Gravitational waves from extreme mass ratio inspirals

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Plan

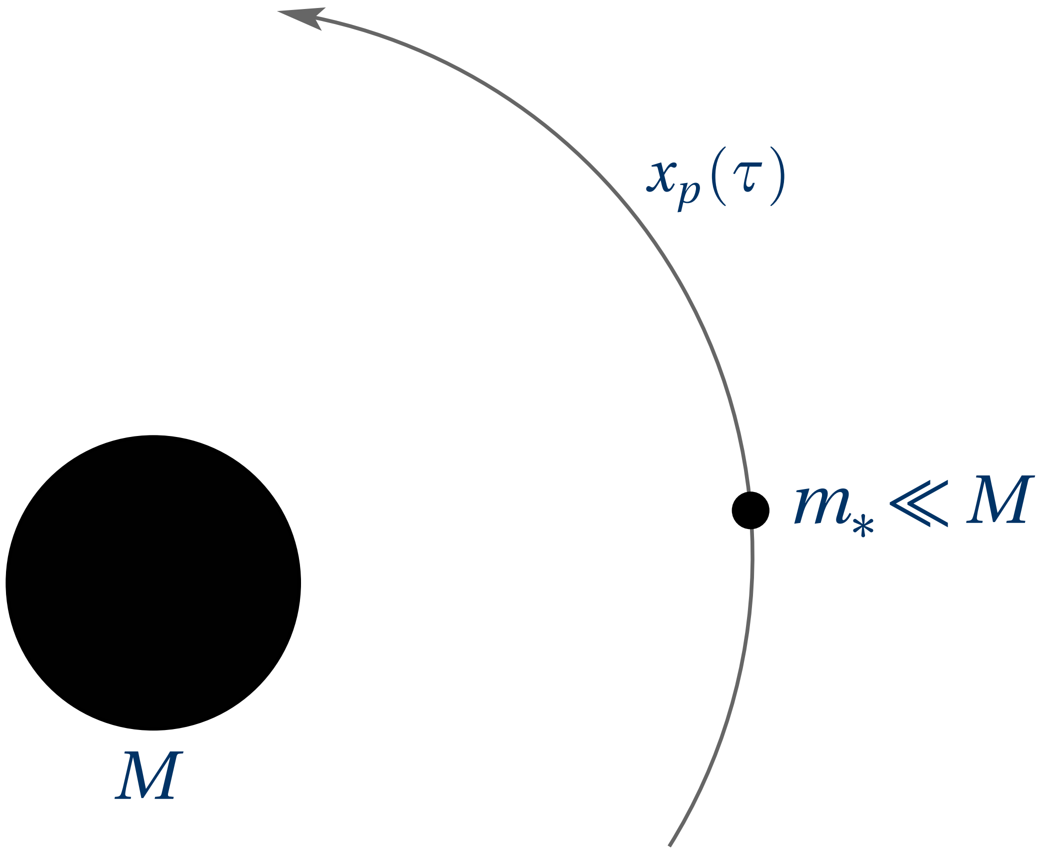
1. **Wave forms in Regge-Wheeler gauge**
   - Perturbation theory
   - Regge-Wheeler gauge
   - Wave equation
   - Numerical implementation
   - Code validation

2. **Self-force computation : radial fall case**
   - Self-force computation
   - Mode-sum regularisation
   - Self-force in RW gauge
   - Action of the SF on the trajectory

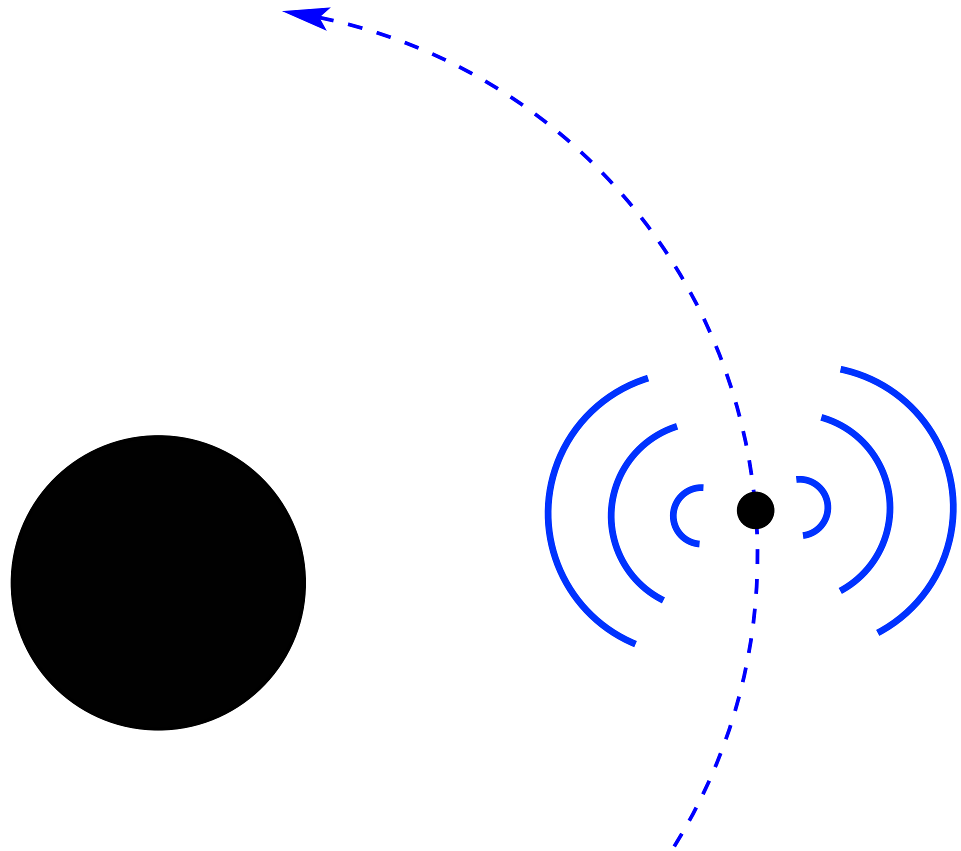
3. **Conclusions**

1
Wave forms in Regge-Wheeler gauge

Perturbation theory



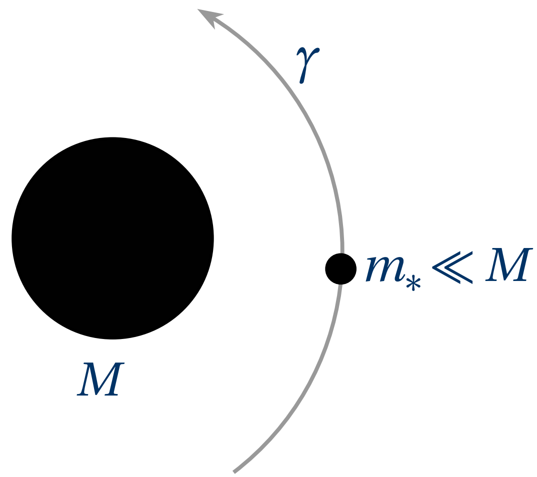
Perturbation theory



Perturbation theory

- **E**xtreme **M**ass **R**atio **I**nspiral

Schwarzschild black hole(-Droste)
+
point particle



- Metric perturbation

$$g^{\text{tot}}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta} \qquad\qquad g_{\alpha\beta} \gg h_{\alpha\beta} \sim O(m_*/M)$$

- Linearised field equations

$$G_{\alpha\beta}[g+h] = 8\pi T_{\alpha\beta}[g+h;\gamma]$$

$$G^{(0)}_{\alpha\beta}[g] + G^{(1)}_{\alpha\beta}[g,h] + O(h^2) = 8\pi T_{\alpha\beta}[g;\gamma] + O(h^2)$$

Perturbation theory

- Linearised field equations

$$G^{(0)}_{\alpha\beta}[g] + G^{(1)}_{\alpha\beta}[g,h] + O(h^2) = 8\pi T_{\alpha\beta}[g;\gamma] + O(h^2)$$

with

$$G^{(0)}_{\alpha\beta}[g] = G_{\alpha\beta}[g] = 0$$

$$T_{\alpha\beta}[g;\gamma] = m_* \int_{-\infty}^{+\infty} \frac{dx_\alpha}{d\tau}\frac{dx_\beta}{d\tau}(-g)^{-1/2}\delta^{(4)}(x - x_p(\tau))\,d\tau$$

$$\sim O(h)$$

Perturbation theory

- Linearised field equations

$$G^{(0)}_{\alpha\beta}[g] + G^{(1)}_{\alpha\beta}[g,h] + O(h^2) = 8\pi T_{\alpha\beta}[g;\gamma] + O(h^2)$$

at first order

$$\boxed{G^{(1)}_{\alpha\beta} = 8\pi T_{\alpha\beta}}$$

with

$$G^{(1)}_{\alpha\beta}[g,h] = -\frac{1}{2}\nabla^\gamma\nabla_\gamma h_{\alpha\beta} + \nabla_\beta\nabla^\gamma h_{\alpha\gamma} + \nabla_\alpha\nabla^\gamma h_{\beta\gamma} - R_{\gamma\alpha\delta\beta}h^{\gamma\delta}$$
$$-\frac{1}{2}\nabla_\beta\nabla_\alpha h - \frac{1}{2}g_{\alpha\beta}\left(\nabla^\delta\nabla^\gamma h_{\delta\gamma} - \nabla^\gamma\nabla_\gamma h\right)$$

Perturbation theory

- Metric multipolar expansion

$$h_{\alpha\beta} = \sum_{\ell,m} \sum_{i=1}^{10} h^{(i)\ell m}(t,r)\, Y^{(i)\ell m}_{\alpha\beta}(\theta,\phi)$$

where

$$\{Y^{(i)\ell m}_{\alpha\beta}\} = \Big\{ \underbrace{Y^{(1)\ell m}_{\alpha\beta}, Y^{(2)\ell m}_{\alpha\beta}, Y^{(3)\ell m}_{\alpha\beta}, \cdots, Y^{(7)\ell m}_{\alpha\beta}}_{\text{even parity}}, \underbrace{Y^{(8)\ell m}_{\alpha\beta}, Y^{(9)\ell m}_{\alpha\beta}, Y^{(10)\ell m}_{\alpha\beta}}_{\text{odd parity}} \Big\}$$

are the 10 Zerilli tensor spherical harmonics

$$\{h^{(i)\ell m}\} \propto \Big\{ \underbrace{H_0^{\ell m}, H_1^{\ell m}, H_2^{\ell m}, h_0^{(e)\ell m}, h_1^{(e)\ell m}, G^{\ell m}, K^{\ell m}}_{\text{even parity}}, \underbrace{h_0^{\ell m}, h_1^{\ell m}, h_2^{\ell m}}_{\text{odd parity}} \Big\}$$

are the 10 Regge-Wheeler metric perturbation functions

Perturbation theory

- Stress-energy tensor multipolar expansion

$$T_{\alpha\beta} = m_* r^{-2} u^t \frac{d x_p^\alpha}{d\tau} \frac{d x_p^\beta}{d\tau} \delta\big(r - r_p(t)\big) \delta\big(\theta - \theta_p(t)\big) \delta\big(\phi - \phi_p(t)\big)$$

$$T_{\alpha\beta} = \sum_{\ell,m} \sum_{i=1}^{10} T^{(i)\ell m}(t,r) Y^{(i)\ell m}_{\alpha\beta}(\theta,\phi)$$

where

$$T^{(i)\ell m}(t,r) = \int_{\mathbb{S}^2} \eta^{\alpha\gamma} \eta^{\beta\delta} T_{\alpha\beta} \left(Y^{(i)\ell m\star}_{\gamma\delta} \right) d\Omega$$

$\eta_{\alpha\beta} \equiv \mathrm{diag}(1, 1, r^2, r^2 \sin^2\theta)$ and $d\Omega = \sin\theta \, d\theta \, d\phi$

Regge-Wheeler gauge

gauge freedom

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} + \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha$$

such that

$$h_2^{\ell m} = 0 \quad \text{et} \quad h_0^{(e)\ell m} = h_1^{(e)\ell m} = G^{\ell m} = 0$$

3 equations for the odd parity perturbations $h_0^{\ell m}$ and $h_1^{\ell m}$:

$$\frac{\partial^2 h_0^{\ell m}}{\partial r^2} - \frac{\partial^2 h_1^{\ell m}}{\partial t \partial r} - \frac{2}{r}\frac{\partial h_1^{\ell m}}{\partial r} + \left[\frac{4M}{r^2} - \frac{\ell(\ell+1)}{r}\right]\frac{h_0^{\ell m}}{r-2M} = \frac{8\pi}{\sqrt{\ell(\ell+1)}}\frac{r^2}{r-2M} T^{(8)\ell m}$$

$$\frac{\partial^2 h_1^{\ell m}}{\partial t^2} - \frac{\partial^2 h_0^{\ell m}}{\partial t \partial r} - \frac{2}{r}\frac{\partial h_0^{\ell m}}{\partial t} + \frac{(\ell-1)(\ell+2)(r-2M)}{r^3} h_1^{\ell m} = -\frac{8\pi(r-2M)}{\sqrt{\ell(\ell+1)/2}} T^{(9)\ell m}$$

$$\frac{\partial}{\partial r}\left[f h_1^{\ell m}\right] - \frac{r}{r-2M}\frac{\partial h_0^{\ell m}}{\partial t} = -\frac{8\pi i r^2}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)/2}} T^{(10)\ell m}$$

Regge-Wheeler gauge

7 equations for the even parity perturbations $K^{\ell m}$, $H_0^{\ell m}$, $H_1^{\ell m}$, $H_2^{\ell m}$

$$f^2 \frac{\partial^2 K}{\partial r^2} + \frac{1}{r} f \left(3 - \frac{5M}{r}\right) \frac{\partial K}{\partial r} - \frac{1}{r} f^2 \frac{\partial H_2}{\partial r} - \frac{1}{r^2} f (H_2 - K) - \frac{\ell(\ell+1)}{2r^2} f (H_2 + K) = -8\pi T^{(1)\ell m}$$

$$\frac{\partial}{\partial t}\left[\frac{\partial K}{\partial r} + \frac{1}{r}(K - H_2) - \frac{M}{r(r-2M)} K\right] - \frac{\ell(\ell+1)}{2r^2} H_1 = -4\sqrt{2}\pi i T^{(2)\ell m}$$

$$f^{-2} \frac{\partial^2 K}{\partial t^2} - \frac{r-M}{r^2 f} \frac{\partial K}{\partial r} - \frac{2}{rf} \frac{\partial H_1}{\partial t} + \frac{1}{r} \frac{\partial H_0}{\partial r} + \frac{H_2 - K}{r^2 f} + \frac{\ell(\ell+1)(K - H_0)}{2r(r-2M)} = -8\pi T^{(3)\ell m}$$

$$\frac{\partial}{\partial r}[f H_1] - \frac{\partial}{\partial t}(H_2 + K) = \frac{8\pi i r}{\sqrt{\ell(\ell+1)/2}} T^{(4)\ell m}$$

$$-\frac{\partial H_1}{\partial t} + f \frac{\partial}{\partial r}(H_0 - K) + \frac{2M}{r^2} H_0 + \frac{1}{r}\left(1 - \frac{M}{r}\right)(H_2 - H_0) = \frac{8\pi(r-2M)}{\sqrt{\ell(\ell+1)/2}} T^{(5)\ell m}$$

$$-f^{-1}\frac{\partial^2 K}{\partial t^2} + f \frac{\partial^2 K}{\partial r^2} + \frac{2}{r} f \frac{\partial K}{\partial r} - f^{-1} \frac{\partial^2 H_2}{\partial t^2} + 2\frac{\partial^2 H_1}{\partial t \partial r} - f \frac{\partial^2 H_0}{\partial r^2} + \frac{2(r-M)}{r^2 f} \frac{\partial H_1}{\partial t} - \frac{r-M}{r^2} \frac{\partial H_2}{\partial r}$$
$$- \frac{r+M}{r^2} \frac{\partial H_0}{\partial r} + \frac{\ell(\ell+1)}{2r^2}(H_0 - H_2) = 8\sqrt{2}\pi T^{(6)\ell m}$$

$$H_0 - H_2 = 16\pi r^2 (\ell(\ell+1)(\ell-1)(\ell+2)/2)^{-1/2} T^{(7)\ell m}$$

where $f = \left(1 - \frac{2M}{r}\right)$

Wave equation

Linear combinations of $h^{(i)\ell m}$ lead to 2 gauge invariant scalar fields (Moncrief 74)

$$\psi^{\ell m}_{\text{even}} = \frac{r}{\lambda+1}\left[K^{\ell m}(r,t) + \frac{r-2M}{\lambda r+3M}\left(H_2^{\ell m}(t,r) - r\partial_r K^{\ell m}(t,r)\right)\right]$$
$$\psi^{\ell m}_{\text{odd}} = \frac{r}{\lambda}\left[r^2\partial_r\left(\frac{h_0^{\ell m}(t,r)}{r^2}\right) - \partial_t h_1^{\ell m}(t,r)\right]$$

The 2 functions satisfy Regge-Wheeler-Zerilli equations

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - V(r)^{\ell}_{e/o}\right]\psi^{\ell m}_{\text{e/o}}(t,r) = P^{\ell m}_{e/o}(t)\frac{\partial}{\partial r}\delta\left(r - r_p(t)\right) + Q^{\ell m}_{e/o}(t)\delta\left(r - r_p(t)\right)$$

$V^{\ell}_{e/o}, P^{\ell m}_{e/o}, Q^{\ell m}_{e/o}$ are known functions
$r^* = r + 2M\ln(r/2M - 1)$ is the tortoise coordinate
$r_p(t)$ particle trajectory
$\lambda = \frac{1}{2}(\ell-1)(\ell+2)$

Knowing $\psi^{\ell m}_{e/o}$, metric reconstruction is still possible : $h^{(i)} = h^{(i)}[\psi, \partial\psi, \partial^2\psi]$.

Wave equation

Properties :

For each multipole (ℓ, m) we have

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^{*2}}-V(r)\right]\psi(t,r)=P(t)\delta'+Q(t)\delta$$

δ' induces a discontinuity at $r=r_p(t)$

$$\boxed{\psi(t,r)=\psi^+(t,r)\,\Theta_1+\psi^-(t,r)\,\Theta_2}$$

Jump of the wave function at the location of the particle

$$\boxed{[\![\psi]\!]_{r_p}=\psi^+(t,r_p(t))-\psi^-(t,r_p(t))}$$

where $\psi^\pm(t,r_p(t))=\lim_{\varepsilon\to 0}\psi(t,r_p\pm\varepsilon)$

$\Theta_1=\Theta\big(r-r_p\big)$, $\Theta_2=\Theta\big(r_p-r\big)$ Heaviside distributions
$\delta=\delta\big(r-r_p\big)$, $\delta'=\frac{\partial}{\partial r}\delta\big(r-r_p\big)$ Dirac distribution and its spatial derivative

Wave equation

Jump conditions :

$$[\![\psi]\!]_{r_p} = \frac{P(t)}{f(r_p)^2 - \dot{r}_p^2}$$

$$[\![\partial_r \psi]\!]_{r_p} = \frac{1}{f(r_p)^2 - \dot{r}_p^2}\left[Q(t) + \left(f(r_p)\frac{df}{dr}(r_p) - \ddot{r}_p\right)[\![\psi]\!]_{r_p} - 2\dot{r}_p \frac{d}{dt}[\![\psi]\!]_{r_p}\right]$$

$$[\![\partial_t \psi]\!]_{r_p} = \frac{d}{dt}[\![\psi]\!]_{r_p} - \dot{r}_p [\![\partial_r \psi]\!]_{r_p}$$

$$\left[\!\left[\partial_r^n \partial_t^m \psi\right]\!\right]_{r_p} \cdots$$

where $\dot{r}_p = \frac{dr_p}{dt}$, $\ddot{r}_p = \frac{d^2 r_p}{dt^2}$ and $f(r_p) = \left(1 - \frac{2M}{r_p}\right)$

Wave equation

For generic orbits $\{t, r_p(t), \theta_p(t), \phi_p(t)\}$

$$P_o^{\ell m}(t) = \frac{8\kappa}{\lambda} r_p\left(\dot r_p^2 - f(r_p)^2\right) A^{\ell m\star}$$

$$Q_o^{\ell m}(t) = -\frac{8\kappa}{\lambda} r_p \dot r_p \frac{dA^{\ell m\star}}{dt} - \frac{8\kappa}{\lambda}\left[\frac{r_p}{u^t}\frac{d}{dt}(u^t \dot r_p) + \left(\dot r_p^2 - f(r_p)\right)\right] A^{\ell m\star}$$

$$V_o^{\ell}(r) = 2f(r)\left[(\lambda+1)r^{-2} - 3Mr^{-3}\right]$$

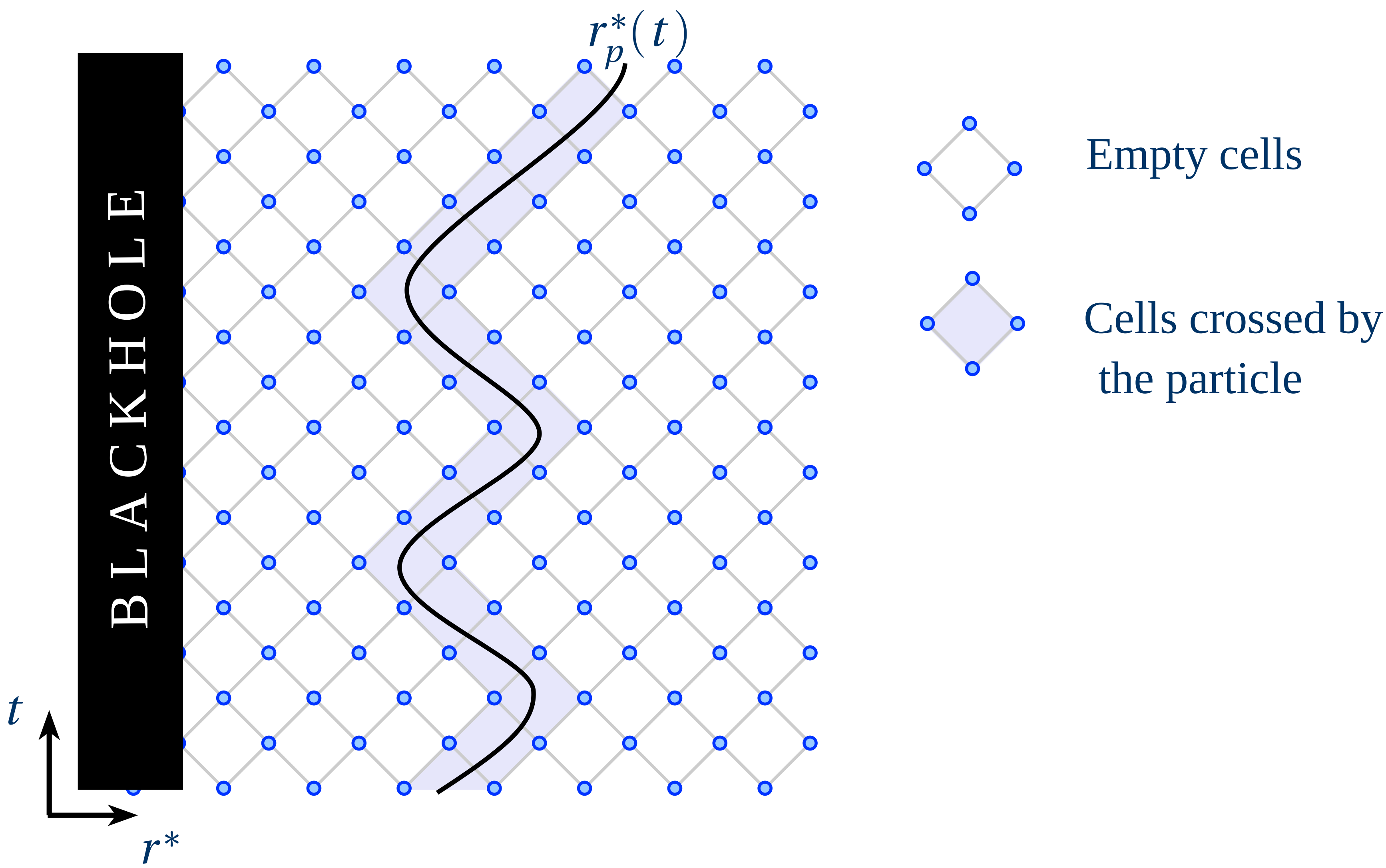
$$P_e^{\ell m}(t) = -8\kappa \frac{r_p f(r_p)\left(\dot r_p^2 - f(r_p)^2\right)}{\lambda r_p + 3M} Y^{\ell m\star}$$

$$Q_e^{\ell m}(t) = 16\kappa \frac{r_p \dot r_p f(r_p)}{\lambda r_p + 3M}\frac{dY^{\ell m\star}}{dt} - 16\frac{\kappa}{\lambda} r_p f(r_p)\dot\theta_p \dot\phi_p \partial_\phi\left(\partial_\theta - \cot\theta_p\right) Y^{\ell m\star} + 8\kappa \frac{r_p^2 f(r_p)^2}{\lambda r_p + 3M}\left(\dot\theta_p^2 + \sin^2\theta_p \dot\phi_p^2\right) Y^{\ell m\star}$$
$$-4\kappa\lambda^{-1} r_p f(r_p)\left(\dot\theta_p^2 - \sin^2\theta_p \dot\phi_p^2\right)\left(\partial_\theta^2 - \cot\theta_p \partial_\theta - \sin^{-2}\theta_p \partial_\phi^2\right) Y^{\ell m\star}$$
$$+8\kappa \frac{\dot r_p^2\left[(\lambda+1)(6r_p M + \lambda r_p^2) + 3M^2\right]}{r_p(\lambda r_p + 3M)^2} Y^{\ell m\star} - 8\kappa \frac{f(r_p)^2\left[r_p^2\lambda(\lambda+1) + 6\lambda r_p M + 15M^2\right]}{r_p(\lambda r_p + 3M)^2} Y^{\ell m\star}$$

$$V_e^{\ell}(r) = 2f(r)\frac{\left[\lambda^2(\lambda+1)r^3 + 3\lambda^2 M r^2 + 9\lambda M^2 r + 9M^3\right]}{r^3(\lambda r + 3M)^2}$$

where $\kappa = (\pi m_* u^t)/(\lambda+1)$, $\lambda = \frac{1}{2}(\ell-1)(\ell+2)$, $f(r) = 1 - 2M/r$ and $A^{\ell m} = \left(\frac{\dot\theta_p}{\sin\theta_p}\partial_\phi - \sin\theta_p \dot\phi_p \partial_\theta\right) Y^{\ell m}$

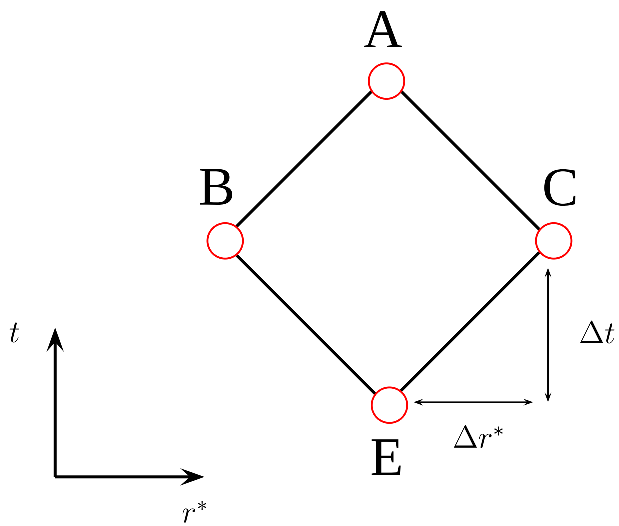
Numerical implementation



Numerical implementation

Empty cells :

2nd order classical finite difference scheme (Lousto-Price, Martel-Poisson)



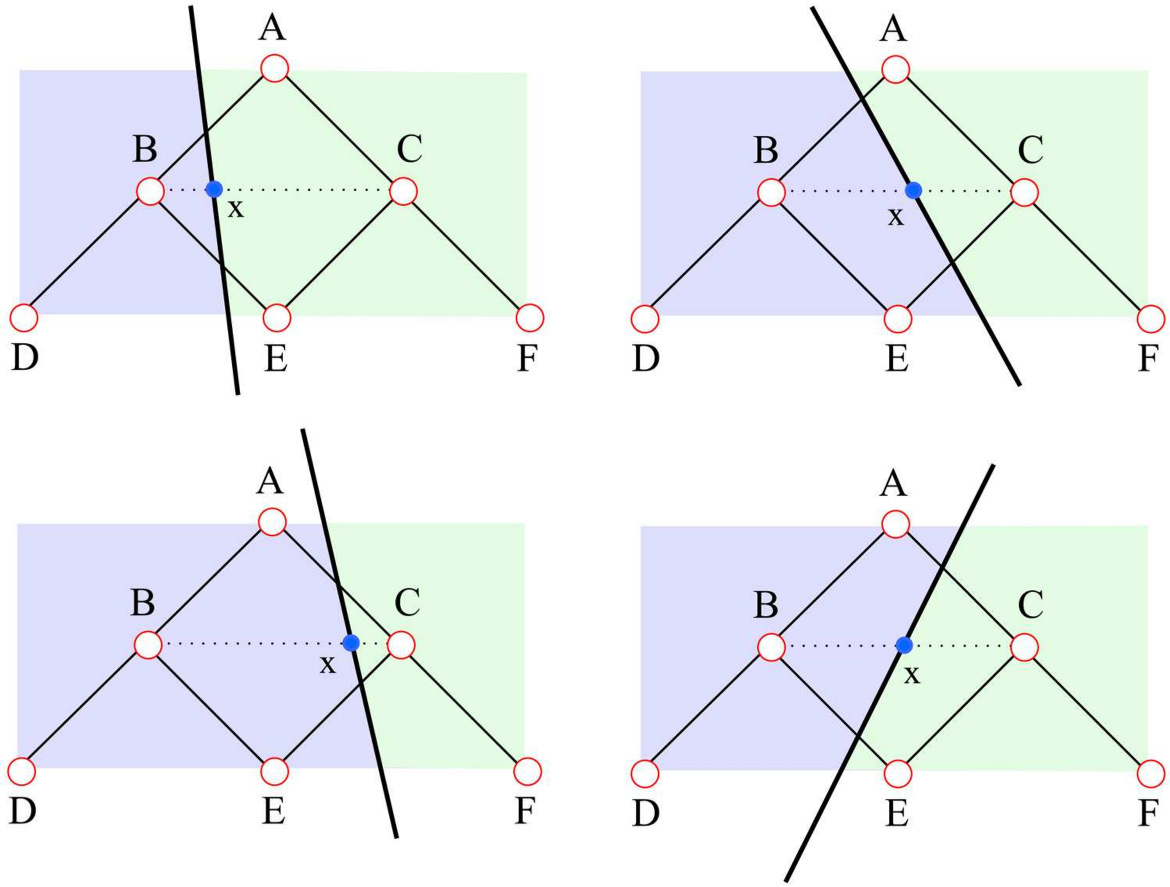
$$\psi_A^{\ell m} = -\psi_E^{\ell m} + \left(\psi_B^{\ell m} + \psi_C^{\ell m}\right)\left(1 - \frac{\Delta r^{*2}}{2} V^\ell(r)\right) + O(\Delta r^{*4})$$

typically $\Delta r^* = \Delta t$

Numerical implementation

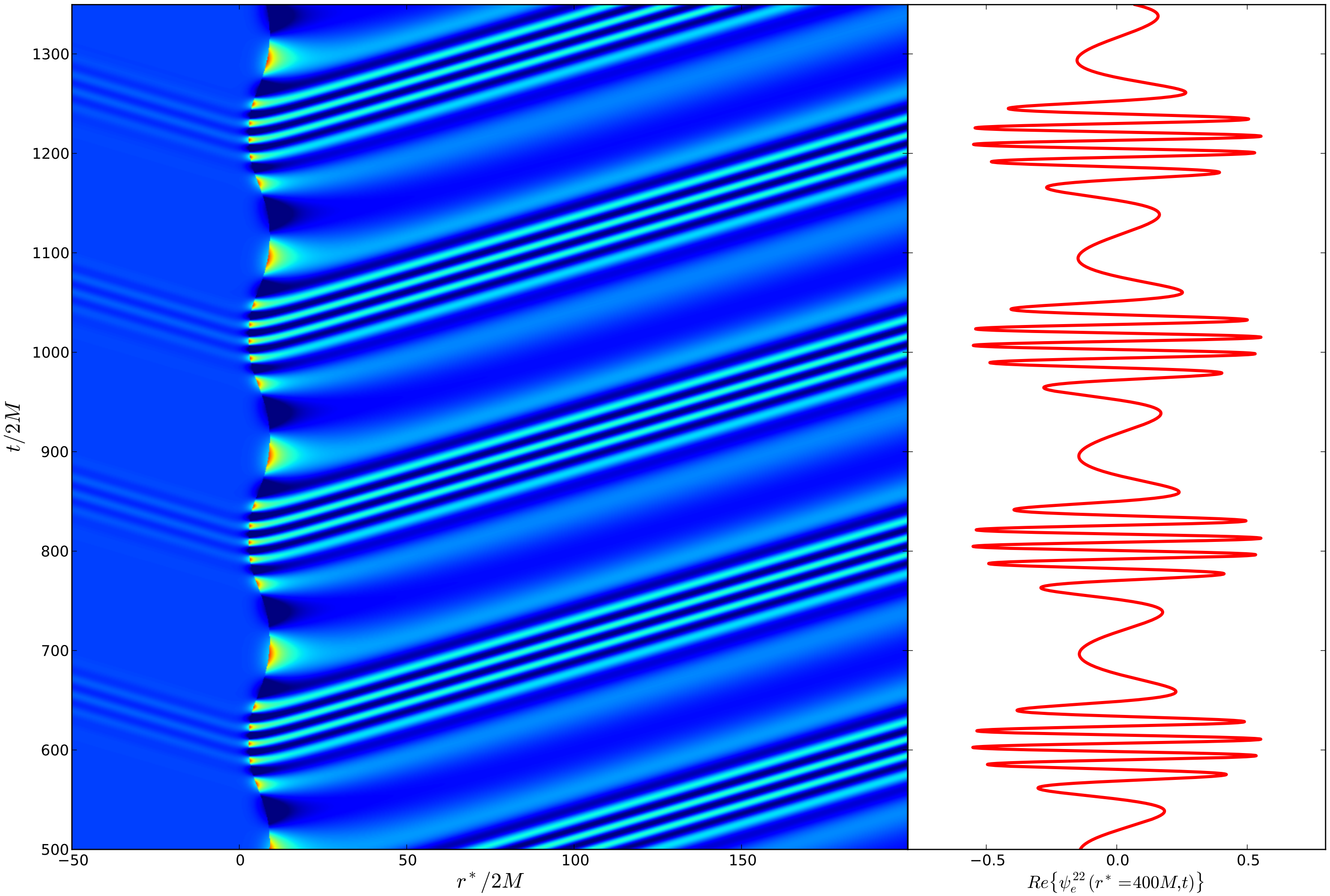
Cells crossed by the world line :

2nd order modified finite difference scheme (Aoudia, Ritter, Spallicci 2013 submitted)



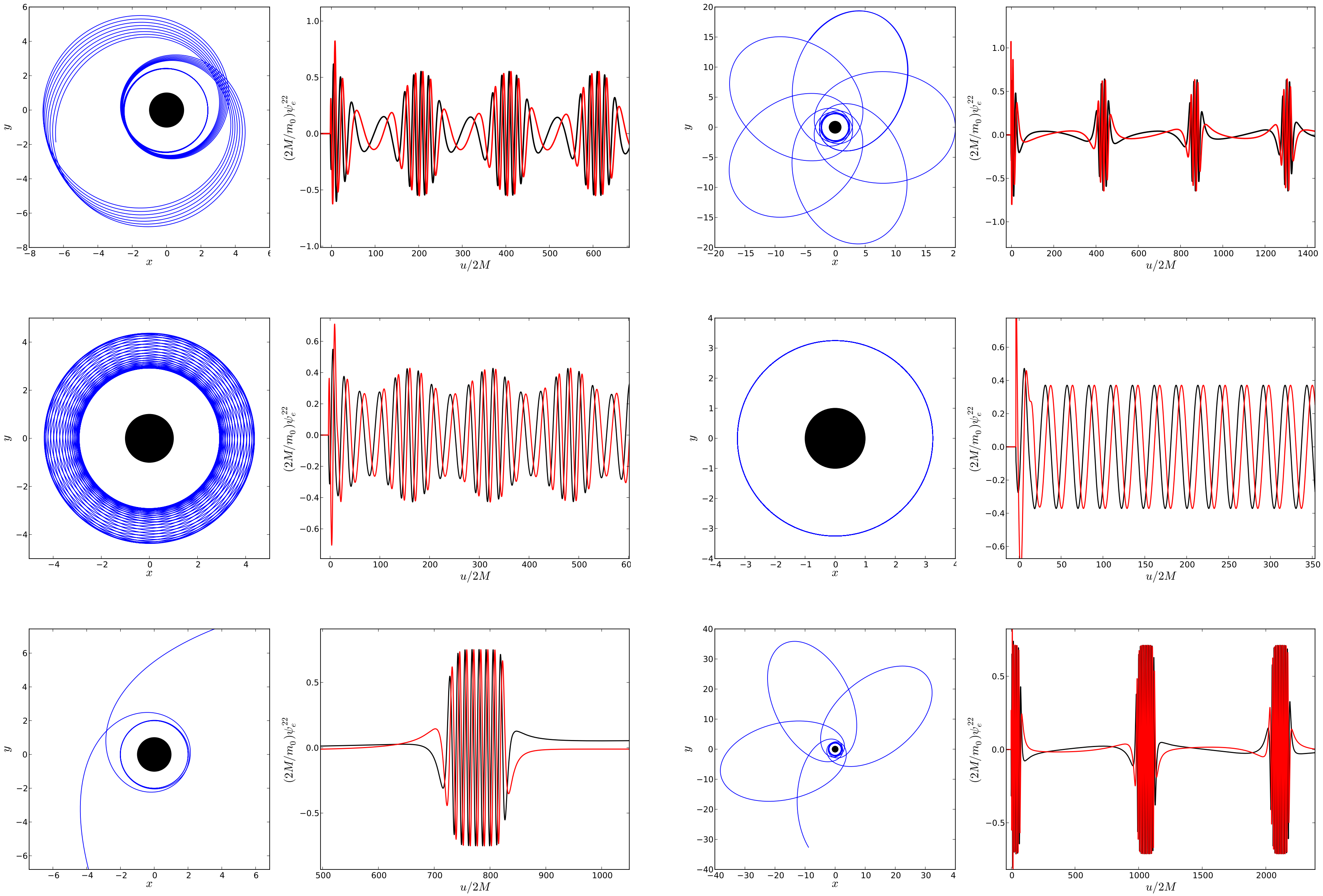
$$\psi_A^{\ell m} = \sum_{n\in\{B,C,D,E,F\}} \alpha_n \psi_n^{\ell m} + \sum_{p+q<3} \beta_{pq} [\![\partial_t^p \partial_{r*}^q \psi^{\ell m}]\!]_{\mathrm{x}} + O(\Delta r^{*3})$$

Numerical implementation



Example : elliptic orbit $(e = 0.5)$ for the quadrupolar mode $(\ell, m) = (2, 2)$

Numerical implementation



Code validation

Isaacson stress-energy tensor

$$T^{OG}_{\alpha\beta} = \frac{1}{64\pi}\left\langle \nabla_\alpha h^{\gamma\delta}\nabla_\beta h_{\gamma\delta}\right\rangle$$

Averaged flux of energy and angular momentum at infinity

$$dE = -\int_\Sigma T^\alpha_\beta \xi^\beta_{(t)} d\Sigma_\alpha \rightarrow \boxed{\frac{dE}{dt} = \frac{1}{64\pi}\sum_{\ell\geq 2,m}\frac{(\ell+2)!}{(\ell-2)!}\left[\left|\frac{d\psi^{\ell m}_e}{dt}\right|^2 + \left|\frac{d\psi^{\ell m}_o}{dt}\right|^2\right]}$$

$$dL = \int_\Sigma T^\alpha_\beta \xi^\beta_{(\phi)} d\Sigma_\alpha \rightarrow \boxed{\frac{dL}{dt} = \frac{im}{64\pi}\sum_{\ell\geq 2,m}\frac{(\ell+2)!}{(\ell-2)!}\left[\psi^{\ell m\star}_e\frac{d\psi^{\ell m}_e}{dt} + \psi^{\ell m\star}_o\frac{d\psi^{\ell m}_o}{dt}\right]}$$

$\xi^\beta_{(t)}$ and $\xi^\beta_{(\phi)}$ are 2 Schwarzschild Killing vectors

Code validation

ℓ	m	$\dot{E}^{\infty}_{\ell m}$ [us]	$\dot{E}^{\infty}_{\ell m}$ [Poisson 97]	$\dot{E}^{\infty}_{\ell m}$ [Martel 04]	$\dot{E}^{\infty}_{\ell m}$ [Barack 05]	$\dot{E}^{\infty}_{\ell m}$ [Sopuerta 06]
2	1	$8.1680.10^{-07}$	$8.1633.10^{-07}$ [0.06%]	$8.1623.10^{-07}$ [0.07%]	$8.1654.10^{-07}$ [0.03%]	$8.1662.10^{-07}$ [0.02%]
2	2	$1.7064.10^{-04}$	$1.7063.10^{-04}$ [0.006%]	$1.7051.10^{-04}$ [0.07%]	$1.7061.10^{-04}$ [0.02%]	$1.7064.10^{-04}$ [<0.001%]
3	1	$2.1757.10^{-09}$	$2.1731.10^{-09}$ [0.1%]	$2.1741.10^{-09}$ [0.07%]	$2.1734.10^{-09}$ [0.1%]	$2.1732.10^{-09}$ [0.1%]
3	2	$2.5203.10^{-07}$	$2.5199.10^{-07}$ [0.02%]	$2.5164.10^{-07}$ [0.2%]	$2.5207.10^{-07}$ [0.01%]	$2.5204.10^{-07}$ [0.002%]
3	3	$2.5471.10^{-05}$	$2.5471.10^{-05}$ [0.001%]	$2.5432.10^{-05}$ [0.2%]	$2.5479.10^{-05}$ [0.03%]	$2.5475.10^{-05}$ [0.02%]
4	1	$8.4124.10^{-13}$	$8.3956.10^{-13}$ [0.2%]	$8.3507.10^{-13}$ [0.7%]	$8.3982.10^{-13}$ [0.2%]	$8.4055.10^{-13}$ [0.08%]
4	2	$2.5099.10^{-09}$	$2.5091.10^{-09}$ [0.03%]	$2.4986.10^{-09}$ [0.5%]	$2.5099.10^{-09}$ [0.002%]	$2.5099.10^{-09}$ [0.002%]
4	3	$5.7750.10^{-08}$	$5.7751.10^{-08}$ [0.001%]	$5.7464.10^{-08}$ [0.5%]	$5.7759.10^{-08}$ [0.02%]	$5.7765.10^{-08}$ [0.03%]
4	4	$4.7251.10^{-06}$	$4.7256.10^{-06}$ [0.01%]	$4.7080.10^{-06}$ [0.4%]	$4.7284.10^{-06}$ [0.07%]	$4.7270.10^{-06}$ [0.04%]
5	1	$1.2632.10^{-15}$	$1.2594.10^{-15}$ [0.3%]	$1.2544.10^{-15}$ [0.7%]	$1.2598.10^{-15}$ [0.3%]	$1.2607.10^{-15}$ [0.2%]
5	2	$2.7910.10^{-12}$	$2.7896.10^{-12}$ [0.05%]	$2.7587.10^{-12}$ [1.2%]	$2.7877.10^{-12}$ [0.1%]	$2.7909.10^{-12}$ [0.003%]
5	3	$1.0933.10^{-09}$	$1.0933.10^{-09}$ [<0.001%]	$1.0830.10^{-09}$ [0.9%]	$1.0934.10^{-09}$ [0.009%]	$1.0936.10^{-09}$ [0.03%]
5	4	$1.2322.10^{-08}$	$1.2324.10^{-08}$ [0.01%]	$1.2193.10^{-08}$ [1.1%]	$1.2319.10^{-08}$ [0.03%]	$1.2329.10^{-08}$ [0.05%]
5	5	$9.4544.10^{-07}$	$9.4563.10^{-07}$ [0.02%]	$9.3835.10^{-07}$ [0.8%]	$9.4623.10^{-07}$ [0.08%]	$9.4616.10^{-07}$ [0.08%]
Total		$2.0293.10^{-04}$	$2.0292.10^{-04}$ [0.005%]	$2.0273.10^{-04}$ [0.096%]	$2.0291.10^{-04}$ [0.009%]	$2.0293.10^{-04}$ [<0.001%]

$\dot{E}$ at infinity in units of M^2/m_*^2. Good agreement with previous litterature (err $< 0.1\%$)

Code validation

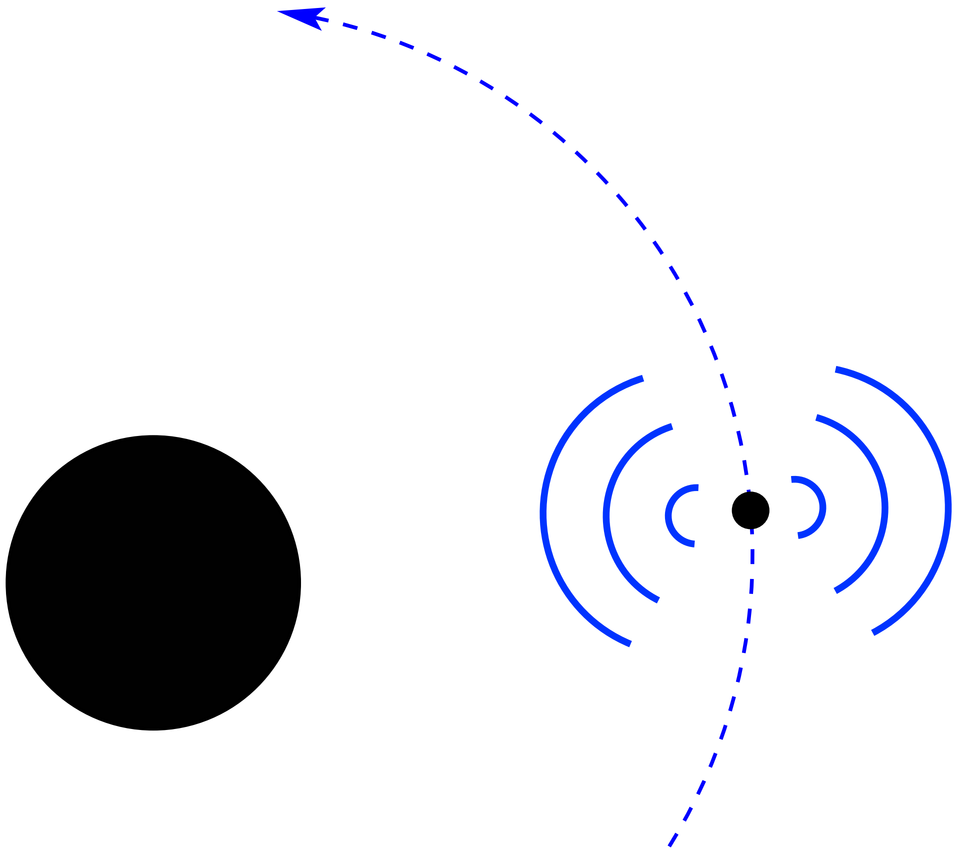
ℓ	m	$\dot{L}^{\infty}_{\ell m}$ [us]	$\dot{L}^{\infty}_{\ell m}$ [Poisson 97]	$\dot{L}^{\infty}_{\ell m}$ [Martel 04]	$\dot{L}^{\infty}_{\ell m}$ [Sopuerta 06]
2	1	$1.8294.10^{-05}$	$1.8283.10^{-05}$ [0.06%]	$1.8270.10^{-05}$ [0.1%]	$1.8289.10^{-05}$ [0.03%]
2	2	$3.8218.10^{-03}$	$3.8215.10^{-03}$ [0.009%]	$3.8164.10^{-03}$ [0.1%]	$3.8219.10^{-03}$ [0.002%]
3	1	$4.8729.10^{-08}$	$4.8670.10^{-08}$ [0.1%]	$4.8684.10^{-08}$ [0.09%]	$4.8675.10^{-08}$ [0.1%]
3	2	$5.6448.10^{-06}$	$5.6439.10^{-06}$ [0.02%]	$5.6262.10^{-06}$ [0.3%]	$5.6450.10^{-06}$ [0.003%]
3	3	$5.7048.10^{-04}$	$5.7048.10^{-04}$ [$<$0.001%]	$5.6878.10^{-04}$ [0.2%]	$5.7057.10^{-04}$ [0.02%]
4	1	$1.8841.10^{-11}$	$1.8803.10^{-11}$ [0.2%]	$1.8692.10^{-11}$ [0.8%]	$1.8825.10^{-11}$ [0.09%]
4	2	$5.6213.10^{-08}$	$5.6195.10^{-08}$ [0.03%]	$5.5926.10^{-08}$ [0.5%]	$5.6215.10^{-08}$ [0.003%]
4	3	$1.2934.10^{-06}$	$1.2934.10^{-06}$ [0.003%]	$1.2933.10^{-06}$ [0.01%]	$1.2937.10^{-06}$ [0.02%]
4	4	$1.0583.10^{-04}$	$1.0584.10^{-04}$ [0.01%]	$1.0518.10^{-04}$ [0.6%]	$1.0586.10^{-04}$ [0.03%]
5	1	$2.8293.10^{-14}$	$2.8206.10^{-14}$ [0.3%]	$2.8090.10^{-14}$ [0.7%]	$2.8237.10^{-14}$ [0.2%]
5	2	$6.2509.10^{-11}$	$6.2479.10^{-11}$ [0.05%]	$6.1679.10^{-11}$ [1.3%]	$6.2509.10^{-11}$ [0.001%]
5	3	$2.4487.10^{-08}$	$2.4486.10^{-08}$ [0.002%]	$2.4227.10^{-08}$ [1.1%]	$2.4494.10^{-08}$ [0.03%]
5	4	$2.7598.10^{-07}$	$2.7603.10^{-07}$ [0.02%]	$2.7114.10^{-07}$ [1.8%]	$2.7613.10^{-07}$ [0.05%]
5	5	$2.1175.10^{-05}$	$2.1179.10^{-05}$ [0.02%]	$2.0933.10^{-05}$ [1.2%]	$2.1190.10^{-05}$ [0.07%]
Total		$4.5449.10^{-03}$	$4.5446.10^{-03}$ [0.007%]	$4.5369.10^{-03}$ [0.2%]	$4.5452.10^{-03}$ [0.005%]

$\dot{L}$ at infinity in units of M/m_*^2. Good agreement with previous litterature (err $< 0.2\%$)

2

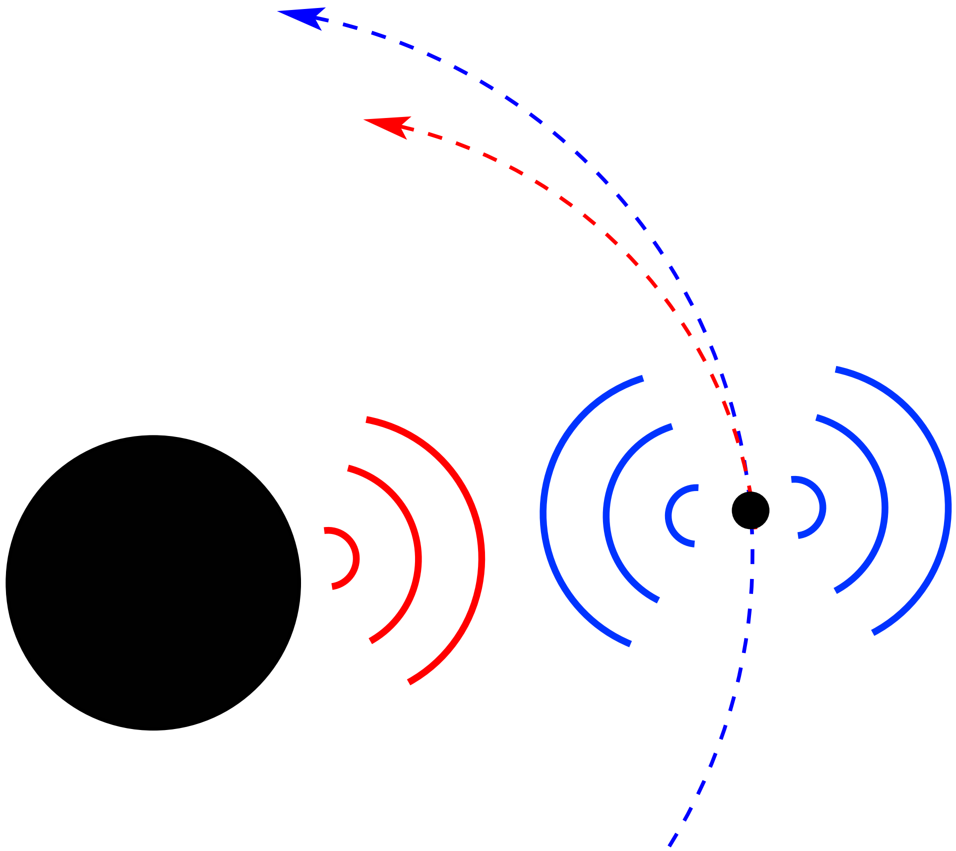
Self-force computation : radial fall case

Self-force



$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{d x^\beta}{d\tau} \frac{d x^\gamma}{d\tau} = 0$$

Self-force



$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = F^\alpha_{\text{Self}}$$

Self-force

$$G^{(1)}_{\alpha\beta}[g_{\alpha\beta}, h_{\alpha\beta}] = 8\pi T_{\alpha\beta}[g_{\alpha\beta}]$$

In harmonic gauge :

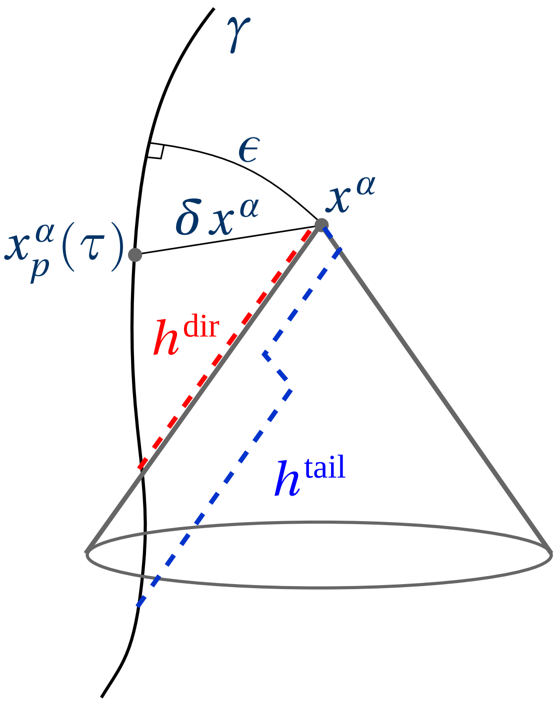
$$\Box \bar{h}^{\rm ret}_{\alpha\beta} + 2R^{\mu\ \nu}_{\ \alpha\ \beta}\bar{h}^{\rm ret}_{\mu\nu} = -16\pi m_* \int_{-\infty}^{+\infty} u_\alpha u_\beta (-g)^{-1/2} \delta^{(4)}(x^\alpha - x^\alpha_p(\tau))\, d\tau$$

$$\nabla^\alpha \bar{h}^{\rm ret}_{\alpha\beta} = 0$$

$\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ wave operator,
$h^{\rm ret}_{\alpha\beta}$ physical retarded solution,
$u^\alpha \equiv \frac{dx^\alpha}{d\tau}$ 4-velocity,
$\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h$ trace reversed metric and $h = g^{\alpha\beta} h_{\alpha\beta}$.

Self-force

In harmonic gauge :



$$h_{\alpha\beta}^{\rm ret} = h_{\alpha\beta}^{\rm dir} + h_{\alpha\beta}^{\rm tail}$$

$h_{\alpha\beta}^{\rm ret}$ and $h_{\mu\nu}^{\rm dir}$ diverge at the coincidence limit $x^\alpha \longrightarrow x_p^\alpha(\tau)$

$h_{\alpha\beta}^{\rm tail}$ is continous and differentiable everywhere

$$F_{\rm Self}^\alpha = F^\alpha[h_{\alpha\beta}^{\rm tail}] = -\frac{1}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\left(2\nabla_\mu h_{\beta\nu}^{\rm tail} - \nabla_\beta h_{\mu\nu}^{\rm tail}\right) u^\mu u^\nu$$

MiSaTaQuWa formulation (Mino, Sasaki, Tanaka, Quinn, Wald 1997)

Self-force

In harmonic gauge :

$$\Box \bar{h}^{\text{ret}}_{\alpha\beta} + 2R^{\mu\ \nu}_{\ \alpha\ \beta}\bar{h}^{\text{ret}}_{\mu\nu} = -16\pi m_* \int_{-\infty}^{+\infty} u_\alpha u_\beta (-g)^{-1/2}\delta^{(4)}(x^\mu - x^\mu_p(\tau))\, d\tau$$

$$u^\beta \nabla_\beta u^\alpha = -\frac{1}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\left(2\nabla_\mu h^{\text{tail}}_{\beta\nu} - \nabla_\beta h^{\text{tail}}_{\mu\nu}\right) u^\mu u^\nu$$

where

$$h^{\text{tail}}_{\alpha\beta} = 4m_* \lim_{\epsilon\to 0}\int_{-\infty}^{\tau-\epsilon}\left(G_{+\alpha\beta\alpha'\beta'} - \frac{1}{2}g_{\alpha\beta}G^{\delta}_{+\delta\alpha'\beta'}\right)\left(x_p(\tau), x_p(\tau')\right) u^{\alpha'} u^{\beta'}\, d\tau'$$

How to compute the self-force ?

Self-force

How to compute the self-force ?

Green function computation

$$G_{+\alpha\beta\alpha'\beta'}(x, x')$$

Mode-sum

$$F^{\alpha}[h^{\mathrm{dir}}] = \sum_{\ell} A^{\alpha} L + B^{\alpha} + C^{\alpha} L^{-1}$$

Effective source

$$\Box h^{\mathrm{reg}}_{\alpha\beta} = 8\pi T_{\alpha\beta} - \Box h^{sing}_{\alpha\beta}$$

Self-force

How to compute the self-force ?

Green function computation

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Effective source

$$\Box h^{\mathrm{reg}}_{\alpha\beta} = 8\pi T_{\alpha\beta} - \Box h^{sing}_{\alpha\beta}$$

OK in RW gauge for radial fall

(Barack-Ori 2001)

Mode-sum regularisation

$$F^{\alpha}_{\text{self}}(x_p) = \lim_{x \to x_p} \left[F^{\alpha}[h^{\alpha\beta}_{\text{ret}}](x) - F^{\alpha}[h^{\alpha\beta}_{\text{dir}}](x) \right]$$

Multipole expansion :

$$F^{\alpha}_{\text{self,ret,dir}} = \sum_{\ell=0}^{\infty} F^{\alpha\ell}_{\text{self,ret,dir}}$$

Regularisation : $F^{\alpha\ell}_{\text{dir}} = A^{\alpha} L + B^{\alpha} + C^{\alpha} L^{-1} + O(L^{-2})$

$$F^{\alpha}_{\text{self}} = \sum_{\ell=0}^{\infty} \left[F^{\alpha\ell}_{\text{ret}} - A^{\alpha} L - B^{\alpha} - C^{\alpha} L^{-1} \right] \equiv \sum_{\ell=0}^{\infty} F^{\alpha\ell}_{\text{self}}$$

with $L = \ell + 1/2$ and A^{α}, B^{α}, C^{α} are regularisation parameters

F^{α}_{self} behaves like $\sim \ell^{-2}$ for large value of ℓ.

Mode-sum regularisation

In Regge-Wheeler gauge

1. Compute ψ, $h^{\rm ret}_{\alpha\beta}$, and $F^\alpha[h^{\rm ret}_{\alpha\beta}]$ for each mode $\ell \le \ell_{\rm max}(\sim 10)$ on the worldline $x_p(t)$

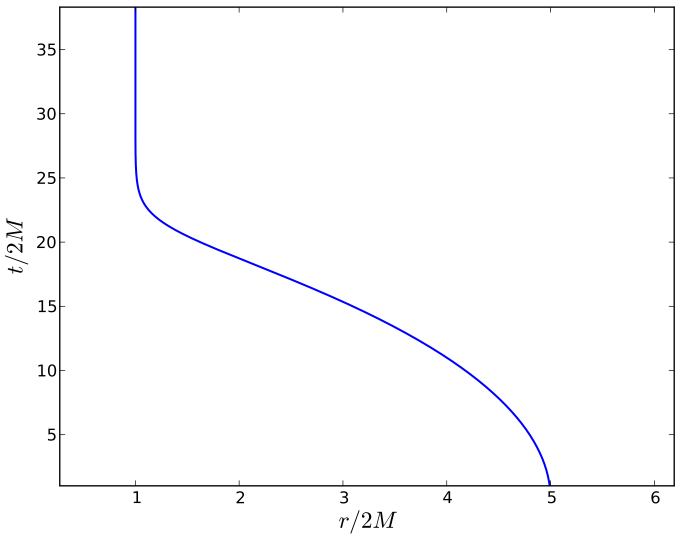
2. Compute regularisation parameters A^α, B^α and C^α

3. Substract mode by mode $F^{\alpha\ell}_{\rm self} = \left[F^{\alpha\ell}_{\rm ret} - A^\alpha L - B^\alpha - C^\alpha L^{-1} \right]$

4. Sum over modes $F^\alpha_{\rm self} = \sum_{\ell=0}^{\ell_{\rm max}} F^{\alpha\ell}_{\rm self} + [\text{extrapolation } \ell > \ell_{\rm max}]$

Self-force in RW gauge

Radial fall case



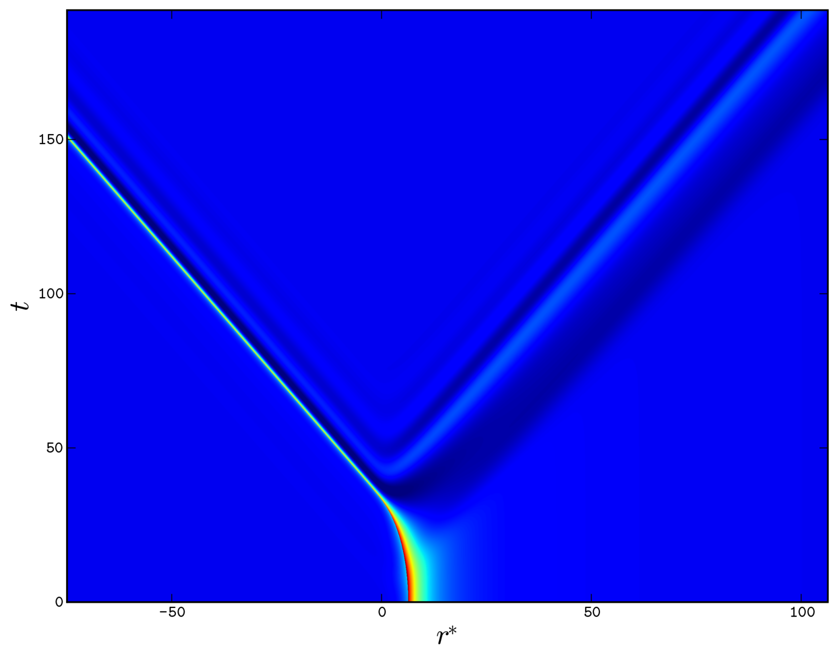
$$\ddot{r}_p = \frac{1}{2} f(r_p) f'(r_p) \left[1 - \frac{3\dot{r}_p^2}{f(r_p)^2} \right]$$

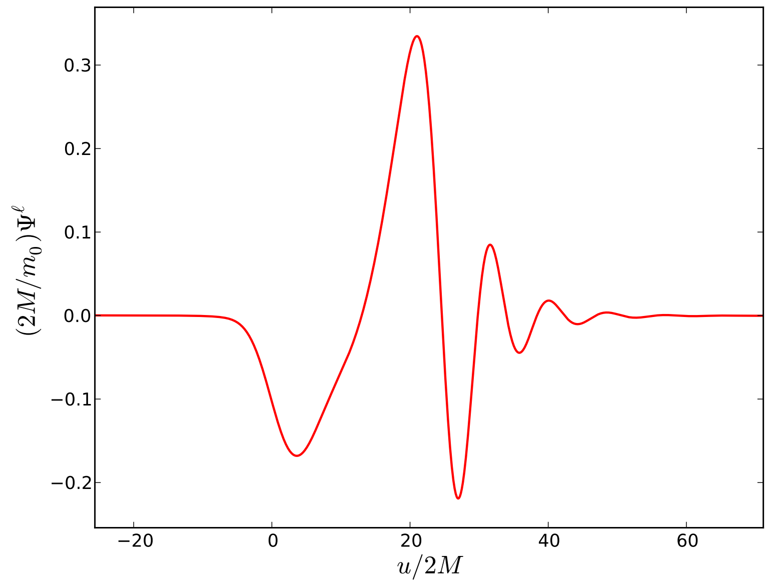
$$\theta_p = 0$$

- Only even perturbations $\psi^\ell \equiv \psi_{\mathrm{e}}^{\ell m=0}$
- $h_{\alpha\beta}^{\mathrm{ret}\ell} = \begin{pmatrix} f H_2^\ell & H_1^\ell \\ H_1^\ell & f^{-1} H_2^\ell \end{pmatrix} Y^{\ell 0}, \quad g_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & 1/f \end{pmatrix}$
- $H_1^\ell(t,r), H_2^\ell(t,r) \in \mathscr{C}^0$
- $H_1^\ell(t,r) = k_0 \partial_t \psi^\ell + k_1 \partial_{rt} \psi^\ell + k_2 \delta' + k_3 \delta$
- $H_2^\ell(t,r) = k_4 \psi^\ell + k_5 \partial_r \psi^\ell + k_6 \partial_r^2 \psi^\ell + k_7 \delta' + k_8 \delta$

Self-force in RW gauge

Radial fall case





Particle falling from $r_p(t=0) = 5(2M)$ with zero initial velocity.

Wave form $\psi^{\ell=2}$.

Self-force in RW gauge

Radial fall case

$$F^{\alpha}[h^{\text{ret}}_{\alpha\beta}] = -\frac{1}{2}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right)\left(2\nabla_{\mu}h^{\text{ret}}_{\beta\nu} - \nabla_{\beta}h^{\text{ret}}_{\mu\nu}\right)u^{\mu}u^{\nu} = \sum_{\ell} F^{\alpha\ell}_{\text{ret}}$$

$$F^{\alpha\ell}_{\text{ret}} = -\frac{1}{2}\left[f^{\alpha}_{0}\left(\frac{\partial H^{\ell}_{2}}{\partial t} - \frac{df}{dr}H^{\ell}_{1}\right) + f^{\alpha}_{1}\left(\frac{\partial H^{\ell}_{1}}{\partial t} - \frac{df}{dr}H^{\ell}_{2}\right) + f^{\alpha}_{2}\frac{\partial H^{\ell}_{2}}{\partial r} + f^{\alpha}_{3}\frac{\partial H^{\ell}_{1}}{\partial r}\right]Y^{\ell 0}$$

where the k_i are functions of r and the f^{α}_{j} are functions of r_p and $\dot{r}_p$

Self-force in RW gauge

Radial fall case

$$F^{\alpha}[h^{\rm ret}_{\alpha\beta}] = -\frac{1}{2}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right)\left(2\nabla_{\mu}h^{\rm ret}_{\beta\nu} - \nabla_{\beta}h^{\rm ret}_{\mu\nu}\right)u^{\mu}u^{\nu} = \sum_{\ell} F^{\alpha\ell}_{\rm ret}$$

$$F^{\alpha\ell}_{\rm ret} = -\frac{1}{2}\left[f^{\alpha}_{0}\left(\frac{\partial H^{\ell}_{2}}{\partial t} - \frac{df}{dr}H^{\ell}_{1}\right) + f^{\alpha}_{1}\left(\frac{\partial H^{\ell}_{1}}{\partial t} - \frac{df}{dr}H^{\ell}_{2}\right) + f^{\alpha}_{2}\frac{\partial H^{\ell}_{2}}{\partial r} + f^{\alpha}_{3}\frac{\partial H^{\ell}_{1}}{\partial r}\right]Y^{\ell 0}$$

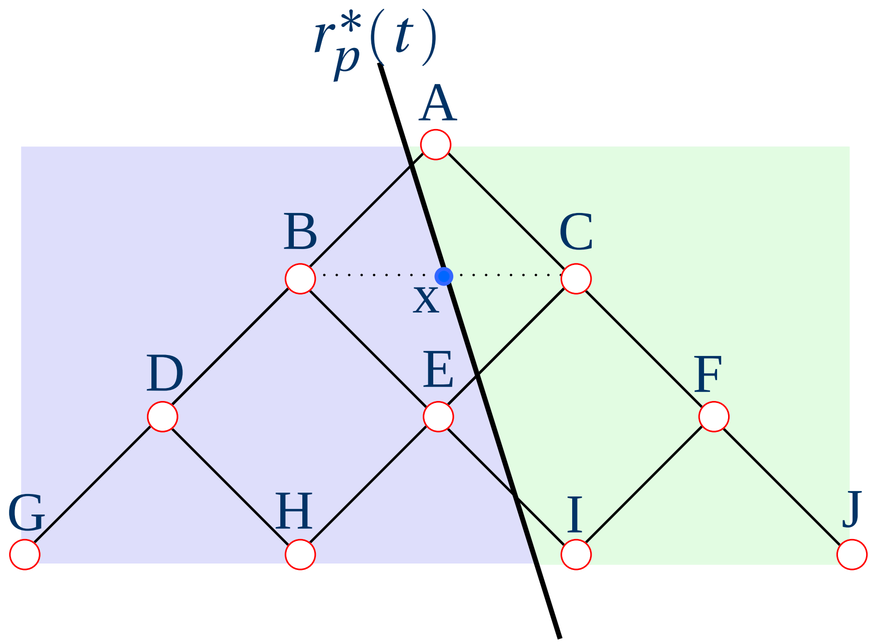
where the k_i are functions of r and the f^{α}_{j} are functions of r_p and $\dot{r}_p$

Need third derivatives of the wave function on the trajectory!

Self-force in RW gauge

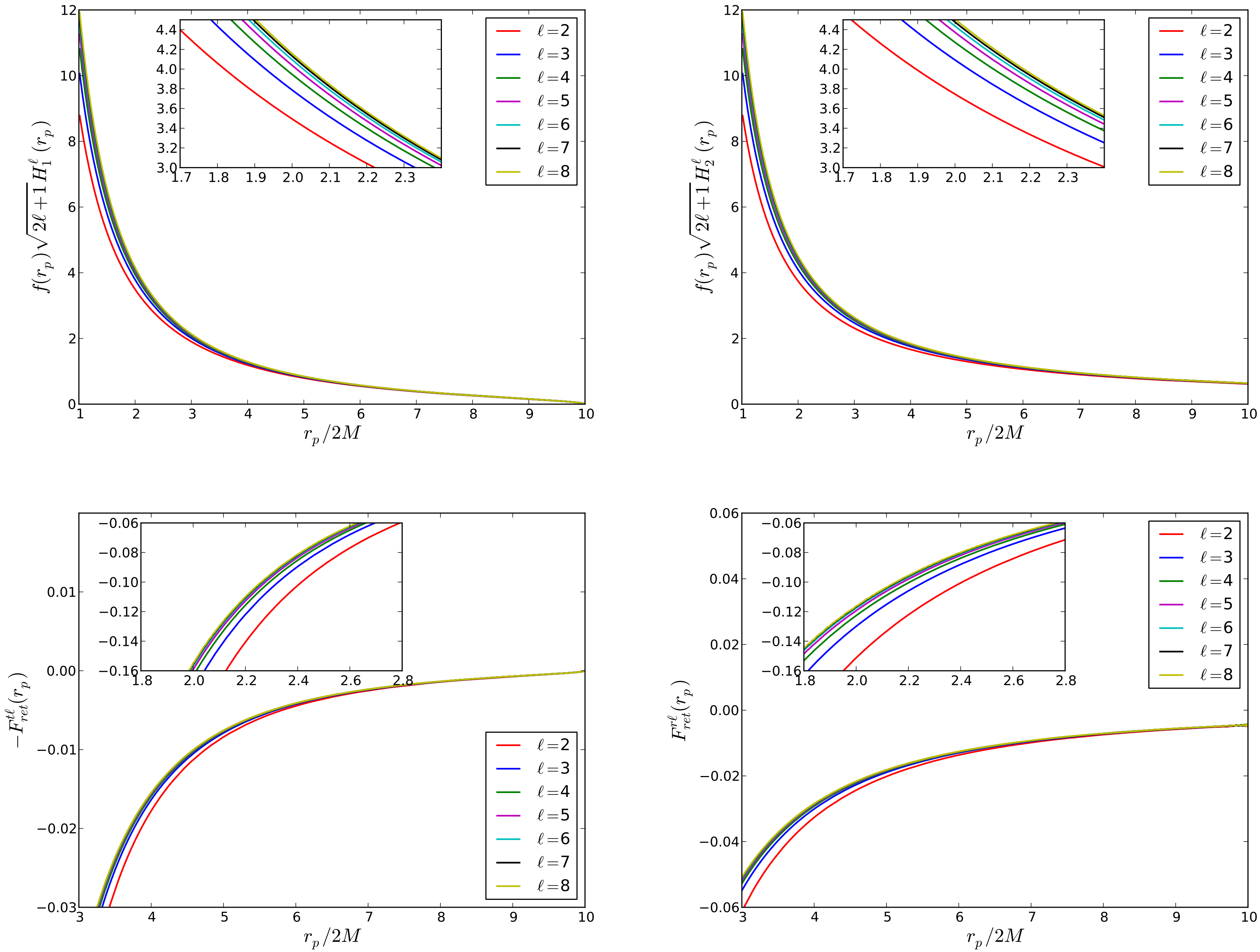
Radial fall case

4th order scheme (Ritter et al. 2012)



$$\psi^\ell_A = \sum_{n\in\{B,\cdots,J\}} \alpha_n \psi^\ell_n + \sum_{p+q<5} \beta_{pq} \left[\!\left[\partial_t^p \partial_{r^*}^q \psi^\ell \right]\!\right]_{\mathrm{x}} + O(\Delta r^{*5})$$

Self-force in RW gauge



Self-force in RW gauge

Regularisation parameters

Recall that $F^{\alpha\ell}[h^{\alpha\beta}_{\rm ret}]$ is function of ψ^ℓ, $\partial\psi^\ell$, $\partial^2\psi^\ell$ and $\partial^3\psi^\ell$.
By a local analysis when $r \to r_p(t)$ and $\ell \to \infty$ we find

$$F^{\alpha\ell\to\infty}[h^{\alpha\beta}_{\rm ret}] \sim F^{\alpha\ell}[h^{\alpha\beta}_{\rm dir}] = A^\alpha L + B^\alpha + C^\alpha L^{-1} + O(L^{-2})$$

with

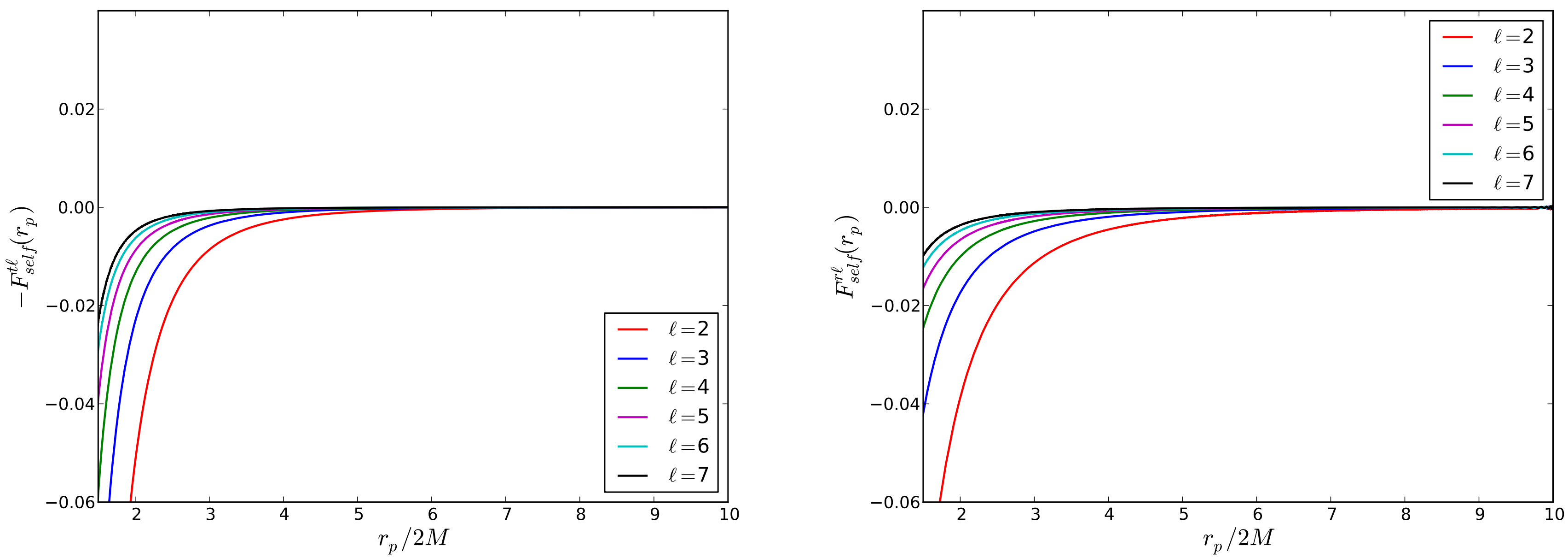
$$A^r = \pm\frac{E}{r_p^2} \qquad A^t = \pm\frac{\dot{r}_p}{f(r_p)r_p^2}$$

$$B^r = -\frac{E^2}{2r_p^2} \qquad B^t = -\frac{E\,\dot{r}_p}{2f(r_p)r_p^2}$$

$$C^\alpha = 0$$

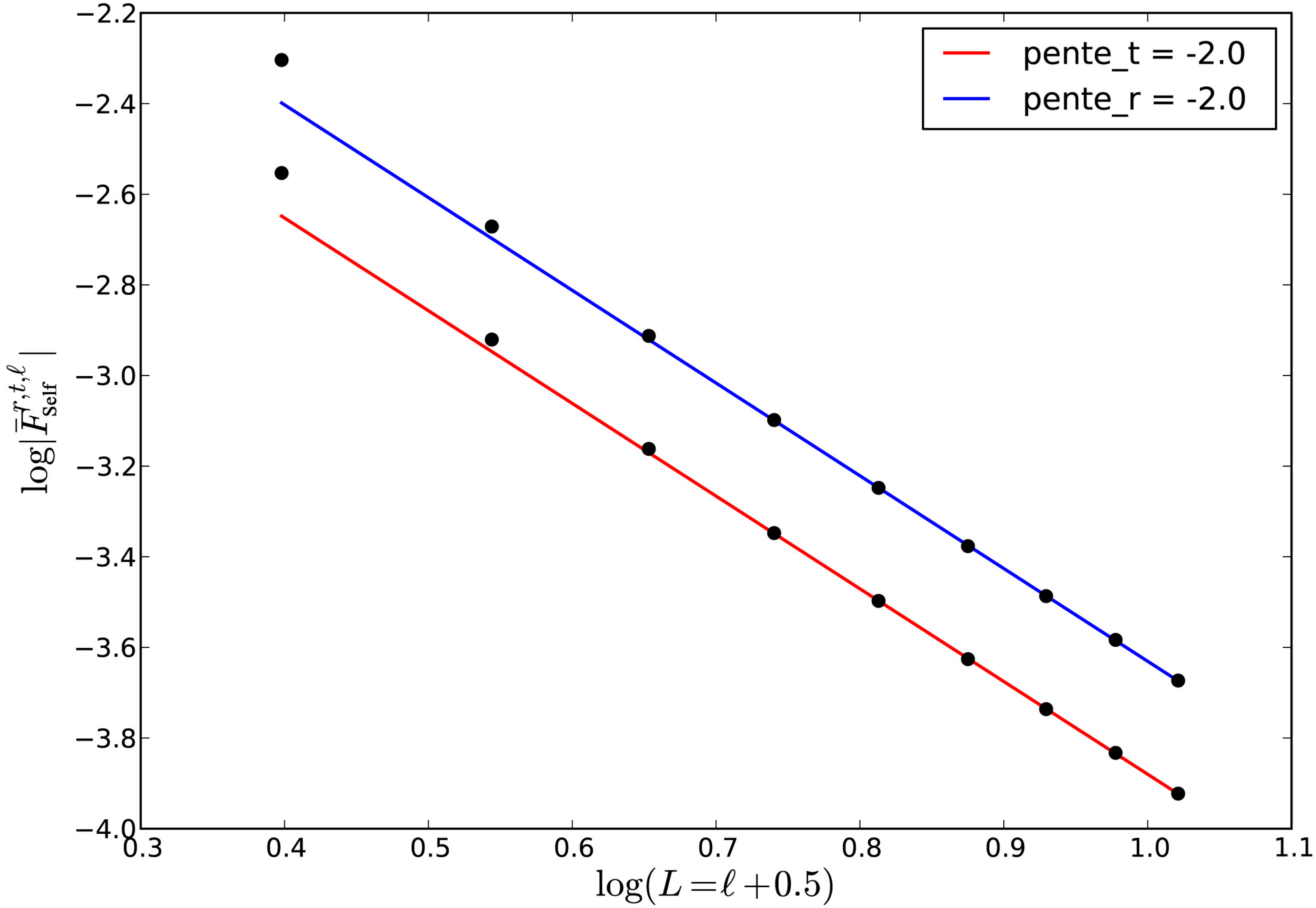
$$F^\alpha_{\rm self} = \sum_{\ell=0}^{\infty}\left[F^{\alpha\ell}_{\rm ret} - A^\alpha L - B^\alpha - C^\alpha L^{-1} + O(L^{-2})\right]$$

Self-force in RW gauge



$$F^{\ell}_{\text{Self}} \xrightarrow{\ell\to\infty} 0 \quad \Rightarrow \quad \sum_{\ell} F^{\ell}_{\text{Self}} \text{ finite}$$

Self-force in RW gauge



Action of the self-force on the trajectory

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = F^\alpha_{\text{Self}}$$

geodesic motion

$$\ddot{r}_p = \frac{1}{2} f(r_p) f'(r_p) \left[1 - \frac{3\dot{r}_p^2}{f(r_p)^2} \right]$$

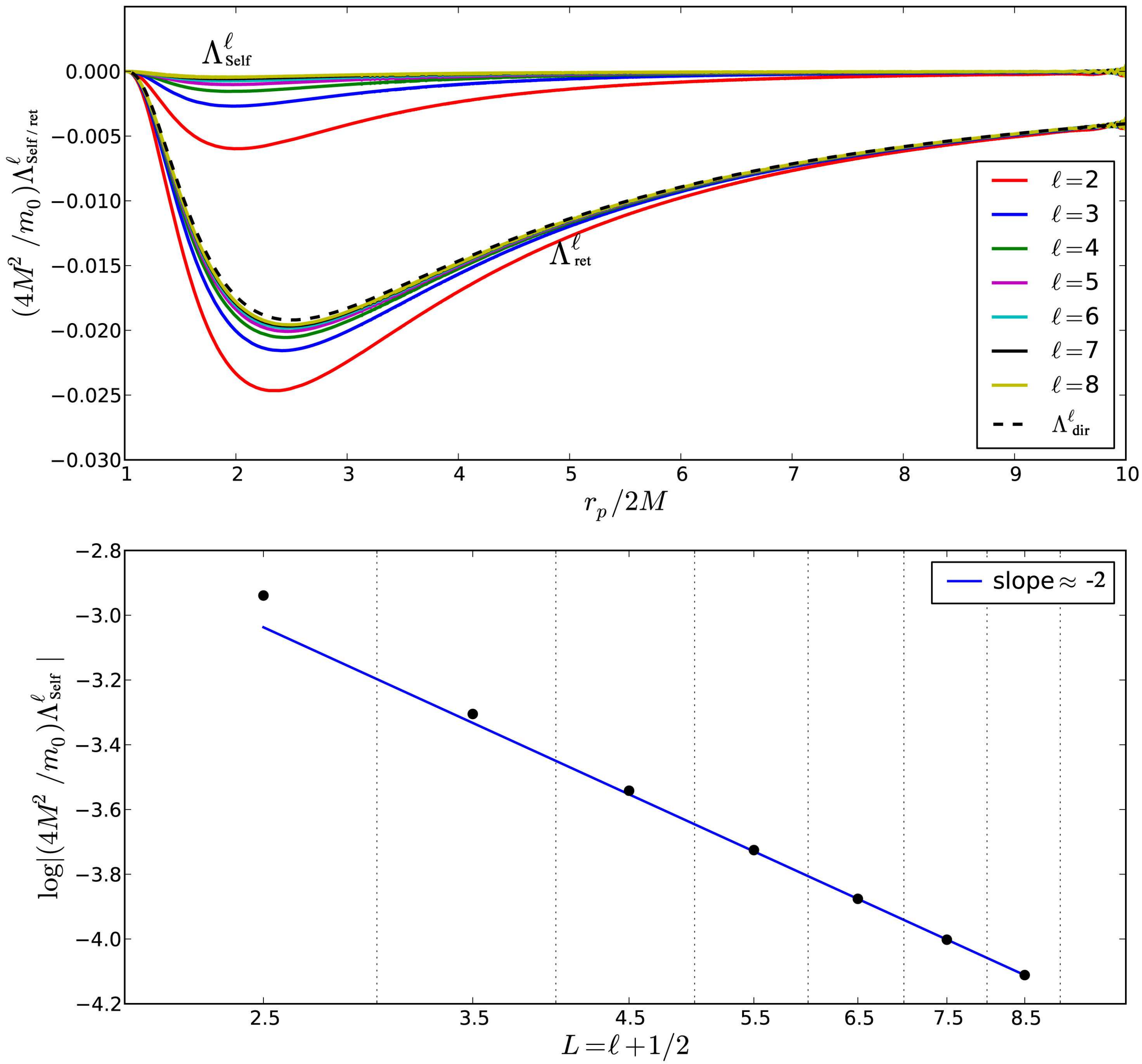
perturbed motion

$$\ddot{r}_p = \frac{1}{2} f(r_p) f'(r_p) \left[1 - \frac{3\dot{r}_p^2}{f(r_p)^2} \right] + \Lambda_{\text{Self}}$$

where

$$\boxed{\Lambda_{\text{Self}} = \sum_\ell \frac{f(r_p)^2}{E^2} \left[F^{r\ell}_{\text{Self}} - \dot{r}_p F^{t\ell}_{\text{Self}} \right]}$$

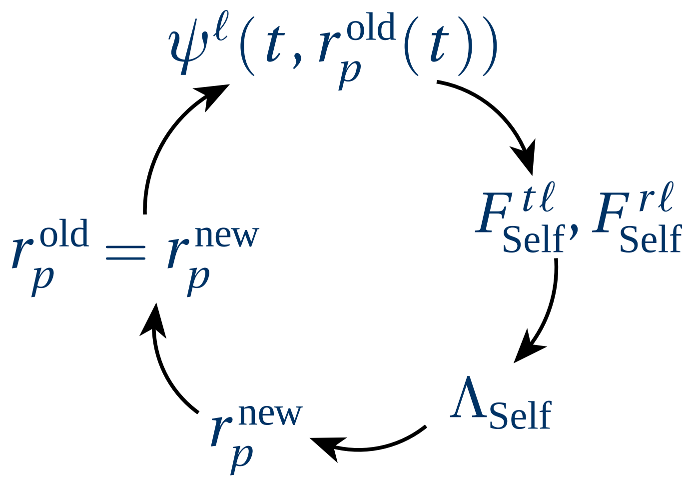
Action of the self-force on the trajectory



Action of the self-force on the trajectory

Self-consistent approach

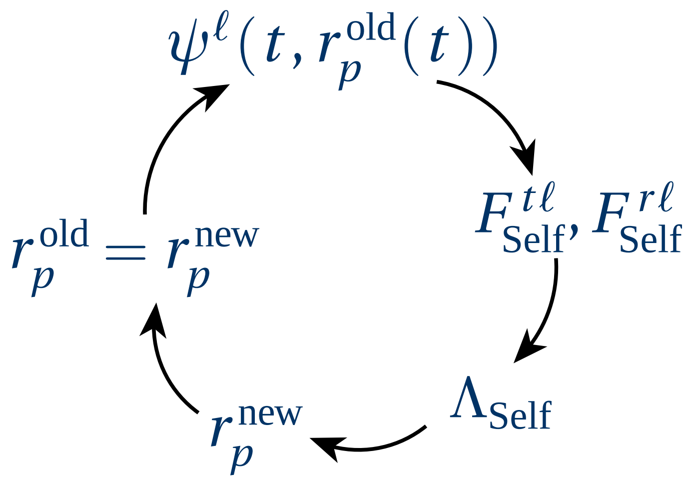
$$\ddot{r}_p = \frac{1}{2} f(r_p) f'(r_p) \left[1 - \dot{r}_p^2 f(r_p)^{-2}\right] + \Lambda_{\mathrm{Self}}(r_p, \dot{r}_p)$$



Action of the self-force on the trajectory

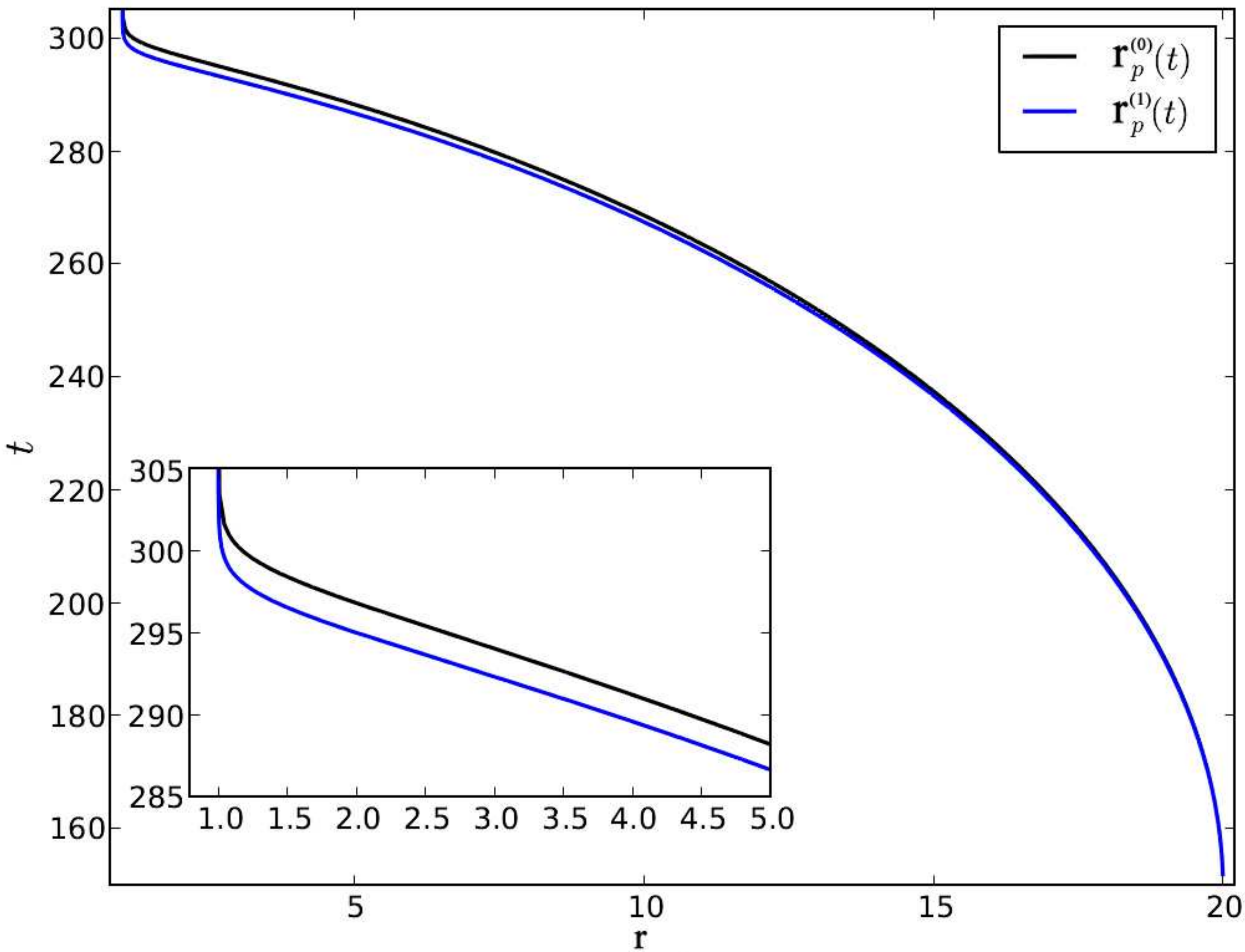
Self-consistent approach

$$\ddot{r}_p = \frac{1}{2} f(r_p) f'(r_p) \left[1 - \dot{r}_p^2 f(r_p)^{-2} \right] + \Lambda_{\rm Self}(r_p, \dot{r}_p)$$



But regularisation parameters A^α, B^α and C^α must be calculated on a geodesic $\rightarrow$ **osculating orbit approach** (Ritter et al. in preparation).

Action of the self-force on the trajectory



Conclusion

Easily implementable numerical scheme based on a jump-conditions method in time domain

Good accuracy in wave-forms and fluxes computations.

Extension to 4th order to treat the orbital evolution in radial fall case

Future?

Find a post-doc...

Scientific exploitation of code and results

Export the code procedure to the harmonic gauge.

Increase the speed by using space-time compactification techniques.

Annexe

$$\psi^{\pm} = \kappa\left[L^{-3} \pm 2EL^{-4}\right] + \mathcal{O}(L^{-5})$$

$$\psi^{\pm}_{,r} = \frac{\kappa}{r} f^{-1}\left[\mp EL^{-2} - \frac{3}{2}E^2L^{-3} \pm \left(\frac{6M}{r} - \frac{9}{4}\right)EL^{-4} + \mathcal{O}(L^{-5})\right]$$

$$\psi^{\pm}_{,rr} = \frac{\kappa}{r^2} f^{-2}\left[E^2L^{-1} \pm \left(2 - \frac{3M}{r}\right)EL^{-2} + \mathcal{O}(L^{-3})\right]$$

$$\psi^{\pm}_{,rrr} = \frac{\kappa}{r^3} f^{-3}\left\{\mp E^3 + E^2\left[\frac{5}{2}E^2 + \frac{9M}{r} - 6\right]L^{-1} \mp 3E\left[\frac{7M}{r}\left(\frac{M}{r} - 1\right) + 2\right]L^{-2} + \mathcal{O}(L^{-3})\right\}$$

$$\psi^{\pm}_{,t} = \frac{\kappa}{r}\left[\pm \dot{r}L^{-2} + \frac{3}{2}E\dot{r}L^{-3} \mp \left(\frac{6M}{r} - \frac{9}{4}\right)\dot{r}L^{-4} + \mathcal{O}(L^{-5})\right]$$

$$\psi^{\pm}_{,tr} = \frac{\kappa}{r^2} f^{-1}\left[-E\dot{r}L^{-1} \pm \left(\frac{3M}{r} - 1\right)\dot{r}L^{-2} + \mathcal{O}(L^{-3})\right]$$

$$\psi^{\pm}_{,trr} = \frac{\kappa}{r^3} f^{-2}\left\{\pm E^2\dot{r} - E\dot{r}\left[\frac{5}{2}E^2 + \frac{9M}{r} - 4\right]L^{-1} \pm \dot{r}\left[\frac{3M}{r}\left(\frac{5M}{r} - 4\right) + 2\right]L^{-2} + \mathcal{O}(L^{-3})\right\}$$

Annexe

$$K^{\pm} = \frac{\kappa}{2r} L^{-1} + \mathcal{O}(L^{-2})$$

$$\partial_t K^{\pm} = \pm \frac{\kappa E \dot{r}}{2f r^2} - \frac{\kappa E^2 \dot{r}}{4 f r^2} L^{-1} + \mathcal{O}(L^{-2})$$

$$\partial_r K^{\pm} = \mp \frac{\kappa E}{2 f r^2} + \frac{\kappa}{2 f r^2}\left(\frac{E^2}{2} - f\right) L^{-1} + \mathcal{O}(L^{-2})$$

$$\partial_{tr} K^{\pm} = -\frac{\kappa E^2 \dot{r}}{2 f^2 r^3} L \pm \frac{\kappa E \dot{r}}{2 f^2 r^3}\Big(5M - 2rE(1-E)\Big) - \frac{\kappa E^2 \dot{r}}{4 f^2 r^4}\Big(17M + 4rE^2 - 11r\Big) L^{-1} + \mathcal{O}(L^{-2})$$

$$H_1^{\pm} = -\frac{\kappa E^2 \dot{r}}{r f^2} L^{-1} + \mathcal{O}(L^{-2})$$

$$\partial_t H_1^{\pm} = \mp \frac{\kappa E}{r^2 f}(E^2 - f) + \frac{\kappa}{2 r^4 f}\Big((5E^4 - 7E^2 + 2) r^2 + (18 M E^2 - 10M) r + 12 M^2\Big) L^{-1} + \mathcal{O}(L^{-2})$$

$$\partial_r H_1^{\pm} = \mp \frac{\kappa E^3 \dot{r}}{r^2 f^3} - \frac{\kappa E^2 \dot{r}}{2 r^4 f^4}\Big((5E^2 - 4) r^2 + (8 - 5E^2) 2 M r - 16 M^2\Big) L^{-1} + \mathcal{O}(L^{-2})$$

$$H_2^{\pm} = \frac{\kappa}{2 r f}\Big(2E^2 - f\Big) L^{-1} + \mathcal{O}(L^{-2})$$

$$\partial_t H_2^{\pm} = \pm \frac{\kappa E \dot{r}}{2 r^2 f^2}(2E^2 - f) - \frac{\kappa E^2 \dot{r}}{4 r^3 f^2}\Big((10E^2 - 9) r + 26M\Big) L^{-1} + \mathcal{O}(L^{-2})$$

$$\partial_r H_2^{\pm} = \mp \frac{\kappa E}{2 r^2 f^2}(2E^2 - f) + \frac{\kappa}{4 r^4 f^2}\Big((10E^4 - 13E^2 + 2) r^2 + (26 M E^2 - 4) 2 M r + 8 M^2\Big) L^{-1} + \mathcal{O}(L^{-2})$$