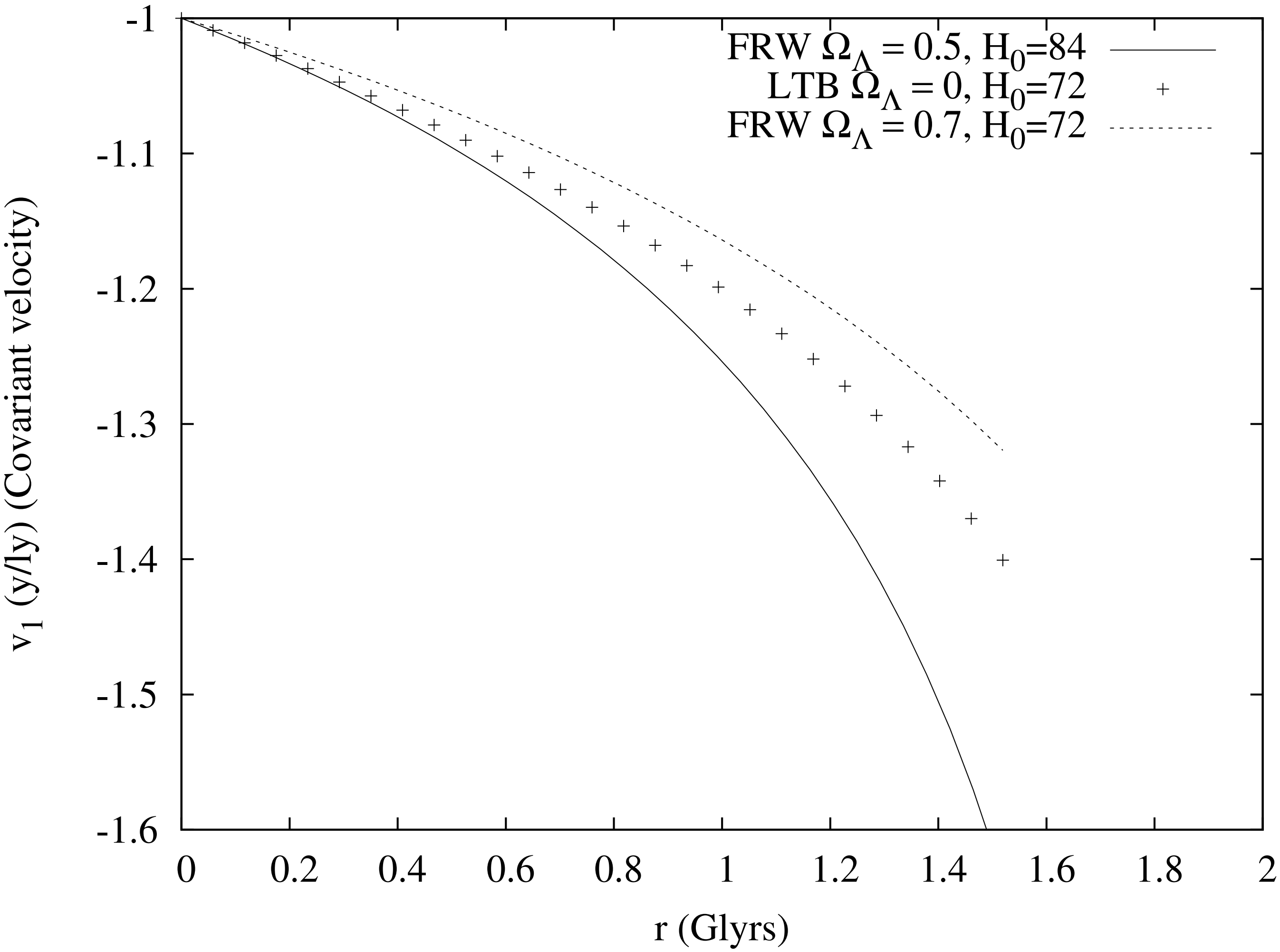




Suppose that we are in a $\Lambda = 0$, LTB universe. In the past, 6 Gyrs ago, could the observational data also have been interpreted as ΛCDM?



FRW $\Omega_\Lambda = 0.5$, H_0=84
LTB $\Omega_\Lambda = 0$, H_0=72
FRW $\Omega_\Lambda = 0.7$, H_0=72
ρ (kg/m^3)
r (Glyrs)
1e-025
9e-026
8e-026
7e-026
6e-026
5e-026
4e-026
3e-026
2e-026
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


FRW Ω_Λ = 0.5, H_0=84
LTB Ω_Λ = 0, H_0=72
FRW Ω_Λ = 0.7, H_0=72
v_1 (y/ly) (Covariant velocity)
r (Glyrs)
-1
-1.1
-1.2
-1.3
-1.4
-1.5
-1.6
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

No: If the universe is LTB, then not only are we at a special place, but also at a special time

Going beyond the point of reconvergence on the past null cone

- Due to the expansion of the universe, the past null cone has a maximum size (where $r = r_{\mathrm{max}}$), and beyond this it reconverges. In EdS, r_{max} is at $z = 1.25$.

- At $r = r_{\mathrm{max}}$ the coordinates are singular, and the code can be used only in a domain $r < r_{\mathrm{max}} - \epsilon$.

- This is a coordinate problem and is resolved by using new coordinates.

Affine radial coordinate

From the geodesic equation

$$\frac{dr}{d\lambda} = e^{-2\beta}. \tag{2}$$

Apply the tensor transformation law to the Bondi-Sachs metric, and re-write it in the form

$$ds^2 = -\left(1 + \frac{\widehat{W}}{\widehat{r}}\right) du^2 - 2du d\lambda + \widehat{r}^2 \{d\theta^2 + \sin^2\theta d\varphi^2\} \tag{3}$$

with: $\widehat{W} = \widehat{W}(u, \lambda)$ and $\widehat{r} = \widehat{r}(u, \lambda)$.

Substitution into the Einstein field equations gives

$$\hat{r}_{,\lambda\lambda} = -\frac{1}{2}\kappa\hat{r}\rho(v_1)^2 \tag{4}$$

$$\hat{r}_{,u\lambda} = \frac{1}{2}\Big\{\widehat{W}_{,\lambda}\hat{r}_{,\lambda} + \hat{r}\hat{r}_{,\lambda\lambda} + \widehat{W}\hat{r}_{,\lambda\lambda} - 2\hat{r}_{,u}\hat{r}_{,\lambda} - 1 \tag{5}$$

$$+ (\hat{r}_{,\lambda})^2 + \frac{1}{2}\kappa\rho\hat{r}^2 - \Lambda\hat{r}^2\Big\}\Big/\hat{r} \tag{6}$$

$$\widehat{W}_{,\lambda\lambda} = \frac{\widehat{W}}{\hat{r}}\hat{r}_{,\lambda\lambda} + 4\hat{r}_{,u\lambda} + 2\kappa\left(v_0 v_1\rho - \frac{1}{2}\rho\right)\hat{r} + 2\Lambda\hat{r} \tag{7}$$

with: $\hat{r}(0) = \widehat{W}(0) = \widehat{W}_{,\lambda}(0) = \hat{r}_{,u}(0) = 0$ and $\hat{r}_{,\lambda}(0) = 1$.

Then substituting the dust stress-tensor ($T_{ab} = \rho v_a v_b$) into the conservation equation, $T^{ab}_{\ \ ;b} = 0$, yields the fluid equations

$$v_{1,u} = \frac{1}{v_1}\left\{\left(\hat{V}_w v_1 - v_0\right) v_{1,\lambda} + \frac{1}{2}(v_1)^2 \hat{V}_{w,\lambda}\right\} \tag{8}$$

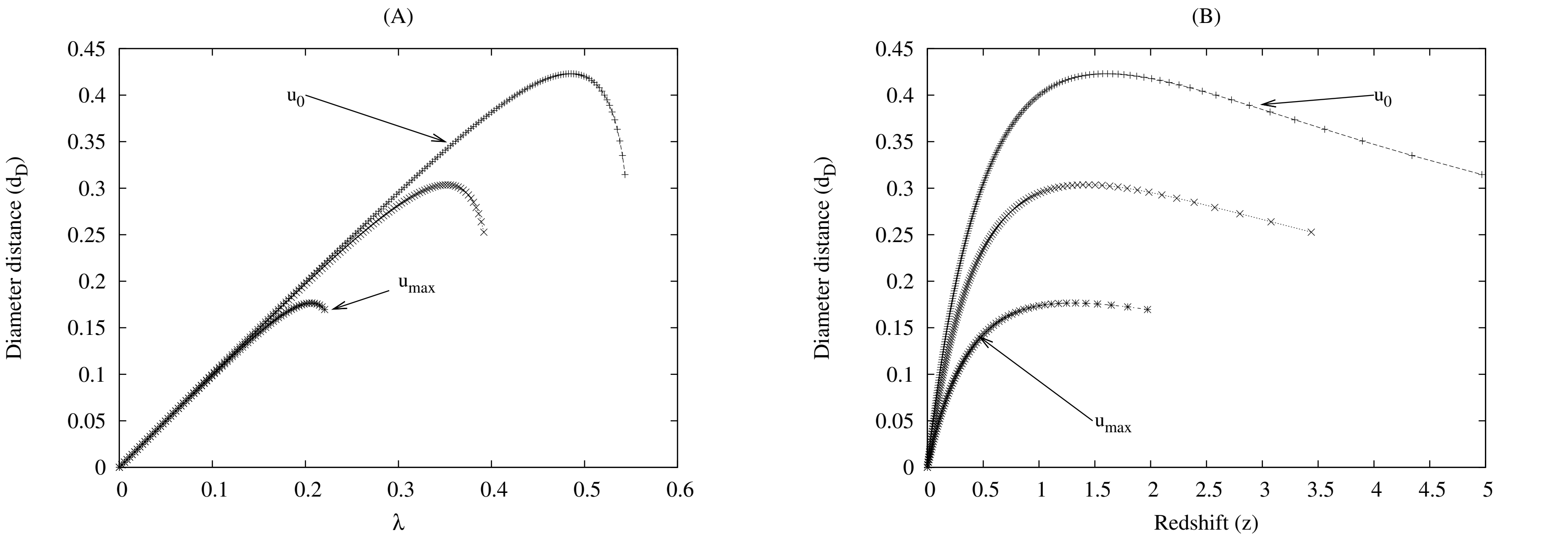
$$\begin{aligned} \rho_{,u} = \frac{1}{v_1}\Bigg\{\rho\Bigg[\hat{V}_w\left(\frac{2v_1}{\hat{r}}\hat{r}_{,\lambda} + v_{1,\lambda}\right) - \left(\frac{2v_0}{\hat{r}}\hat{r}_{,\lambda} + v_{0,\lambda}\right) + \hat{V}_{w,\lambda} v_1 \\ - \left(\frac{2\hat{r}_{,u}}{\hat{r}}\right) v_1\Bigg] + \rho_{,\lambda}\left(\hat{V}_w v_1 - v_0\right) - \rho v_{1,u}\Bigg\} \end{aligned} \tag{9}$$

with: $\hat{V}_w = 1 + \frac{\hat{W}}{\hat{r}}$.

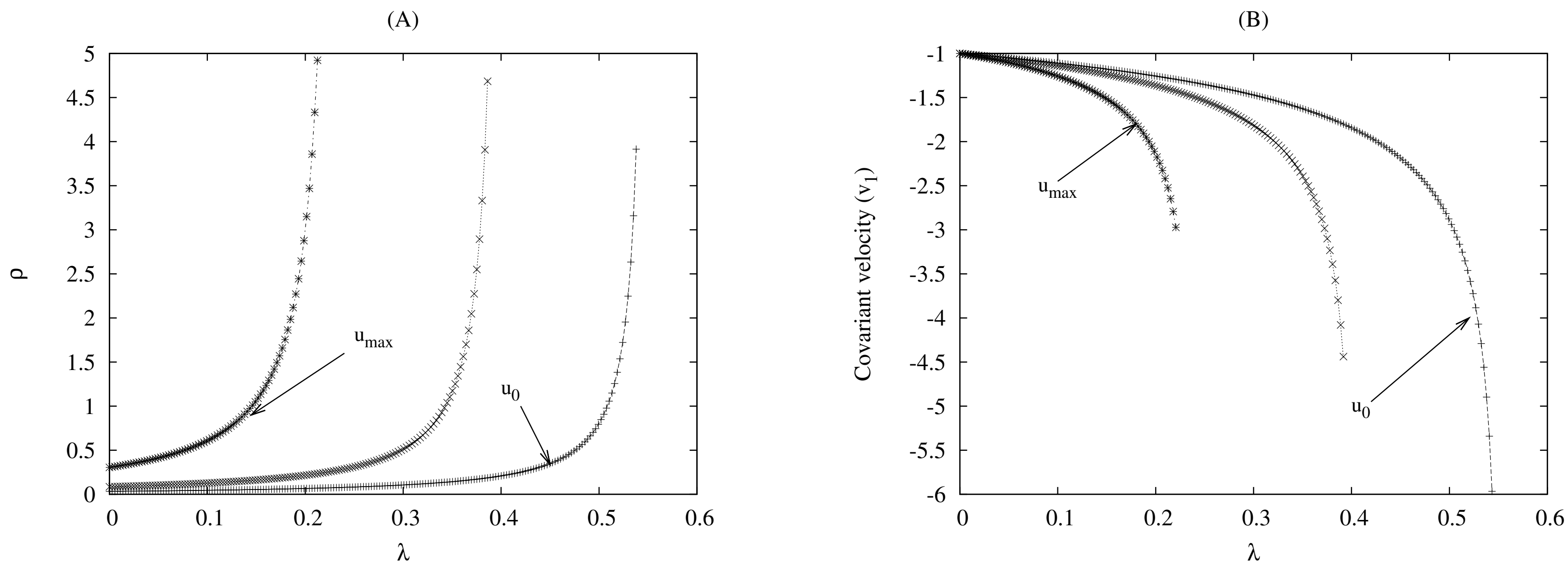
Using the condition $g^{ab} v_a v_b = -1$, v_0 can be written as

$$v_0 = \frac{1}{2}\hat{V}_w v_1 + \frac{1}{2} v_1^{-1}. \tag{10}$$

ΛCDM with $\Omega_\Lambda = 0.7$



Diameter distance against λ (A) and against of z (B) on PNCs at different proper times (u) evolved from a local PNC up to $z = 5$.



Density distribution (A) and covariant velocity (B) on PNCs at different proper times (u) evolved from a local PNC up to $z = 5$.

Conclusions

- The characteristic formalism can be used for numerical evolutions in vacuum, or with matter.
- The formalism is use for the extraction of gravitational radiation from a "3+1" simulation.
- Linearized solutions are used for code-testing, for gravitational wave calculations, and for the calculation of quasi-normal modes.
- The characteristic code can be used, in principle, to compute the past behaviour of the universe from observations.
- If the universe is $\Lambda = 0$ LTB rather than ΛCDM, then not only are we in a special position, but we are also at a special time.
- In order to get past the reconvergence of the past null cone, the code has been reformulated using an affine, rather than a surface area, radial parameter.

THANK YOU