Mapping gravitational-wave backgrounds of arbitrary polarisation using pulsar timing arrays

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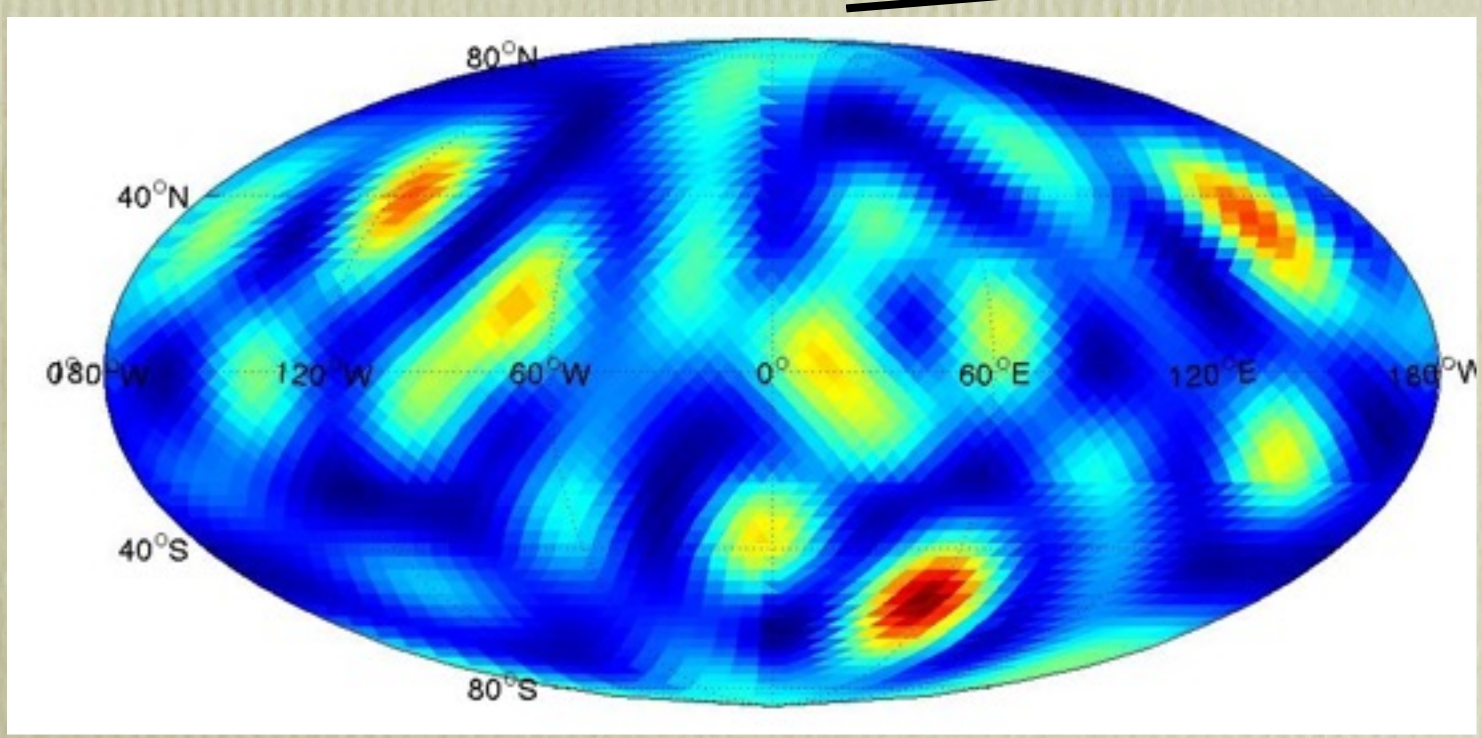
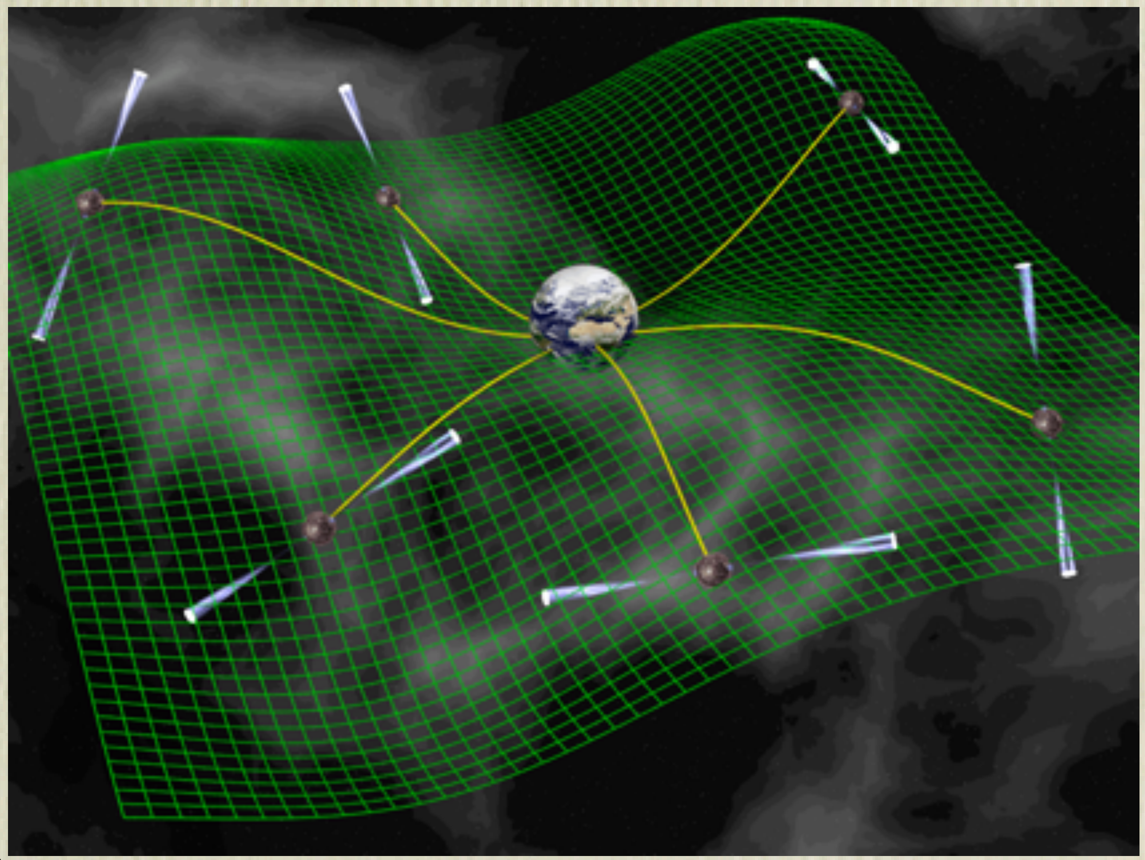


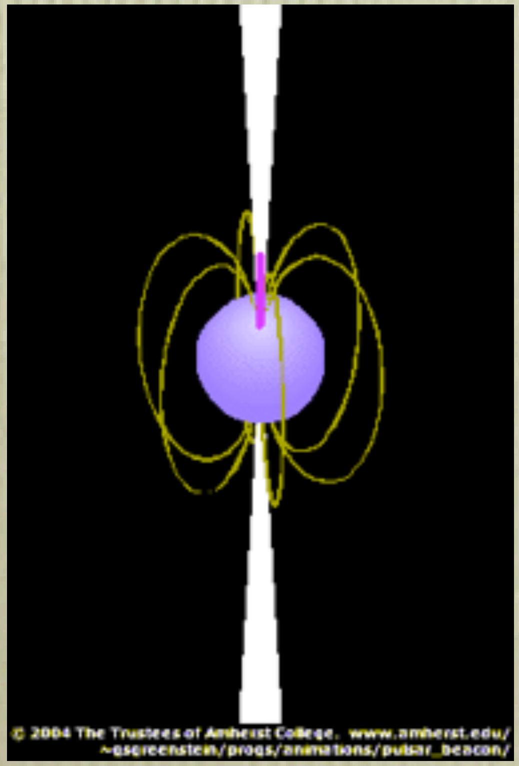


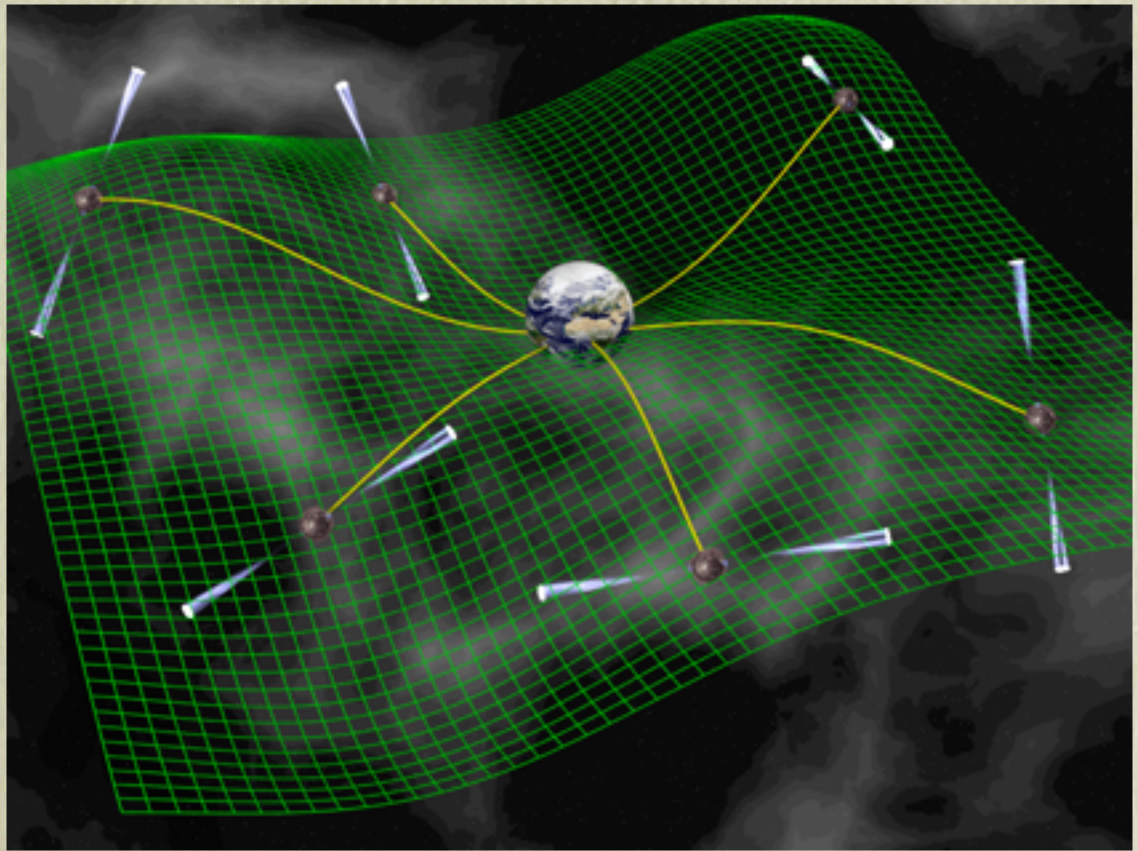
Talk Outline

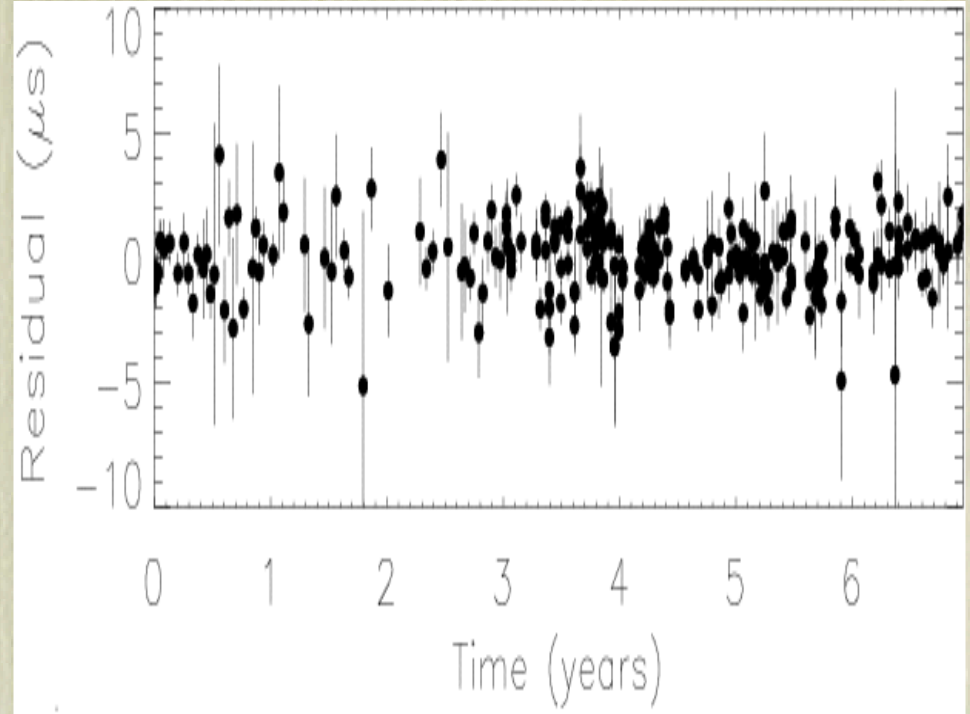
- Introduction to pulsar timing arrays
- Formalism for CMB polarisation analysis
- PTA response to individual modes of the background
- Recovery of Hellings and Downs correlation curve for an isotropic uncorrelated background
- Characterising and measuring general backgrounds
- Implications of a measurement of the coefficients inconsistent with expectations
- Extension to ground-based interferometers

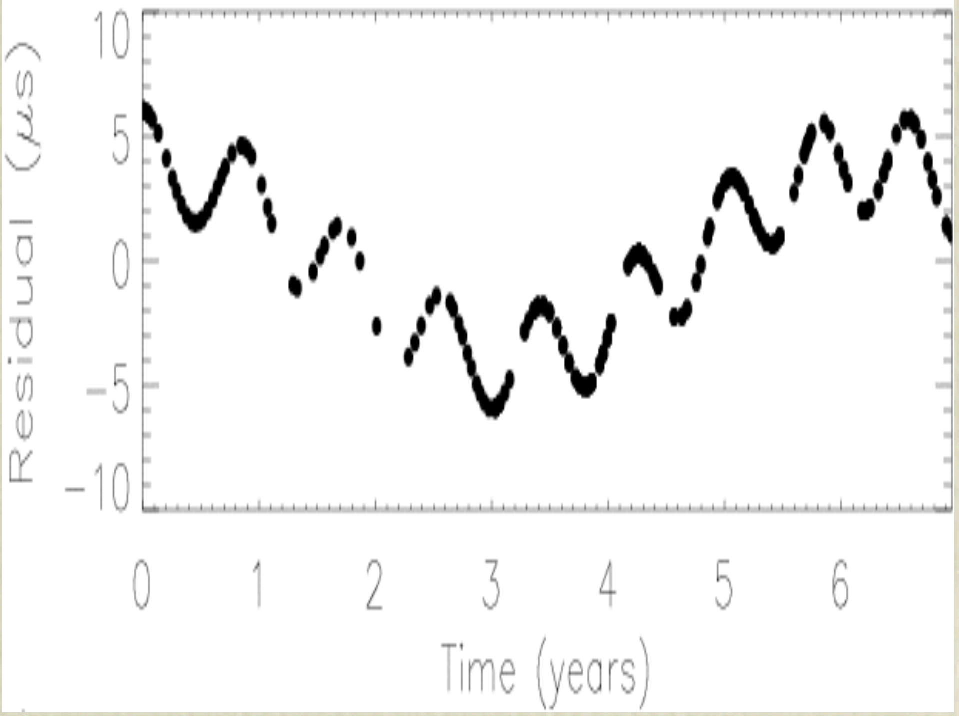
Pulsar timing arrays

- Pulsars are very accurate clocks.
  - GW passing between source and observer induces periodic change in pulse time of arrival.
  - Use a network (array) of pulsars to increase signal to noise.
  - Ongoing international effort using various radio telescopes - EPTA, PPTA and NANOGrav.



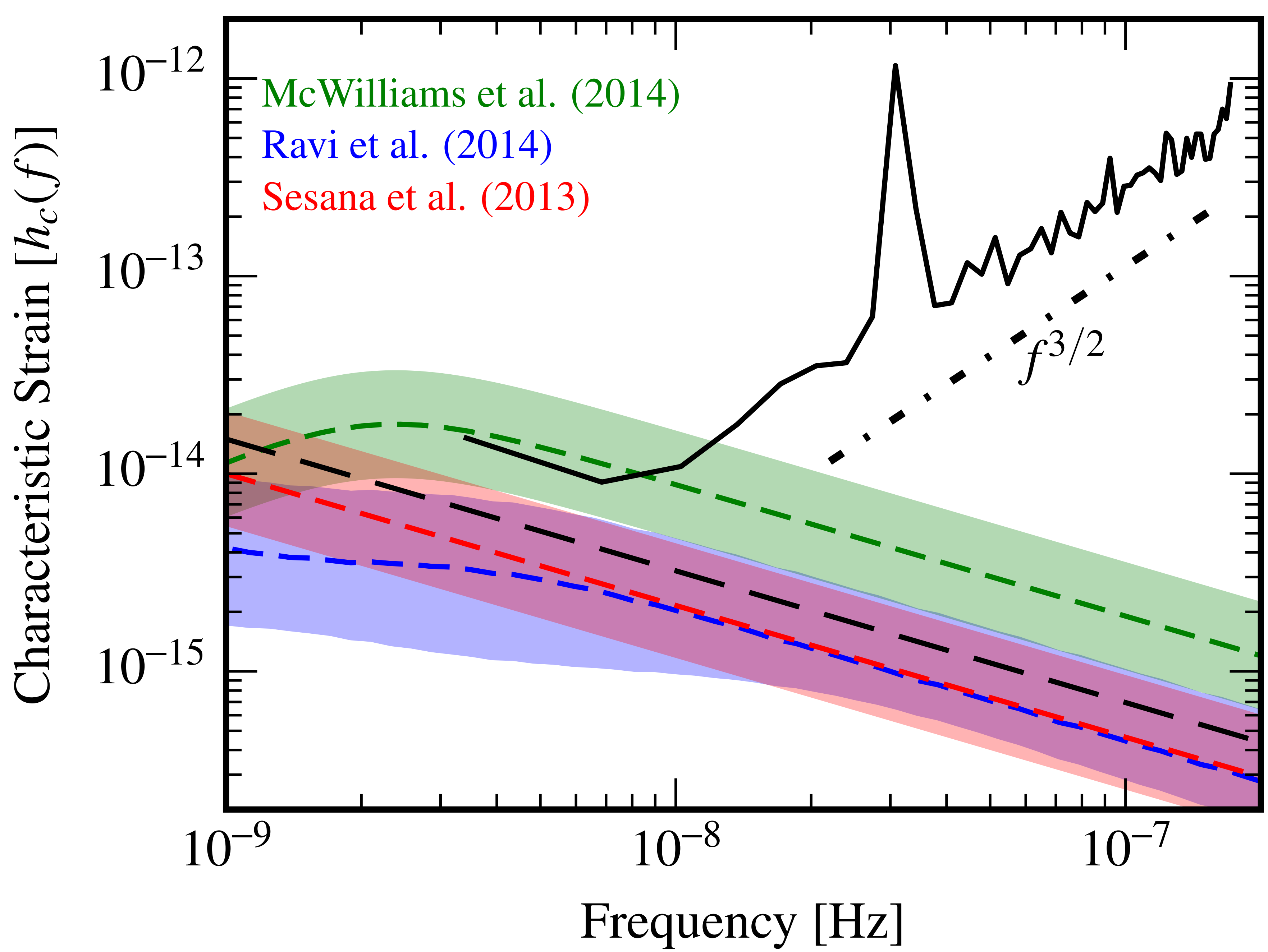






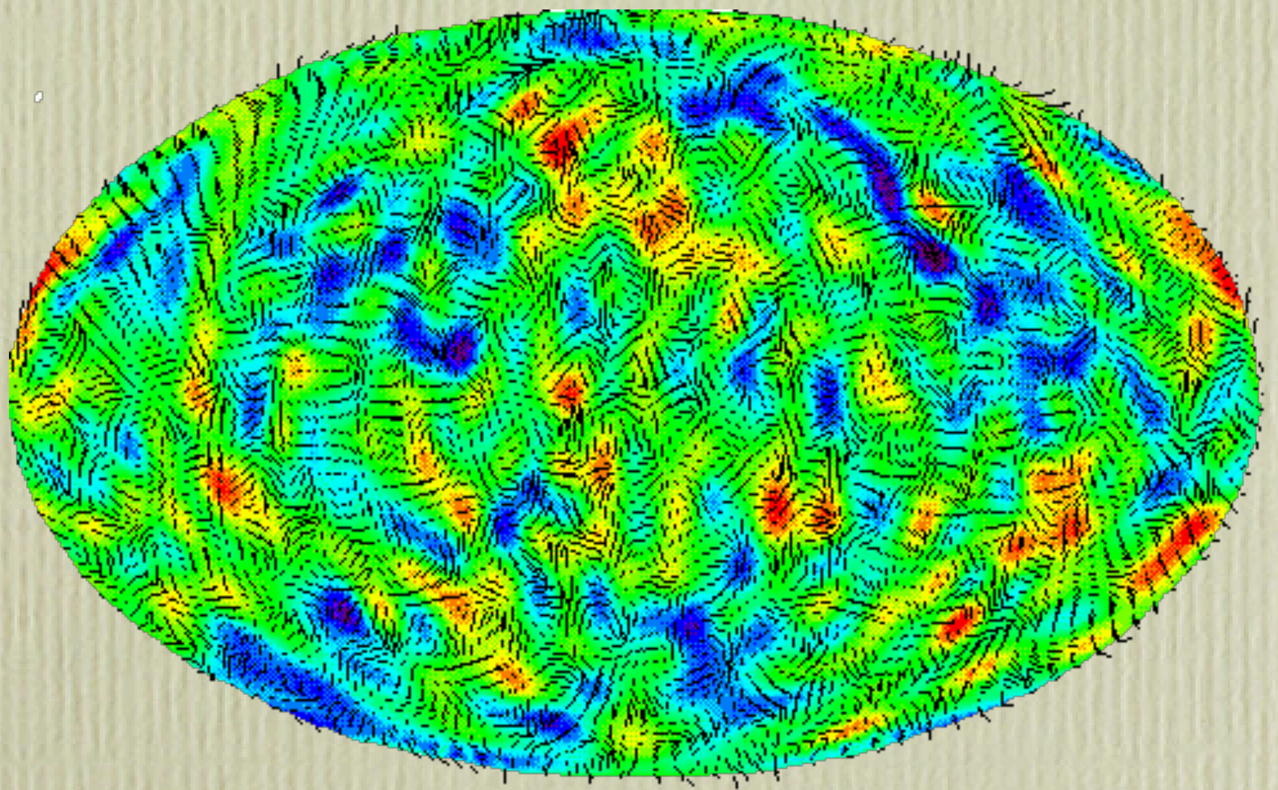
Pulsar timing arrays

- No detection yet, but recent limits are starting to become astrophysical interesting.



NANOGrav 9-year results [Arzoumanian et al. (2015)]

Lessons from the CMB



- CMB community measure temperature and polarisation maps
- Polarisation is described in terms of Stokes parameters Q and U that give the polarisation tensor

$$\mathcal{P} = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}$$

- This is a transverse-traceless tensor on the sky, c.f. the GW field

$$h_{ab}^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}$$

Spin-weighted functions

- A spin-weighted function $f(\hat{k},\hat{l},\hat{m})$ maps a point $\hat{k}$ and an orthonormal basis $(\hat{l},\hat{m})$ on the sphere onto $\mathbb{C}$ and has the property

$$f(\hat{k},\cos\psi\hat{l}-\sin\psi\hat{m},\sin\psi\hat{l}+\cos\psi\hat{m})=\mathrm{e}^{is\psi}f(\hat{k},\hat{l},\hat{m})$$

- where s is the spin weight.
- Under such a rotation

$$h_+\rightarrow h_+\cos 2\psi+h_\times\sin 2\psi \qquad h_\times\rightarrow -h_+\sin 2\psi+h_\times\cos 2\psi$$

- so the quantities $m^a_\pm m^b_\pm h_{ab}(\hat{k})$, where $\hat{m}^a_\pm=\hat{l}^a\pm i\hat{m}^a$ have spin-weight ± 2. A spin-weight s function can be expanded in terms of

$${}_sY_{lm}(\theta,\phi)=\sqrt{\frac{(l-s)!}{(l+s)!}}\check{\eth}^sY_{lm}(\theta,\phi)$$

$$\check{\eth}^s\eta=-(\sin\theta)^s\left[\frac{\partial}{\partial\theta}+i\csc\theta\frac{\partial}{\partial\phi}\right](\sin\theta)^{-s}\eta$$

Grad and curl spherical harmonics

- Can decompose any transverse-traceless tensor field on the sky as a superposition of gradients and curls of spherical harmonics

$$Y^G_{(lm)ab} = N_l \left(Y_{(lm);ab} - \frac{1}{2} g_{ab} Y_{(lm);c}{}^c \right)$$

$$Y^C_{(lm)ab} = \frac{N_l}{2} \left(Y_{(lm);ac}\epsilon^c{}_b + Y_{(lm);bc}\epsilon^c{}_a \right)$$

$$N_l = \sqrt{\frac{2(l-2)!}{(l+2)!}}$$

- NB we have modes with $l \geq 2$ only. Using standard polarisation tensors on the sky

$$e^+_{ab}(\hat{k}) = \hat{\theta}_a \hat{\theta}_b - \hat{\phi}_a \hat{\phi}_b \qquad e^\times_{ab}(\hat{k}) = \hat{\theta}_a \hat{\phi}_b + \hat{\phi}_a \hat{\theta}_b$$

- we have

$$Y^G_{(lm)ab}(\hat{k}) = \frac{N_l}{2} \left[W_{(lm)}(\hat{k}) e^+_{ab}(\hat{k}) + X_{(lm)}(\hat{k}) e^\times_{ab}(\hat{k}) \right]$$

$$Y^C_{(lm)ab}(\hat{k}) = \frac{N_l}{2} \left[W_{(lm)}(\hat{k}) e^\times_{ab}(\hat{k}) - X_{(lm)}(\hat{k}) e^+_{ab}(\hat{k}) \right]$$

Grad and curl spherical harmonics

- The W and X functions are related to spin-2 spherical harmonics

$$_{\pm 2}Y_{(lm)}(\hat{k}) = \frac{N_l}{\sqrt{2}}\left[W_{(lm)}(\hat{k}) \pm i X_{(lm)}(\hat{k})\right]$$

- and can be written in terms of associated Legendre polynomials

$$W_{(lm)}(\hat{k}) = e^{im\phi} \times \{\text{combinations of } P_l^m\text{'s}\}$$

$$iX_{(lm)}(\hat{k}) = m\, e^{im\phi} \times \{\text{combinations of } P_l^m\text{'s}\}$$

- In terms of these grad and curl harmonics, a general GW background with GR polarisation can be written

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}} \left\{\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left[a^G_{(lm)}(f) Y^G_{(lm)ab}(\hat{k}) + a^C_{(lm)}(f) Y^C_{(lm)ab}(\hat{k})\right]\right\} e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

PTA response

- A plane gravitational wave induces a redshift in a pulsar signal

$$z(t,\hat{k}) \equiv \frac{\Delta v(t)}{\nu_0} = \frac{1}{2}\frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}}\Delta h_{ab}(t,\hat{k})$$

- The redshift induced by a GW background can be written as

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} \mathrm{d}f \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}}\, \frac{1}{2}\frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}} h_{ab}(f,\hat{k}) \left[1 - e^{-i2\pi f L(1+\hat{k}\cdot\hat{u})/c}\right] e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)} \\ &= \int_{-\infty}^{\infty} \mathrm{d}f \sum_{(lm)}\sum_{P} R^P_{(lm)}(f) a^P_{(lm)}(f) e^{i2\pi f t} \end{aligned}$$

- where the response functions for individual modes are given by

$$R^P_{(lm)}(f) = \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}}\, \frac{1}{2}\frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}} Y^P_{(lm)ab}(\hat{k}) e^{-i2\pi f\hat{k}\cdot\vec{x}/c}\left[1 - e^{-i2\pi f L(1+\hat{k}\cdot\hat{u})/c}\right]$$

- We make the simplifying assumption that $\vec{x} \approx 0$. We will use the notation $y \equiv 2\pi f L/c$ and often assume $y \gg 0$.

Alternative polarisation states

- GR admits two transverse and traceless (TT) polarisations

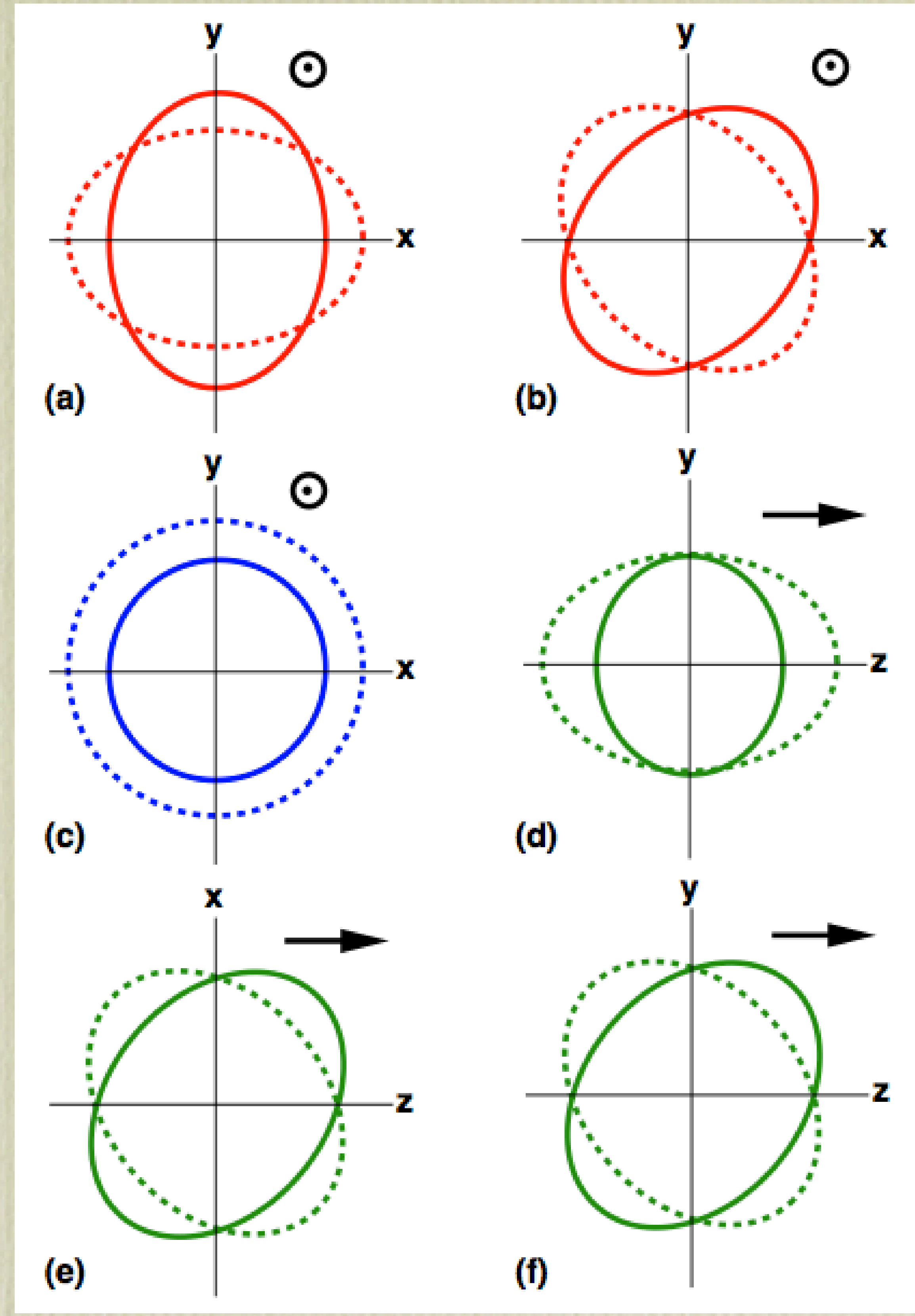
$$e^{+}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e^{\times}_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- In other metric theories of gravity, can have up to four additional states

$$e^{B}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e^{L}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e^{X}_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad e^{Y}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- Can use a similar approach to map non-GR polarisation backgrounds.



Extensions - alternative polarisations

- For the scalar modes (B, L) the quantities

$$\hat{m}^a_\pm \hat{m}^b_\pm h^B_{ab}(\hat{k}) \qquad\qquad \hat{m}^a_\pm \hat{m}^b_\pm h^L_{ab}(\hat{k})$$

- where $\hat{m}^a_\pm = \hat{l}^a \pm i\hat{m}^a$ as before, are invariant under rotations, i.e., they are spin-weight zero.

- Expand in terms of **standard spherical harmonics**, e.g.,

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int_{S^2} \mathrm{d}^2\hat{\Omega}_{\hat{k}}\, h_B(f,\hat{k}) e^B_{ab}(\hat{k}) \mathrm{e}^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

$$h_B(f,\hat{k}) = \frac{1}{\sqrt{2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a^B_{(lm)}(f) Y_{lm}(\hat{k})$$

$$R^B_{(lm)}(f) = \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}} \left[\frac{1}{2\sqrt{2}} \frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}} e^B_{ab}(\hat{k})\, Y_{lm}(\hat{k}) \left(1 - \mathrm{e}^{-iy(1+\hat{k}.\hat{u})}\right)\right]$$

Extensions - alternative polarisations

- For the vector modes (X, Y) the quantities

$$\hat{m}^a_\pm \hat{m}^b_\pm h^X_{ab}(\hat{k}) \qquad\qquad \hat{m}^a_\pm \hat{m}^b_\pm h^Y_{ab}(\hat{k})$$

- transform like spin-weight ± 1 objects under rotations. **Expand in terms of spin-weight ± 1 spherical harmonics**

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}} \left[h_X(f,\hat{k}) e^X_{ab}(\hat{k}) + h_Y(f,\hat{k}) e^Y_{ab}(\hat{k})\right] \mathrm{e}^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

$$h_X(f,\hat{k}) = \frac{1}{2\sqrt{2}} \sum_{lm} \left[v^G_{(lm)}(f)\left({}_{-1}Y_{lm}(\hat{k}) - {}_{1}Y_{lm}(\hat{k})\right) - i v^C_{(lm)}(f)\left({}_{-1}Y_{lm}(\hat{k}) + {}_{1}Y_{lm}(\hat{k})\right)\right],$$

$$h_Y(f,\hat{k}) = \frac{1}{2\sqrt{2}} \sum_{lm} \left[v^C_{(lm)}(f)\left({}_{-1}Y_{lm}(\hat{k}) - {}_{1}Y_{lm}(\hat{k})\right) + i v^G_{(lm)}(f)\left({}_{-1}Y_{lm}(\hat{k}) + {}_{1}Y_{lm}(\hat{k})\right)\right]$$

- Can write

$$Y^{V_G}_{(lm)a}(\hat{k}) = \frac{1}{2\sqrt{2}}\left[\left({}_{-1}Y_{lm}(\hat{k}) - {}_{1}Y_{lm}(\hat{k})\right)\hat{l}_a + i\left({}_{-1}Y_{lm}(\hat{k}) + {}_{1}Y_{lm}(\hat{k})\right)\hat{m}_a\right],$$

$$Y^{V_C}_{(lm)a}(\hat{k}) = \frac{1}{2\sqrt{2}}\left[\left({}_{-1}Y_{lm}(\hat{k}) - {}_{1}Y_{lm}(\hat{k})\right)\hat{m}_a - i\left({}_{-1}Y_{lm}(\hat{k}) + {}_{1}Y_{lm}(\hat{k})\right)\hat{l}_a\right]$$

Extensions - alternative polarisations

- Then we define

$$Y^{V_G}_{(lm)ab} = Y^{V_G}_{(lm)a}\hat{k}_b + Y^{V_G}_{(lm)b}\hat{k}_a\,,$$
$$Y^{V_C}_{(lm)ab} = Y^{V_C}_{(lm)a}\hat{k}_b + Y^{V_C}_{(lm)b}\hat{k}_a\,,$$

- so that

$$h_{ab}(t,\vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}} \left\{ \sum_{(lm)} \left[a^{V_G}_{(lm)}(f) Y^{V_G}_{(lm)ab}(\hat{k}) + a^{V_C}_{(lm)}(f) Y^{V_C}_{(lm)ab}(\hat{k}) \right] \right\} \mathrm{e}^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

- and work with the responses

$$R^{V_G}_{(lm)}(f) = \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}} \left[\frac{1}{2} \frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}} \left(1 - e^{-iy(1+\hat{k}\cdot\hat{u})}\right) Y^{V_G}_{(lm)ab}(\hat{k}) \right],$$
$$R^{V_C}_{(lm)}(f) = \int_{S^2} \mathrm{d}^2\Omega_{\hat{k}} \left[\frac{1}{2} \frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}} \left(1 - e^{-iy(1+\hat{k}\cdot\hat{u})}\right) Y^{V_C}_{(lm)ab}(\hat{k}) \right]$$

Pulsar response functions

- Compute response in *computational frame*, in which pulsar is in the z-direction. Expansion coefficients transform under a rotation in a similar way to spherical harmonic coefficients.

$$Y^P_{(lm)ab}(\theta,\phi) = \sum_{m'=-l}^{l} \left[{D^l}_{mm'}(\chi_I,\zeta_I,0)\right]^* Y^P_{(lm')\bar{a}\bar{b}}(\bar{\theta},\bar{\phi})\mathbf{R}(\chi_I,\zeta_I,0)^{\bar{a}}{}_a\mathbf{R}(\chi_I,\zeta_I,0)^{\bar{b}}{}_b$$

- Deduce that the response functions in the cosmic frame for a pulsar in direction $\hat{u}^a_I = (\sin\zeta_I\cos\chi_I, \sin\zeta_I\sin\chi_I, \cos\zeta_I)$ take the form

$$R^P_{I(lm)}(f) = Y_{lm}(\hat{u}_I)\mathcal{R}^P_l(y_I)$$

- for all polarisation states.

Response to tensor modes

- In the computational frame we have

$$\frac{1}{2}\frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}}e^{+}_{ab}(\hat{k}) = \frac{1}{2}(1-\cos\theta), \qquad \frac{1}{2}\frac{\hat{u}^a\hat{u}^b}{1+\hat{k}\cdot\hat{u}}e^{\times}_{ab}(\hat{k}) = 0$$

- Recall

$$Y^{G}_{(lm)ab}(\hat{k}) = \frac{N_l}{2}\left[W_{(lm)}(\hat{k})e^{+}_{ab}(\hat{k}) + X_{(lm)}(\hat{k})e^{\times}_{ab}(\hat{k})\right]$$

$$Y^{C}_{(lm)ab}(\hat{k}) = \frac{N_l}{2}\left[W_{(lm)}(\hat{k})e^{\times}_{ab}(\hat{k}) - X_{(lm)}(\hat{k})e^{+}_{ab}(\hat{k})\right]$$

- We deduce that, in this frame

$$R^{C}_{(lm)} = -\int_0^{2\pi}\mathrm{d}\bar{\phi}\int_{-1}^{1}\mathrm{d}\cos\bar{\theta}\,\frac{N_l}{2}F^{+}(\bar{\theta},\bar{\phi})X_{(l0)}(\bar{\theta},\bar{\phi})\left[1-\mathrm{e}^{-iy(1+\cos\bar{\theta})}\right] = 0$$

- We have **zero sensitivity to curl modes** in any frame.

Response to tensor modes

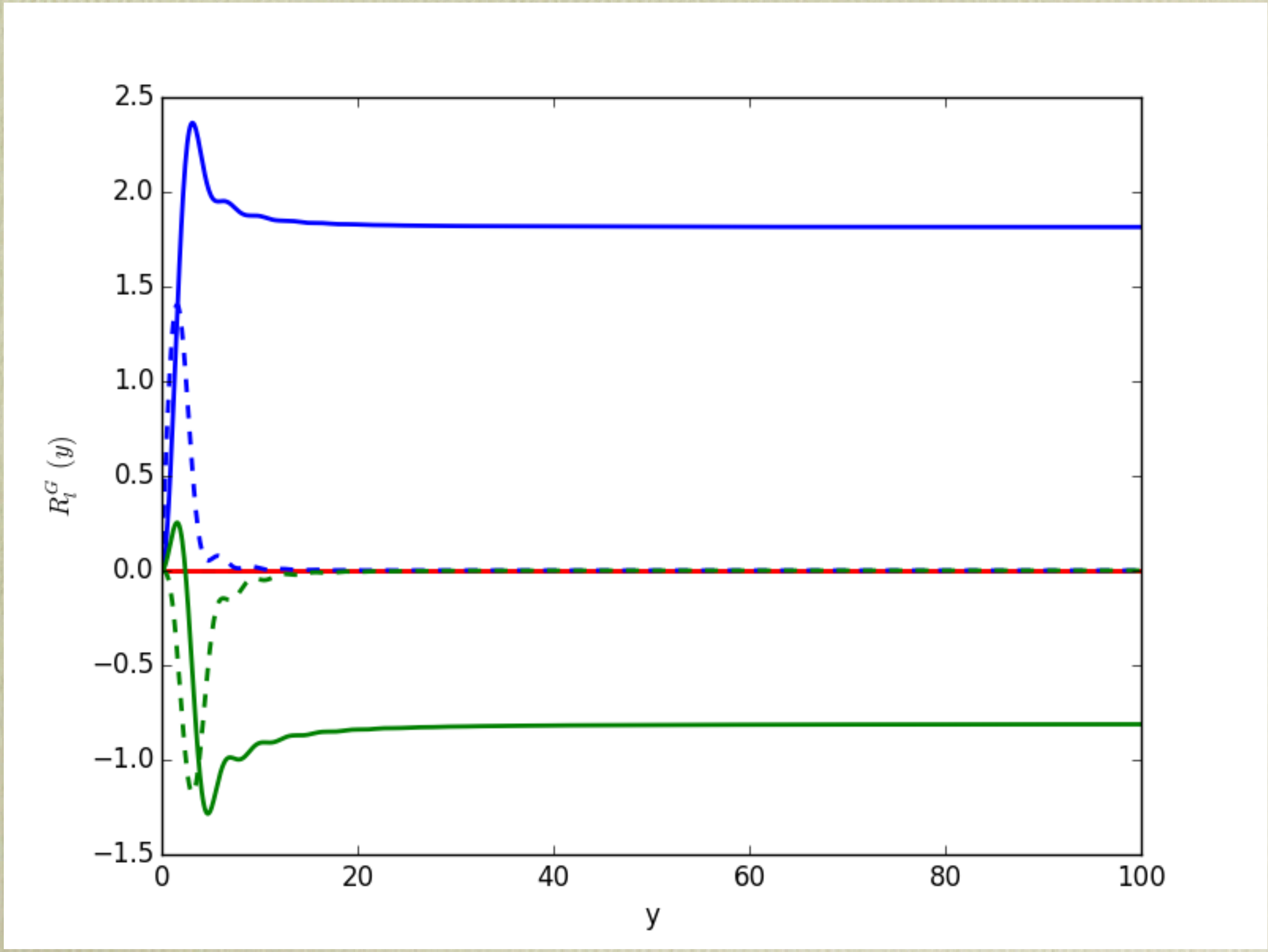
- For the grad modes we have

$$
\begin{aligned}
R^G_{(lm)} &= \int_0^{2\pi} \mathrm{d}\bar{\phi} \int_{-1}^{1} \mathrm{d}\cos\bar{\theta}\, \frac{N_l}{4}(1-\cos\bar{\theta}) W_{(l0)}(\bar{\theta},\bar{\phi}) \left[1 - \mathrm{e}^{-iy(1+\cos\bar{\theta})}\right] \\
&= \frac{\sqrt{(2l+1)\pi}}{2} N_l (-i)^l e^{-iy} \left[(2 - 2iy + y^2) j_l(y) - i(6 + 4iy + y^2)\frac{\mathrm{d}j_l}{\mathrm{d}y} \right. \\
&\qquad \left. -(6iy - y^2)\frac{\mathrm{d}^2 j_l}{\mathrm{d}y^2} - iy^2 \frac{\mathrm{d}^3 j_l}{\mathrm{d}y^3}\right]
\end{aligned}
$$

- The y-dependent terms are the contributions from the pulsar term and are negligible for $y \gtrsim 10$, the regime in which PTAs operate.
- Deduce that the response functions in the cosmic frame for a pulsar in direction $\hat{u}^a_I = (\sin\zeta_I \cos\chi_I, \sin\zeta_I \sin\chi_I, \cos\zeta_I)$ are

$$
R^G_{I(lm)} = 2\pi(-1)^l N_l Y_{(lm)}(\hat{u}_I) \qquad\qquad R^C_{(lm)} = 0
$$

Response to tensor modes



Response to scalar modes

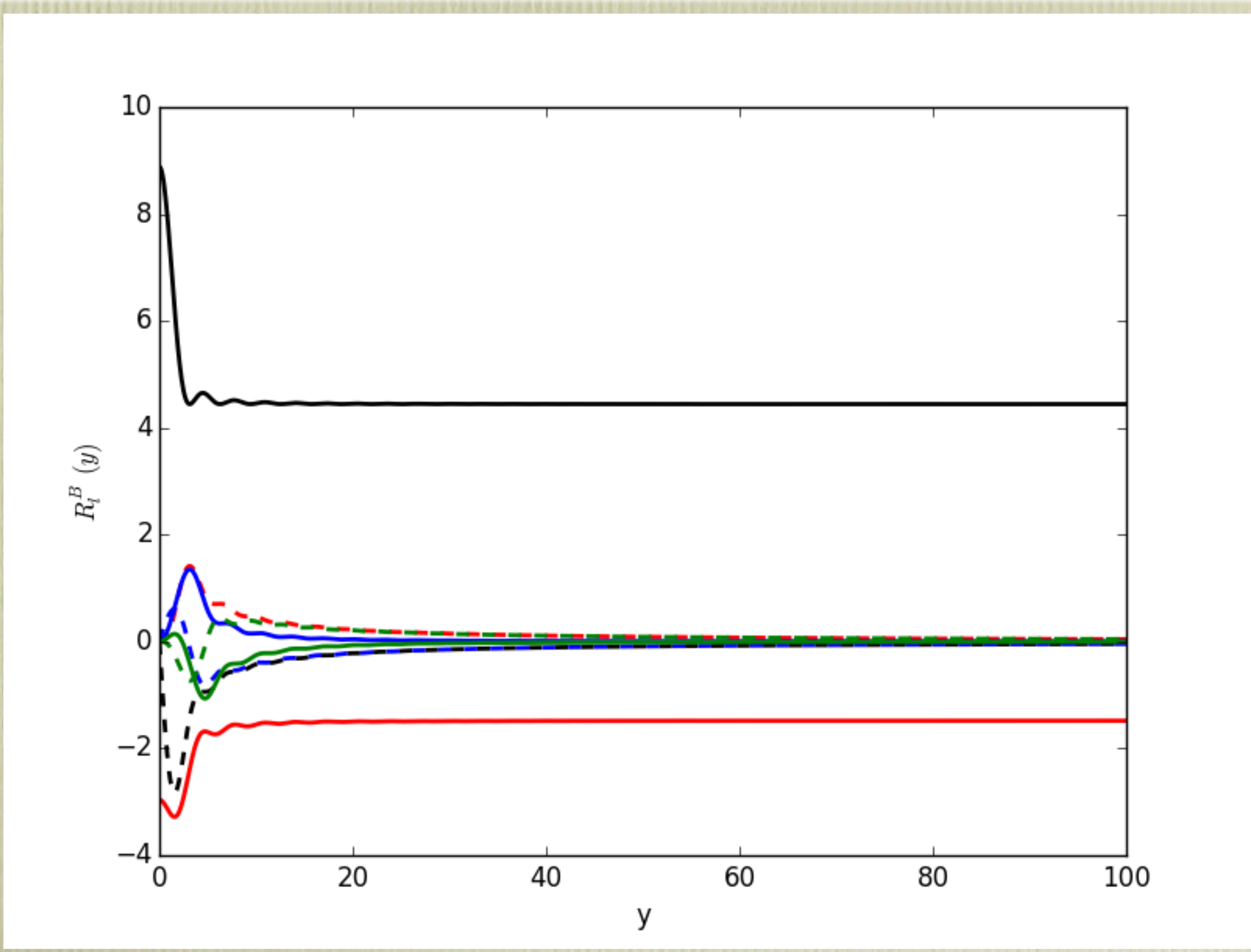
- For the breathing mode, we find

$$\mathcal{R}_l^B(y) = 2\pi\frac{1}{\sqrt{2}}\int_{-1}^{1}\mathrm{d}x\,\frac{1}{2}(1-x)P_l(x)\left(1-\mathrm{e}^{-i(1+x)y}\right)$$
$$= 2\pi\frac{1}{\sqrt{2}}\left\{\delta_{l0}-\frac{1}{3}\delta_{l1}-(-i)^l\mathrm{e}^{-iy}\left[\left(1-i\frac{l}{y}\right)j_l(y)+ij_{l+1}(y)\right]\right\}$$

- and in the limit in which we can ignore the pulsar term, this becomes

$$R^B_{I(lm)} = 4\pi N_l^0\sqrt{\frac{\pi}{2l+1}}Y_{lm}(\hat{u}_I)\left[\delta_{l0}-\frac{1}{3}\delta_{l1}\right]$$

Response to breathing modes



Response to scalar modes

- For the scalar longitudinal modes $R^L_{I(lm)}(f) = Y_{lm}(\hat{u}_I)\mathcal{R}^L_l(y_I)$

$$\mathcal{R}^L_l(y) \equiv 2\pi \int_{-1}^{1} \mathrm{d}x \, \frac{1}{2} \frac{x^2}{1+x} P_l(x) \left(1 - \mathrm{e}^{-iy(1+x)}\right)$$

$$= 2\pi \int_{-1}^{1} \mathrm{d}x \, \frac{1}{2} \left[-1 + x + \frac{1}{1+x}\right] P_l(x) \left(1 - \mathrm{e}^{-iy(1+x)}\right)$$

$$= 2\pi \left\{-\delta_{l0} + \frac{1}{3}\delta_{l1} + (-i)^l \mathrm{e}^{-iy} \left[\left(1 - i\frac{l}{y}\right) j_l(y) + i j_{l+1}(y)\right] + \frac{1}{2} H_l(y)\right\}$$

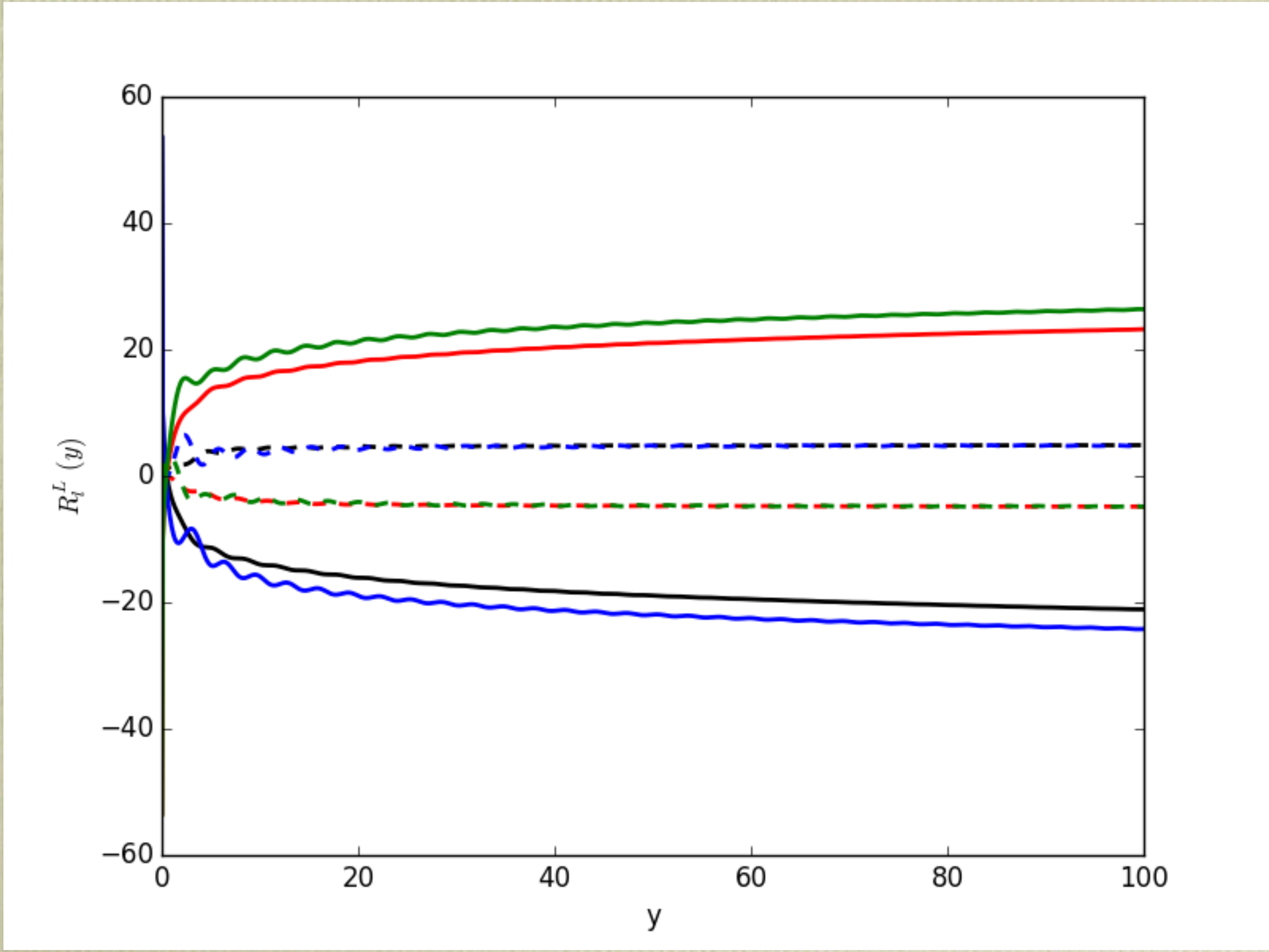
- in which

$$H_l(y) = \int_{-1}^{1} \mathrm{d}x \, \frac{1}{(1+x)} P_l(x) \left(1 - \mathrm{e}^{-iy(1+x)}\right)$$

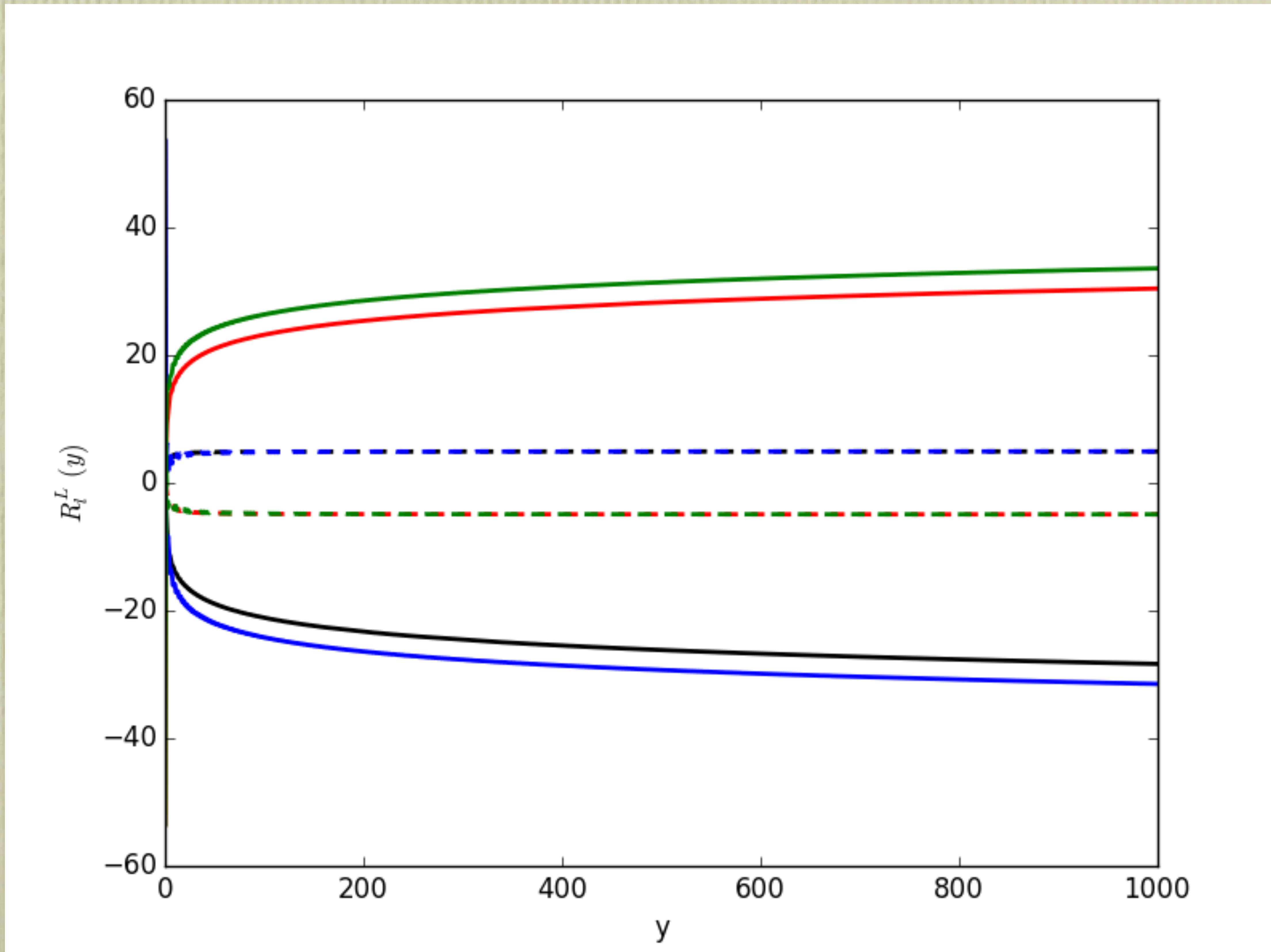
- and for large *y*, we have

$$R^L_{I(lm)}(f) \approx 2\pi Y_{lm}(\hat{u}_I) \left[-\delta_{l0} + \frac{1}{3}\delta_{l1} + \frac{1}{2} H_l(y_I)\right]$$

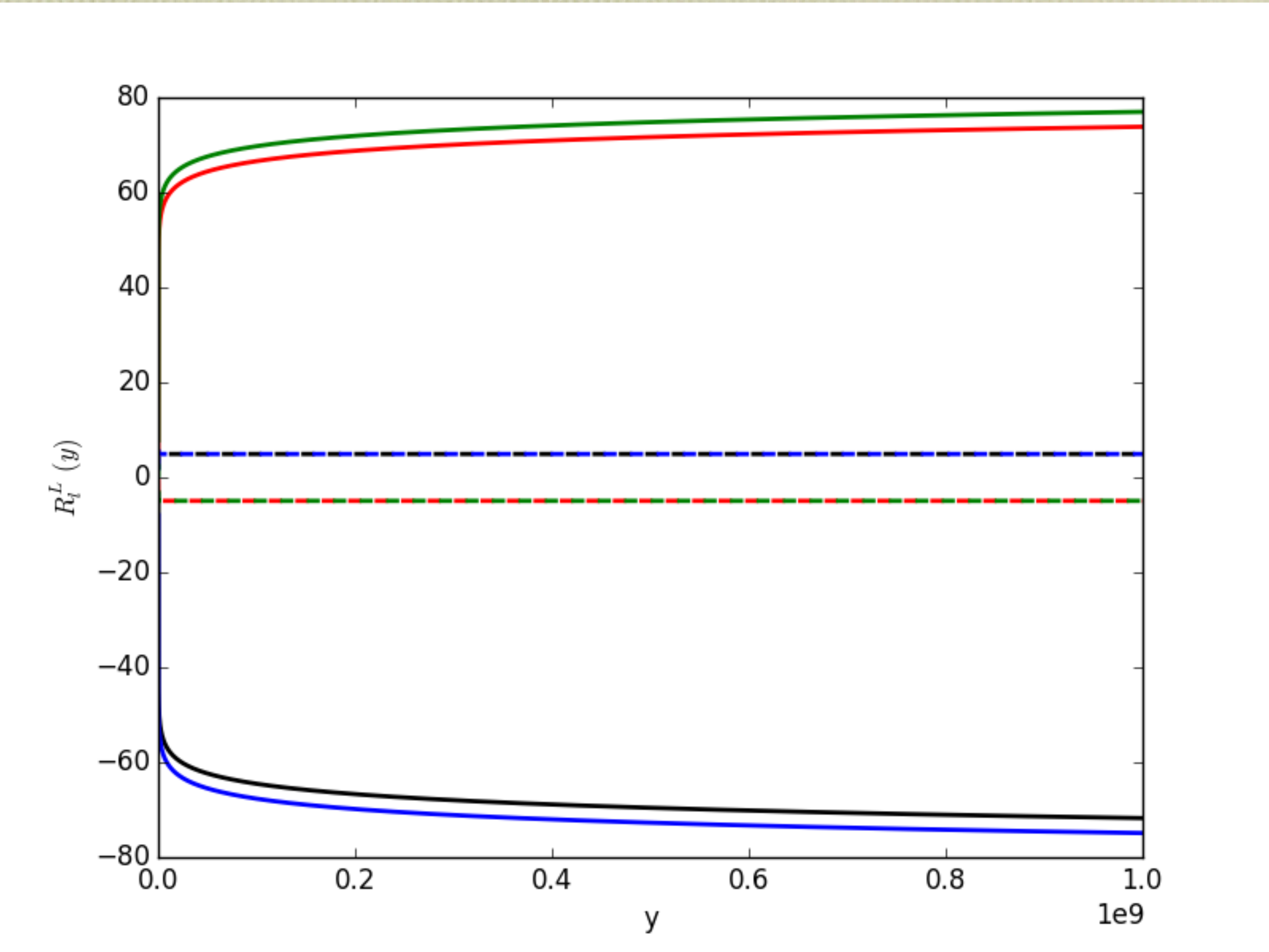
Response to scalar-longitudinal modes



Response to scalar-longitudinal modes



Response to scalar-longitudinal modes



Response to vector modes

- For the vector-longitudinal modes we have

$$R^{V_G}_{I(lm)}(f) = Y_{lm}(\hat{u}_I)\mathcal{R}^{V_G}_l(y_I)\,, \qquad R^{V_C}_{I(lm)}(f) = 0$$

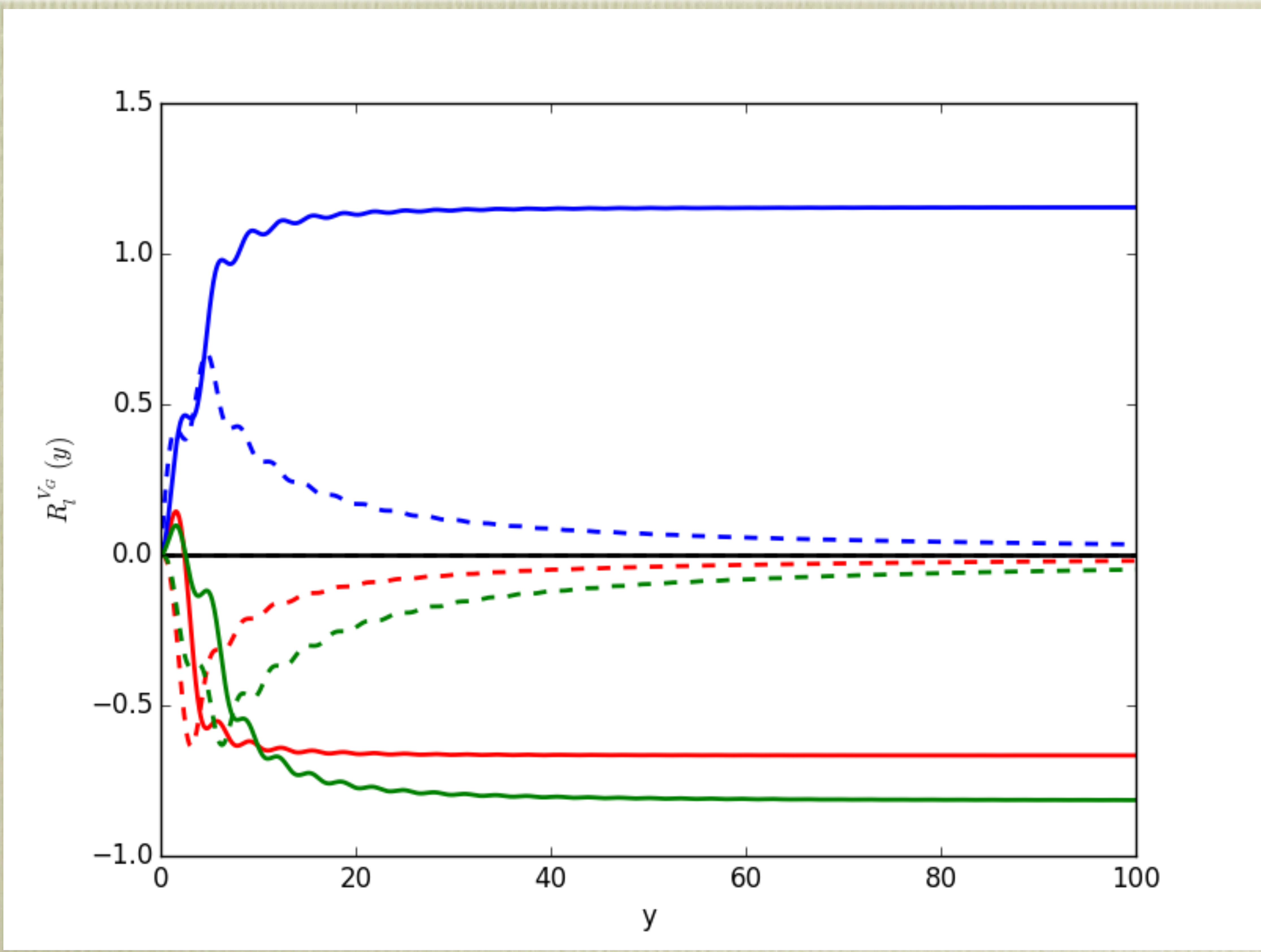
- Once again we find zero response of a PTA to the curl components of the background, but for grad modes

$$\begin{aligned}
\mathcal{R}^{V_G}_l(y) &= \pi\,{}^{(1)}N_l \int_{-1}^{1} \mathrm{d}x \left[x(1-x)\left(1-\mathrm{e}^{-iy(1+x)}\right)\frac{\mathrm{d}P_l}{\mathrm{d}x}\right] \\
&= \pi\,{}^{(1)}N_l \left[-2\delta_{l0} + \frac{4}{3}\delta_{l1} + (-1)^l \mathrm{e}^{-iy}\int_{-1}^{1} \mathrm{d}x\,(1+(2+iy)x+iyx^2)\mathrm{e}^{iyx}P_l(x)\right] \\
&= \pi\,{}^{(1)}N_l \left\{\frac{4}{3}\delta_{l1} + 2(-i)^l \mathrm{e}^{-iy}\left[\left(1-\frac{il}{y}\right)(l+1)j_l(y) - (y-i(2l+3))j_{l+1}(y) - iyj_{l+2}(y)\right]\right\}
\end{aligned}$$

- For large *y*, this can be approximated by

$$R^{V_G}_{I(lm)}(f) \approx 2\pi Y_{lm}(\hat{u}_I)\sqrt{\frac{2(l-1)!}{(l+1)!}}\left[\frac{2}{3}\delta_{l1} + (-1)^l\right]$$

Response to vector modes



Summary of response functions

- To summarise, the full set of response functions are

$$\mathcal{R}_l^G(y) = \pi\,{}^{(2)}N_l(-i)^l e^{-iy}\left[(2-2iy+y^2)j_l(y) - i(6+4iy+y^2)\frac{\mathrm{d}j_l}{\mathrm{d}y} - (6iy-y^2)\frac{\mathrm{d}^2 j_l}{\mathrm{d}y^2} - iy^2\frac{\mathrm{d}^3 j_l}{\mathrm{d}y^3}\right]$$

$$\mathcal{R}_l^C(y) = 0$$

$$\mathcal{R}_l^B(y) = 2\pi\frac{1}{\sqrt{2}}\left\{\delta_{l0} - \frac{1}{3}\delta_{l1} - (-i)^l \mathrm{e}^{-iy}\left[\left(1 - i\frac{l}{y}\right)j_l(y) + ij_{l+1}(y)\right]\right\}$$

$$\mathcal{R}_l^L(y) = 2\pi\left\{-\delta_{l0} + \frac{1}{3}\delta_{l1} + (-i)^l \mathrm{e}^{-iy}\left[\left(1 - i\frac{l}{y}\right)j_l(y) + ij_{l+1}(y)\right] + \frac{1}{2}H_l(y)\right\}$$

$$H_l(y) = \int_{-1}^{1}\mathrm{d}x\,\frac{1}{(1+x)}P_l(x)\left(1 - \mathrm{e}^{-iy(1+x)}\right)$$

$$R_l^{V_G}(y) = \pi\,{}^{(1)}N_l\left\{\frac{4}{3}\delta_{l1} + 2(-i)^l\mathrm{e}^{-iy}\left[\left(1 - \frac{il}{y}\right)(l+1)j_l(y) - (y - i(2l+3))j_{l+1}(y) - iyj_{l+2}(y)\right]\right\}$$

$$R_l^{V_C}(y) = 0$$

- We have **zero sensitivity to tensor and vector curl modes**.
- Without the pulsar term, we have **no sensitivity to structure beyond dipole in scalar-tensor** (breathing mode) **backgrounds**.

Why zero curl response?

- PTAs have a common origin (the Solar System) for all pulsar lines of sight. Curl mode metric perturbation vanishes at the origin.
- Analogous to separation between odd and even modes, e.g., waves on a string

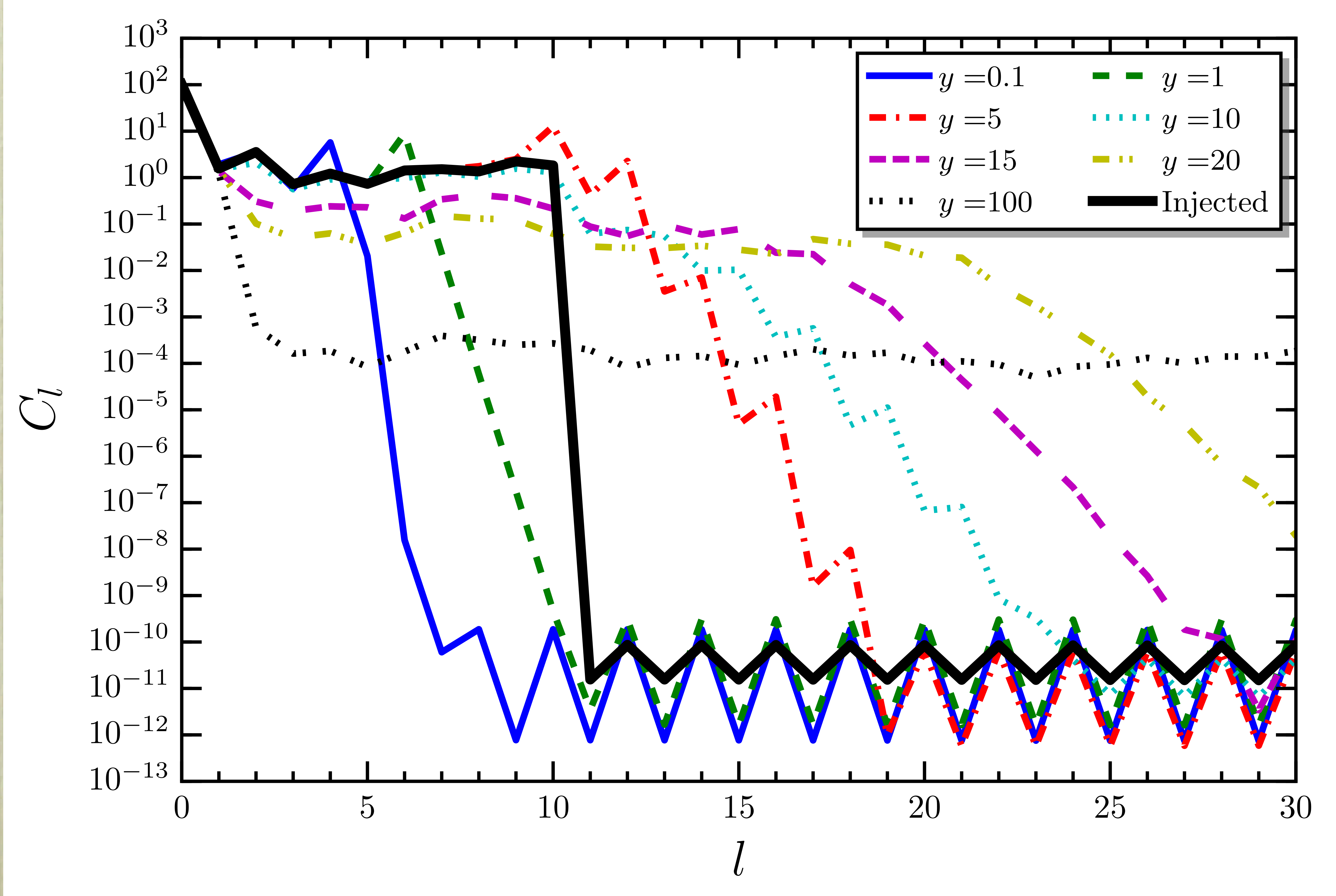
$$A\cos(x-t) + B\sin(x-t) + C\cos(x+t) + D\sin(x+t)$$

- Measurement at x=0 can only determine A+C and B+D. Break degeneracy by adding a measurement at $x_1 \neq 0$ or using point-able detector that can distinguish left and right propagating modes.
- GW detectors are non-point-able and over a year

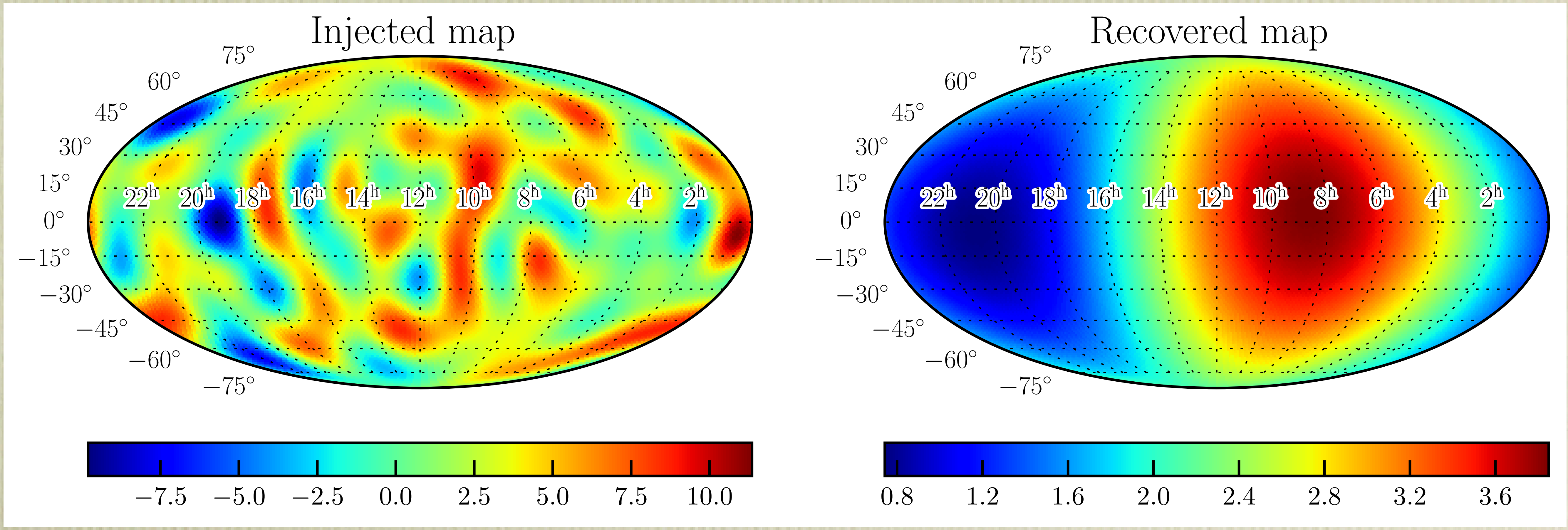
$$\Delta(f\hat{k}\cdot\vec{x}/c) \sim 0.0005$$

- for a GW frequency $f = 10^{-6}$Hz.
- No curl sensitivity because PTA moves by much less than a GW wavelength over typical observation durations.

Scalar-tensor background recovery



Scalar-tensor background recovery



Isotropic, uncorrelated backgrounds

- An isotropic, uncorrelated and unpolarised background is described by the two-point functions

$$\langle h_+(f,\hat{k})h_+^*(f',\hat{k}')\rangle = \langle h_\times(f,\hat{k})h_\times^*(f',\hat{k}')\rangle = \frac{1}{2}H(f)\delta^2(\hat{k},\hat{k}')\delta(f-f')$$

$$\langle h_+(f,\hat{k})h_\times^*(f',\hat{k}')\rangle = \langle h_\times(f,\hat{k})h_+^*(f',\hat{k}')\rangle = 0$$

- or in terms of the grad and curl expansion coefficients

$$\langle a^G_{(lm)}(f)a^{G*}_{(l'm')}(f')\rangle = \langle a^C_{(lm)}(f)a^{C*}_{(l'm')}(f')\rangle = \delta_{ll'}\delta_{mm'}H(f)\delta(f-f')$$

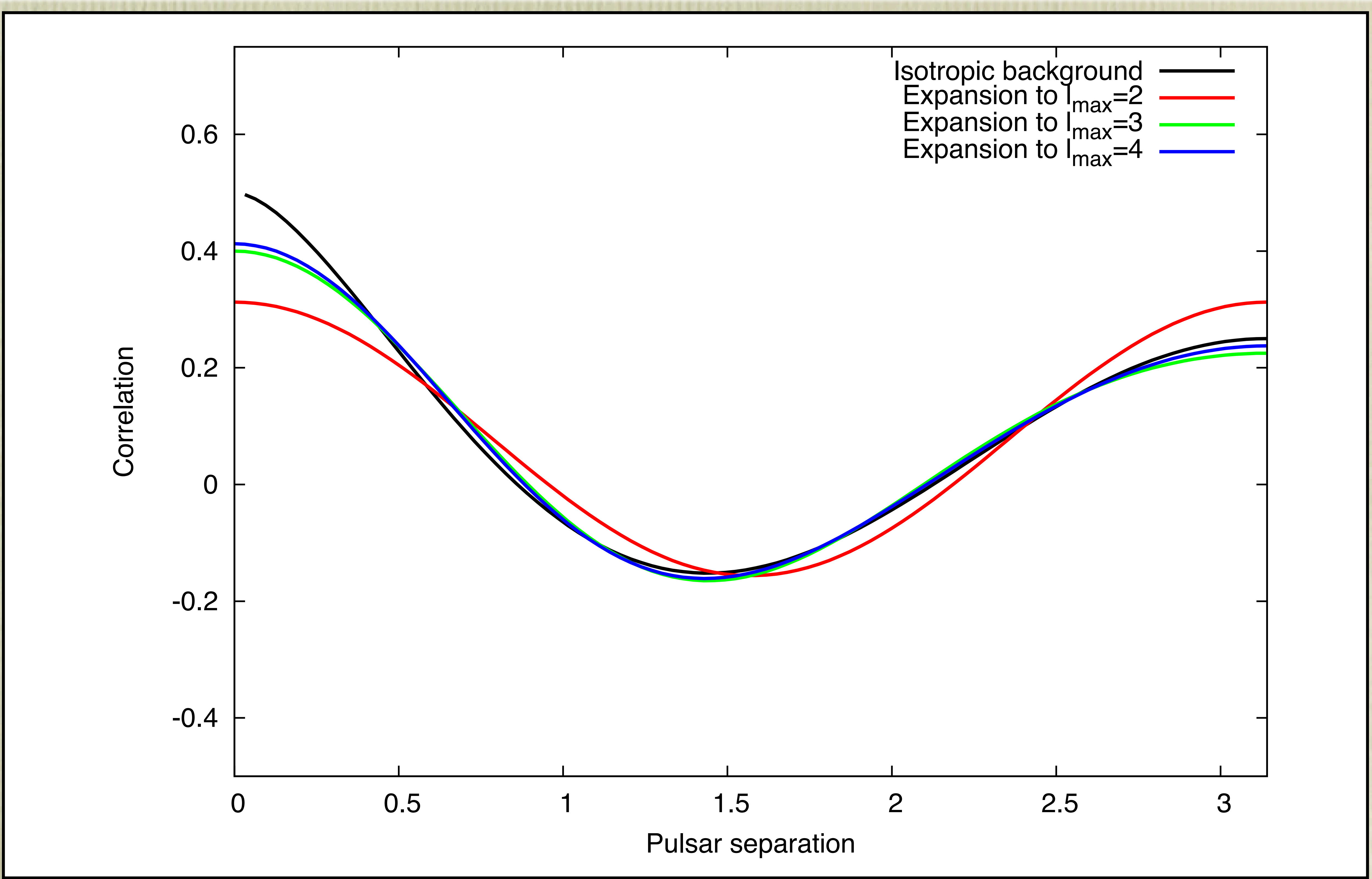
$$\langle a^G_{(lm)}(f)a^{C*}_{(l'm')}(f')\rangle = \langle a^C_{(lm)}(f)a^{G*}_{(l'm')}(f')\rangle = 0$$

- The expected correlation between the response of two pulsars for such a background is

$$\langle h_1(t)h_2(t')\rangle = \int_{-\infty}^{\infty} \mathrm{d}f\, e^{i2\pi f(t-t')}H(f)\Gamma_{12}(f)$$

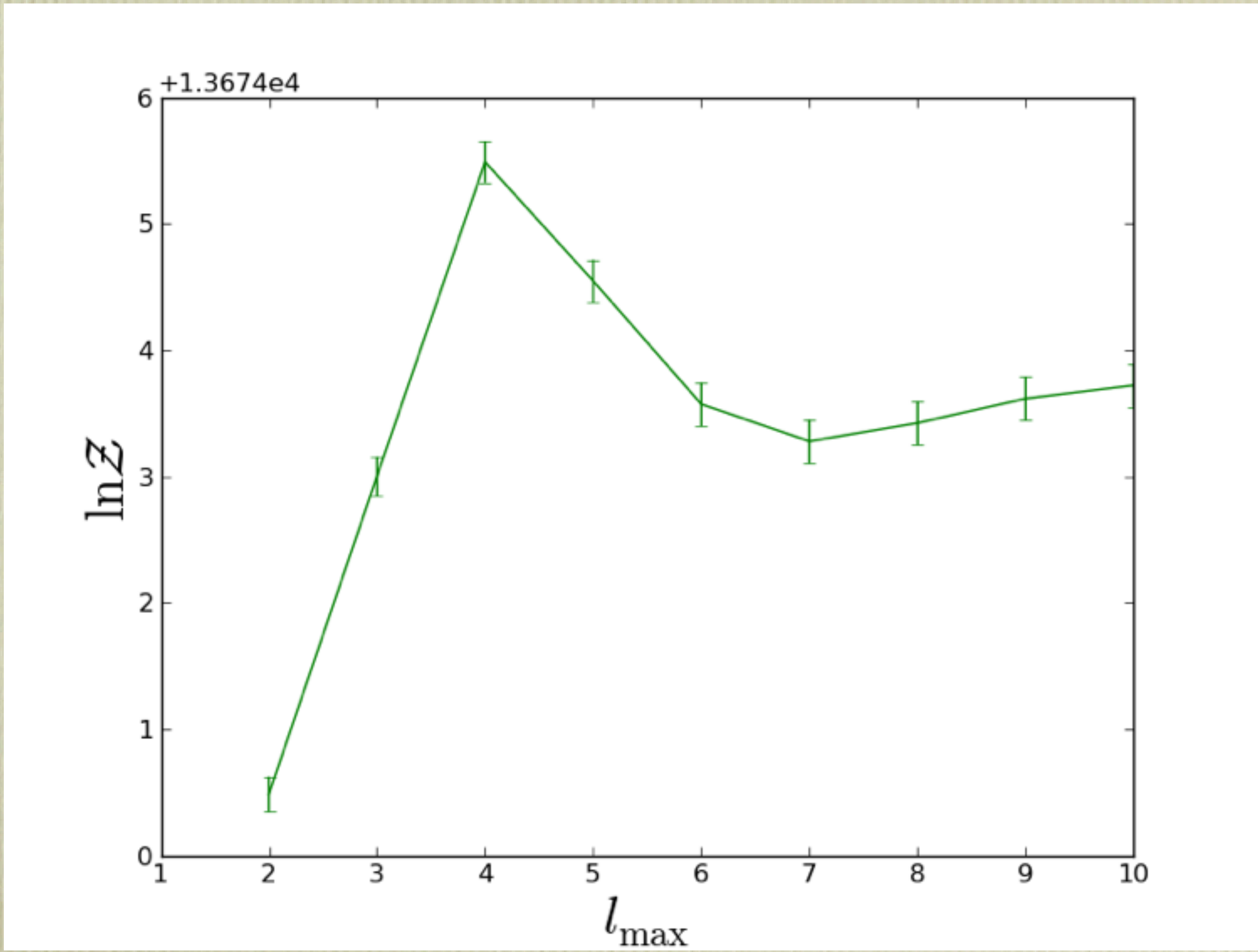
$$\Gamma_{12}(f) = \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\sum_{P} R^P_{1(lm)}(f)R^{P*}_{2(lm)}(f) = \sum_{l=2}^{\infty}(N_l)^2(2l+1)\pi P_l(\hat{u}_1\cdot\hat{u}_2)$$

Isotropic, uncorrelated backgrounds



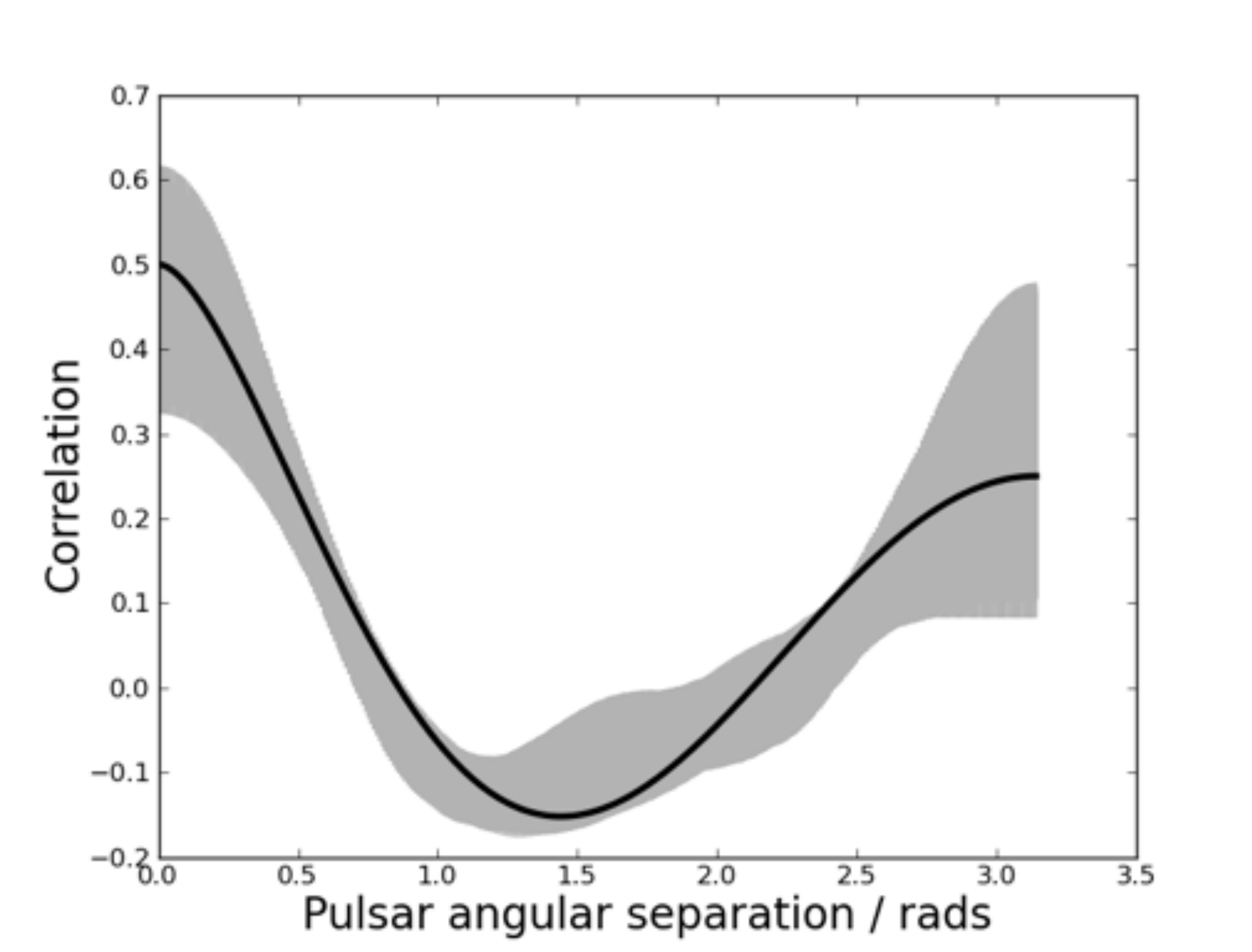
- This agrees with the standard Hellings and Downs correlation.

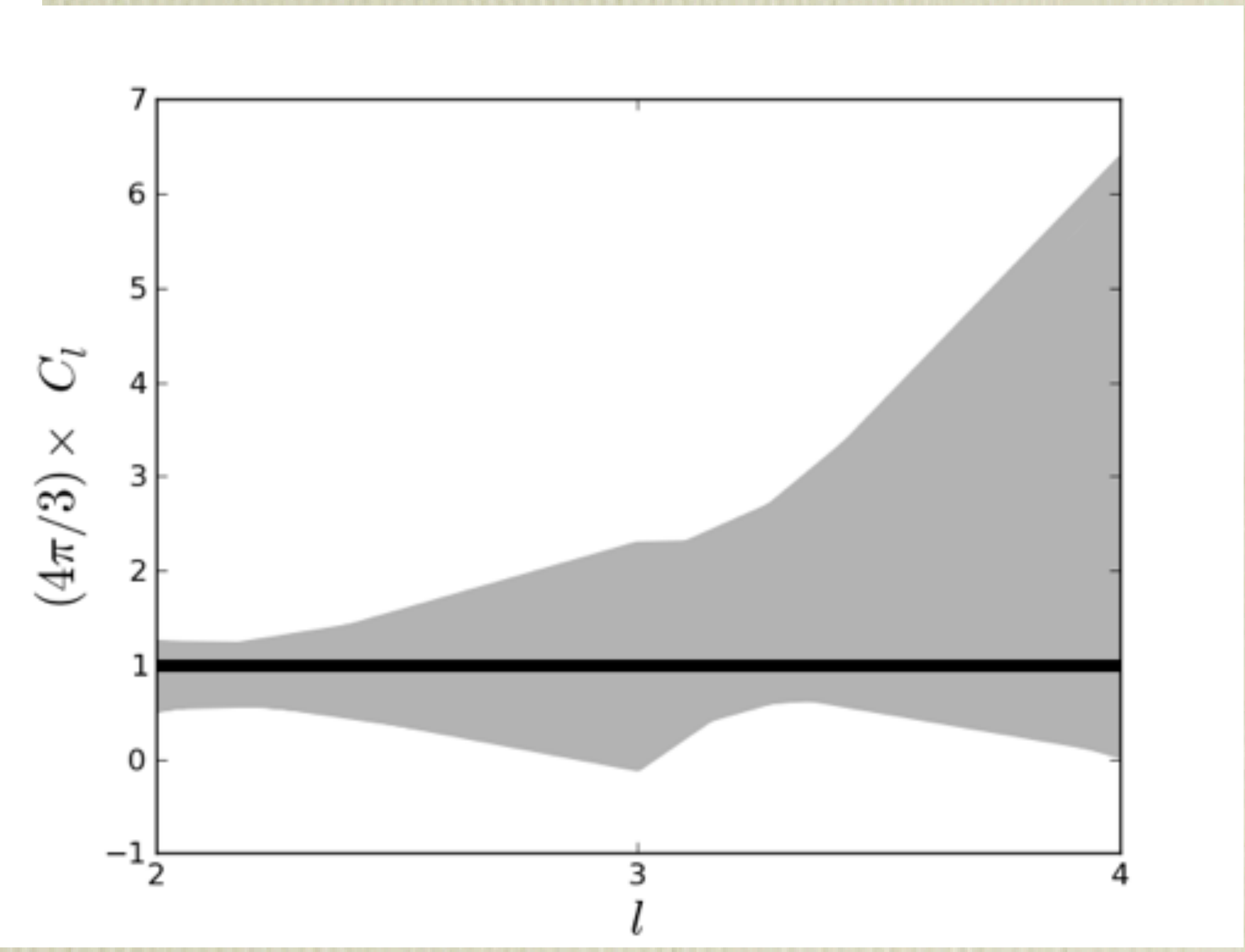
Isotropic, uncorrelated backgrounds



- Including three modes in the expansion is enough to characterise an isotropic background.

Isotropic, uncorrelated backgrounds





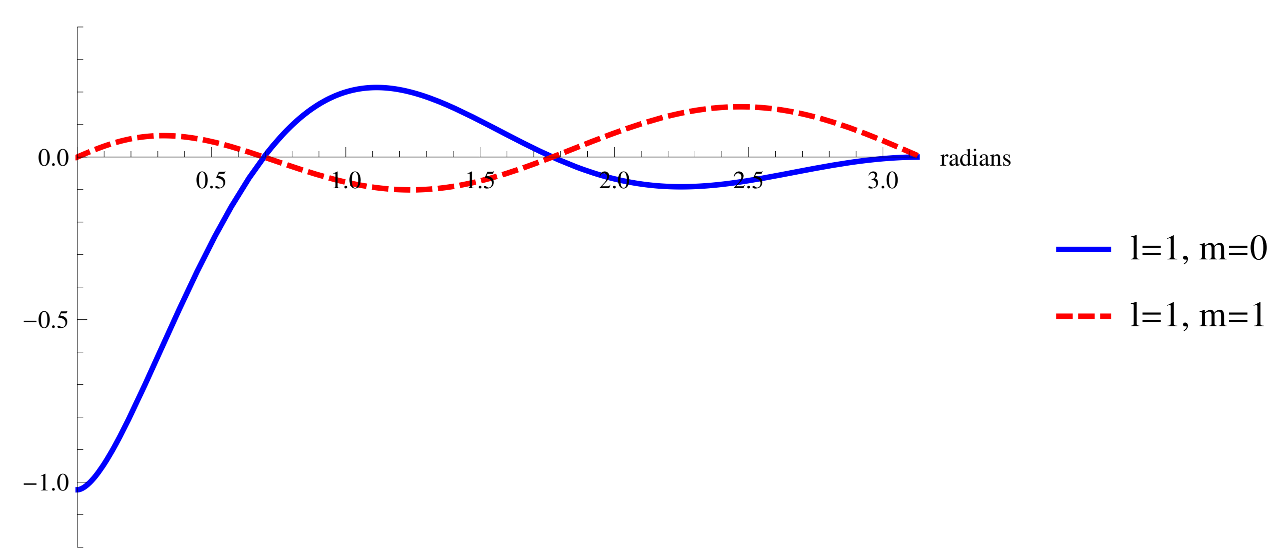
General backgrounds

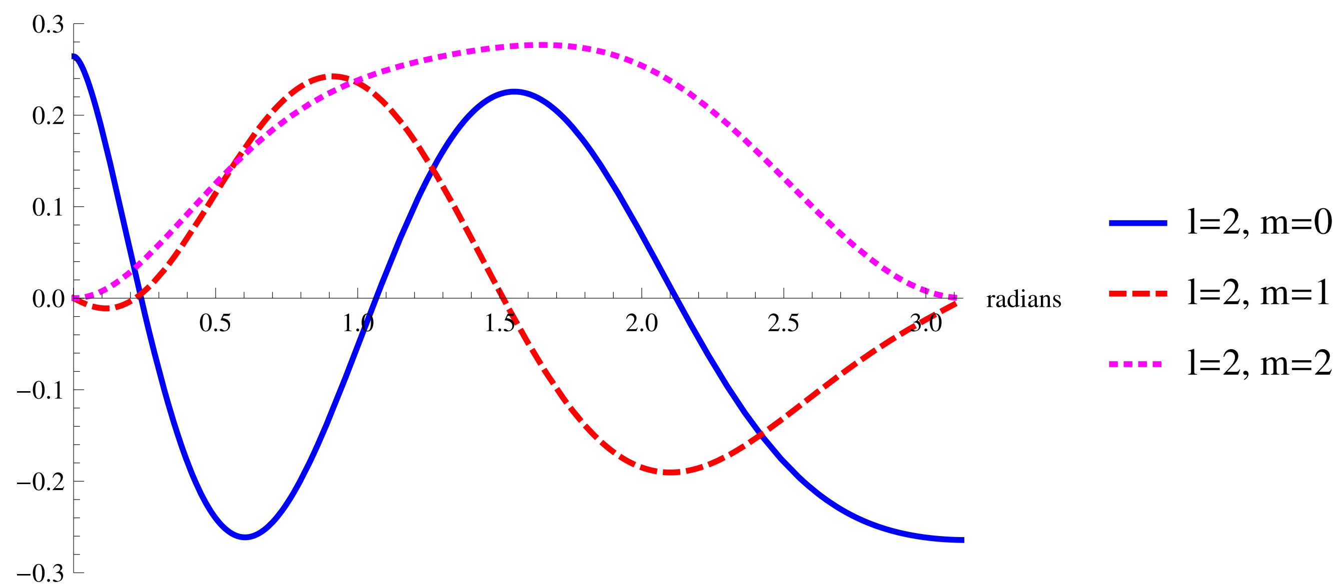
- Can characterise any kind of background in this formalism

$$\langle a^{P}_{(lm)}(f) a^{P'*}_{(l'm')}(f') \rangle = C^{PP'}_{lml'm'} H(f)\, \delta(f - f')$$

$$\Gamma_{12}(f) = 4\pi^2 \sum_{(lm)} \sum_{(l'm')} (-1)^{l+l'} N_l N_{l'} C^{GG}_{lml'm'} Y_{(lm)}(\hat{u}_1) Y^{*}_{(l'm')}(\hat{u}_2)$$

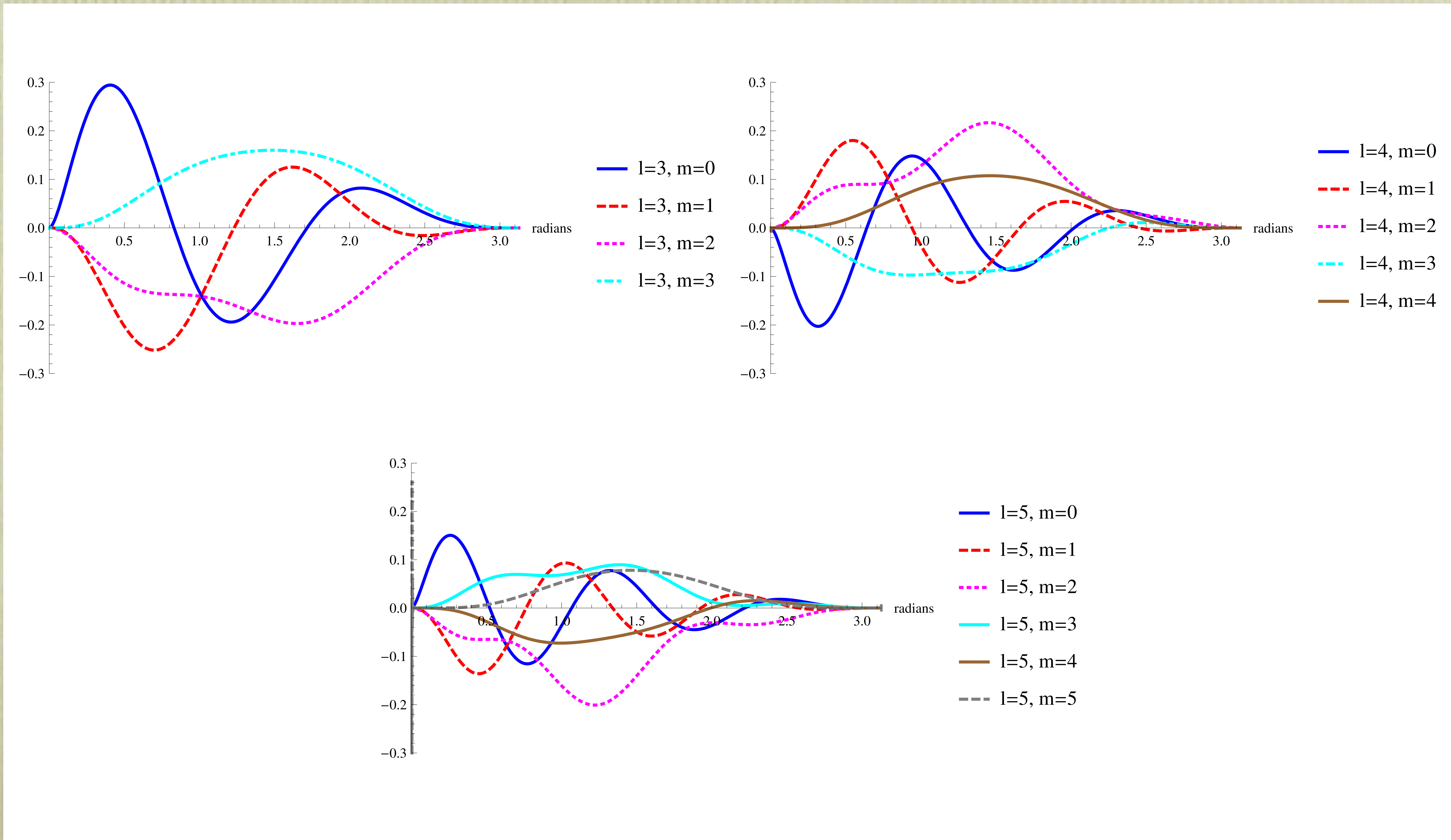
- For example, recover overlap reduction functions for anisotropic, uncorrelated backgrounds (Mingarelli et al. 2013)





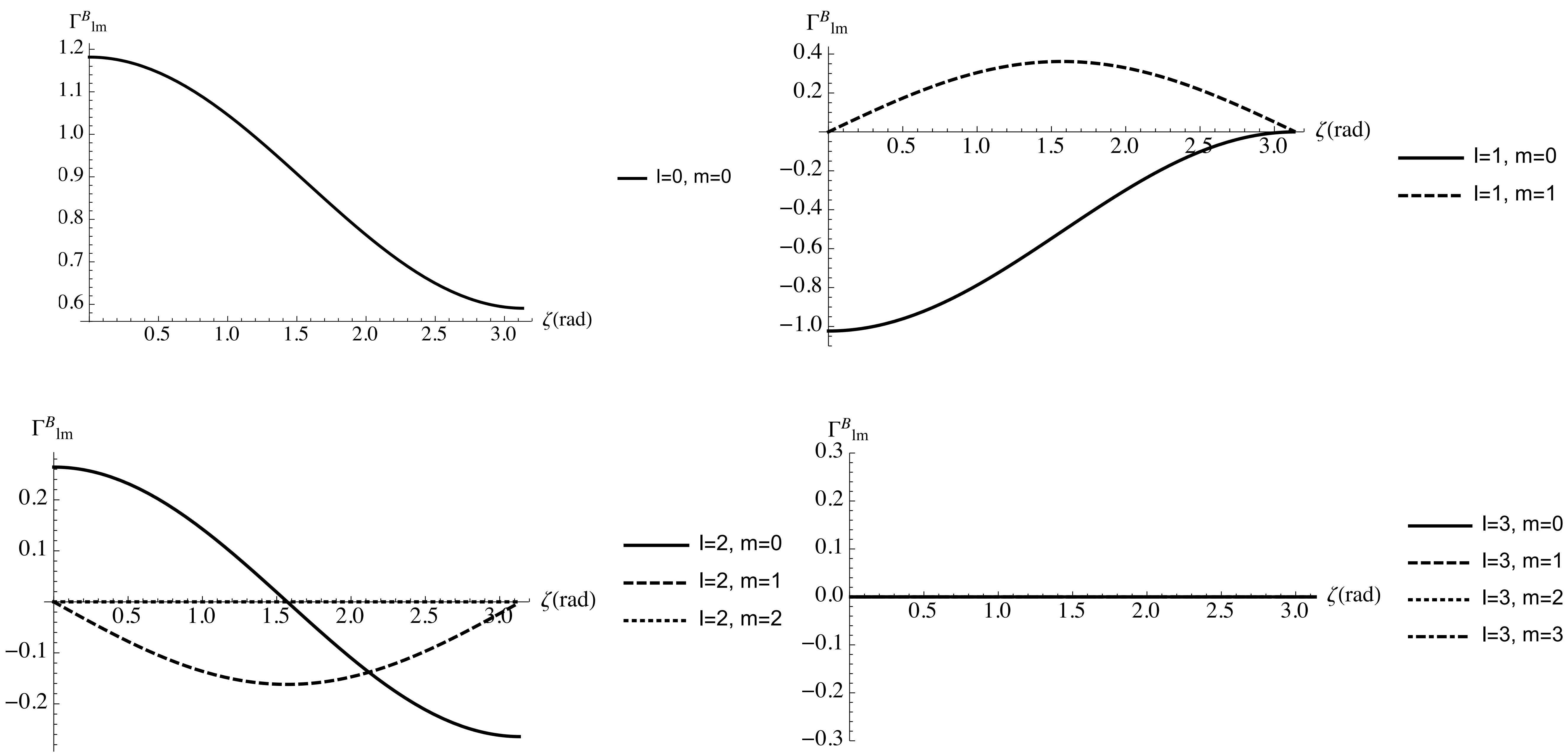
General backgrounds

- and extend these results analytically to arbitrary multipole order



General backgrounds

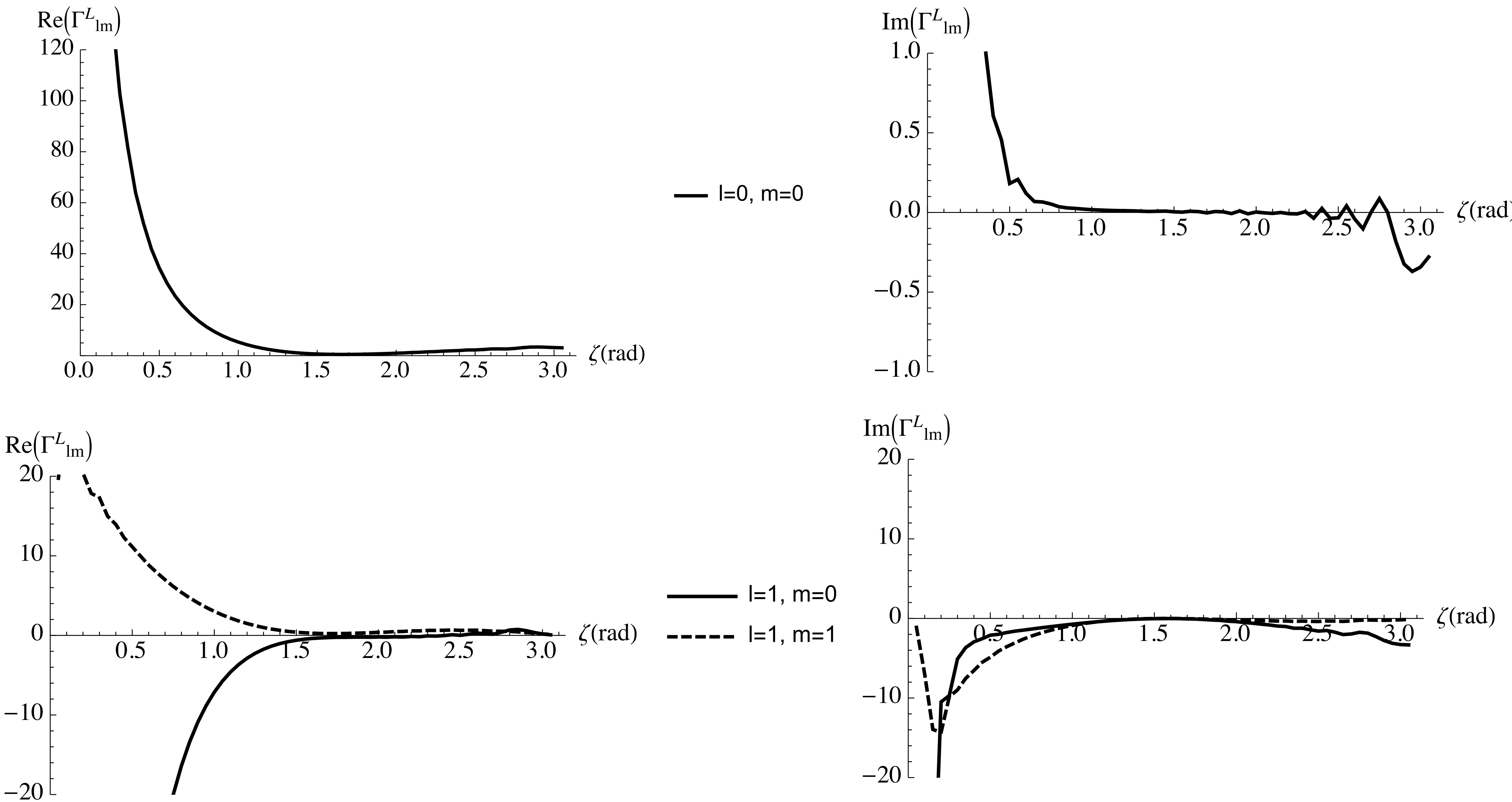
- Also obtain similar results for other polarisation states.



Scalar-transverse (breathing) modes

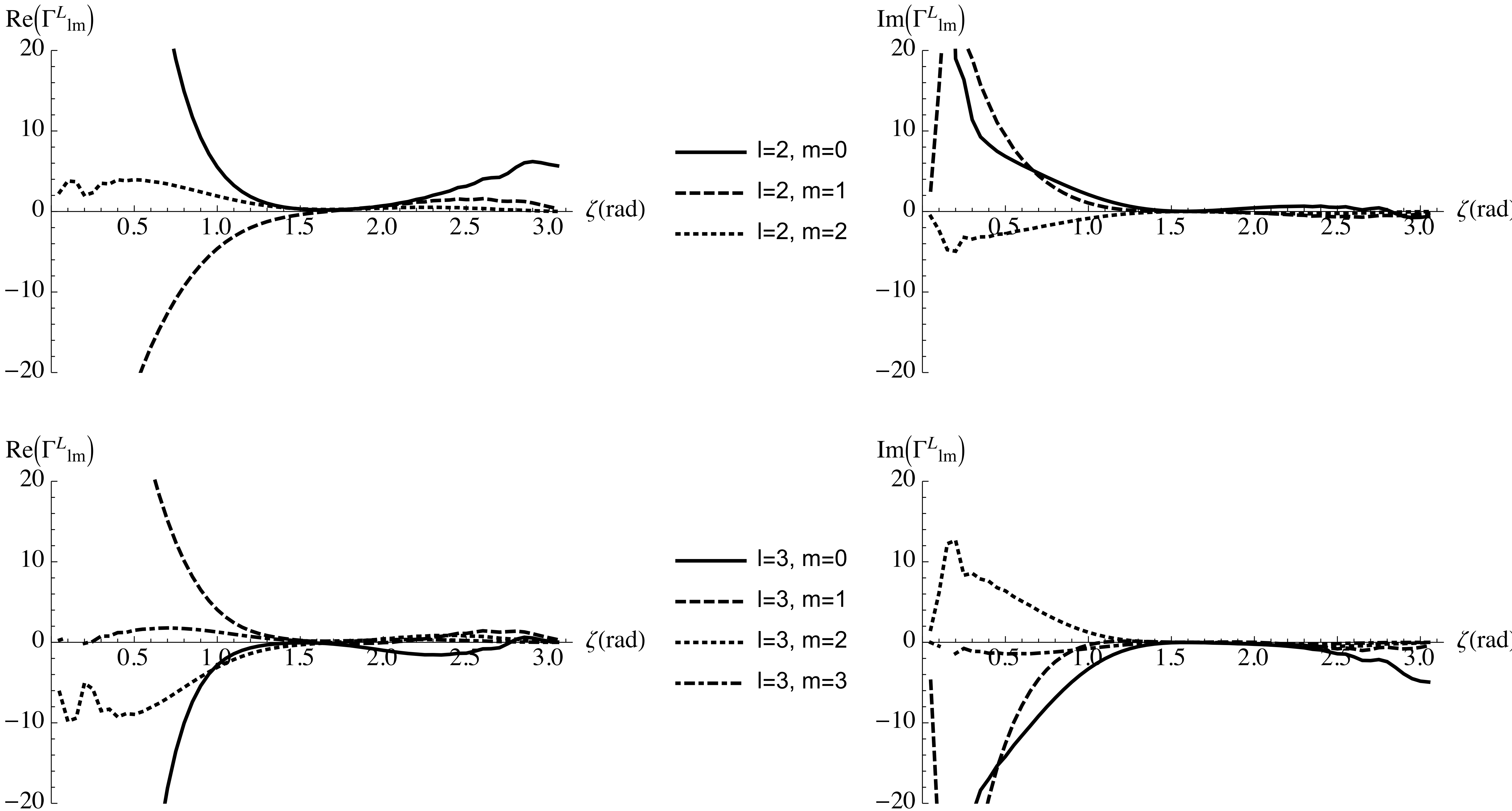
General backgrounds

Scalar-longitudinal modes



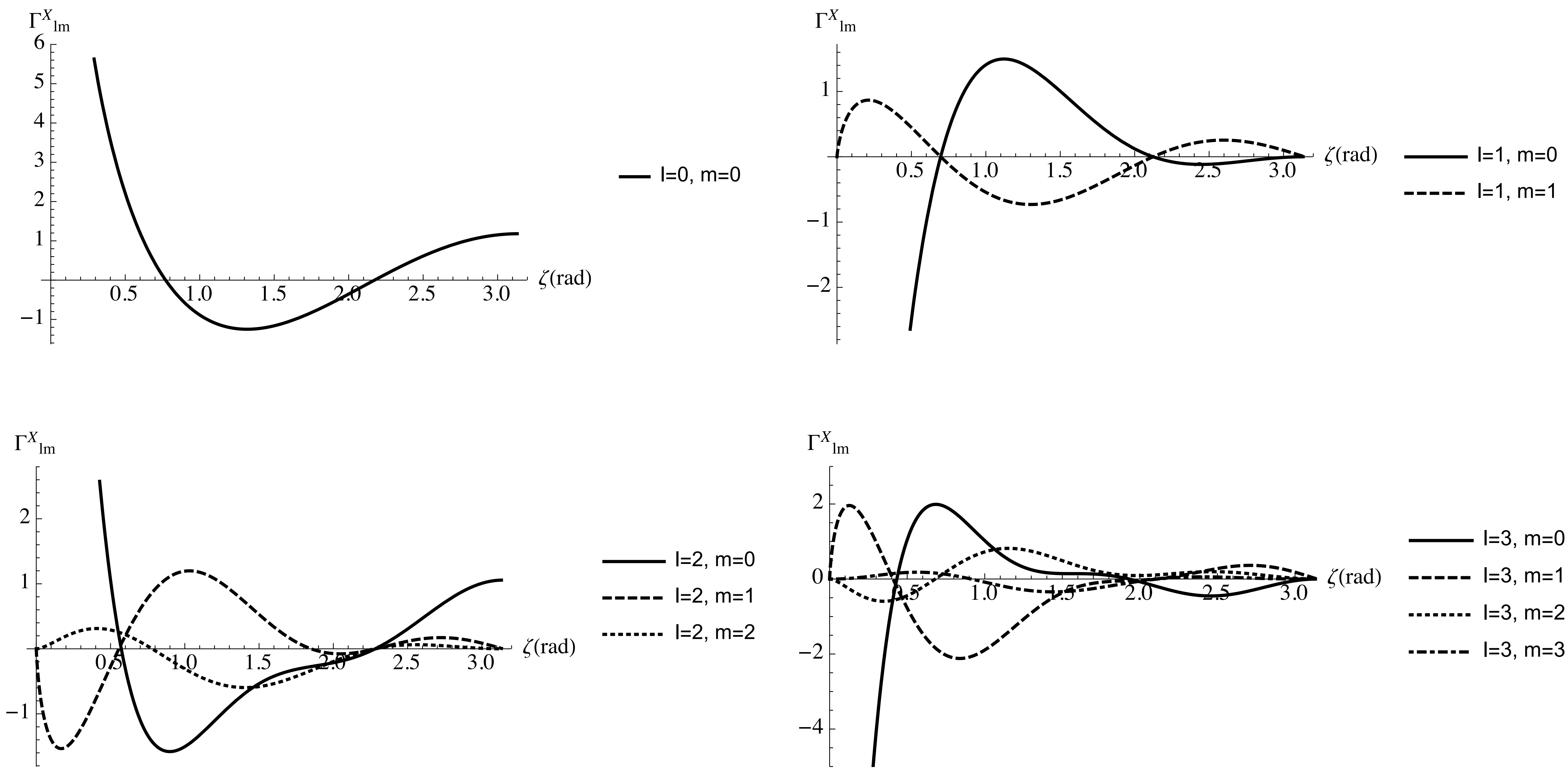
General backgrounds

Scalar-longitudinal modes

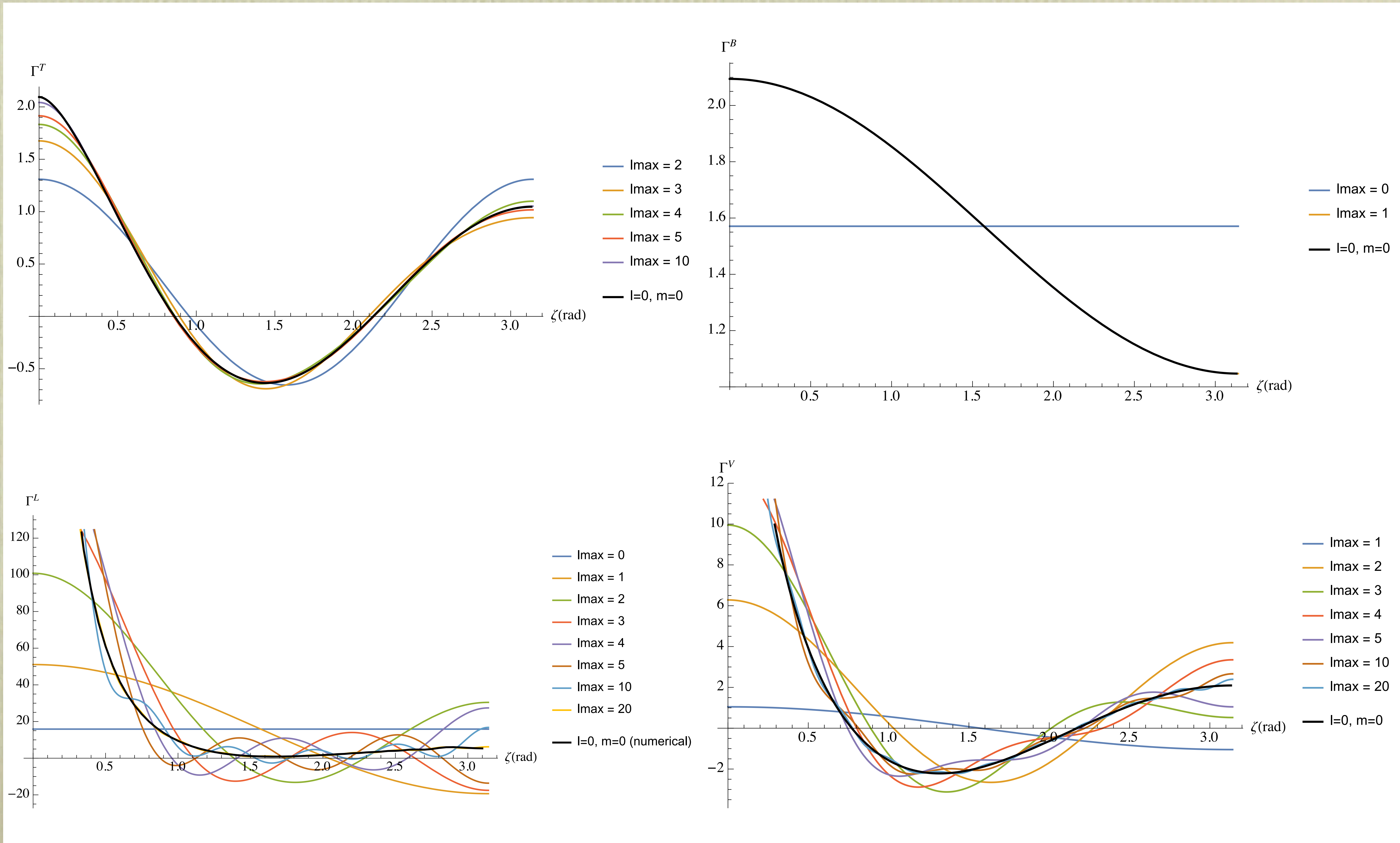


General backgrounds

Vector-longitudinal modes



Isotropic, uncorrelated backgrounds of arbitrary polarisation



Background mapping

- We can use observed timing residuals, ***s***, to infer the coefficients, ***a***, of the background. The likelihood takes the form

$$p(s|F,\vec{a}) \propto \exp\left[-\frac{1}{2}\left(\vec{s}-H\vec{a}\right)^{\dagger} F^{-1}\left(\vec{s}-H\vec{a}\right)\right]$$

- At a given frequency we make only $2N_p$ measurements - an amplitude and phase for each of the N_p pulsars. Can only hope to recover N_p combinations of the (complex) $a^G_{(lm)}$'s.

- This shows up in a singular-value decomposition of H

$$H = U\Sigma V^{\dagger}$$

- The rectangular matrix Σ has at most N_p non-zero elements on the diagonal.

- We can write $U = [H_{\rm range} H_{\rm null}]$ where the N_p columns of $H_{\rm range}$ span the range of H.

Background mapping - GR modes

- In a search we can replace $H\vec{a}$ by $H_{\rm range}\vec{b}$ in the likelihood. The value of $\vec{a}$ corresponding to a given value of $\vec{b}$ is given in terms of the pseudo-inverse of Σ, Σ^+, by $\vec{a} = V\Sigma^+\vec{b}$.
- A similar analysis can be performed in a real space representation (Cornish & van Haasteren 2014).
- Which components do we expect to be able to measure? Since

$$R^G_{I(lm)} \sim \frac{1}{l^{\frac{3}{2}}} \qquad \text{as } l \to \infty$$

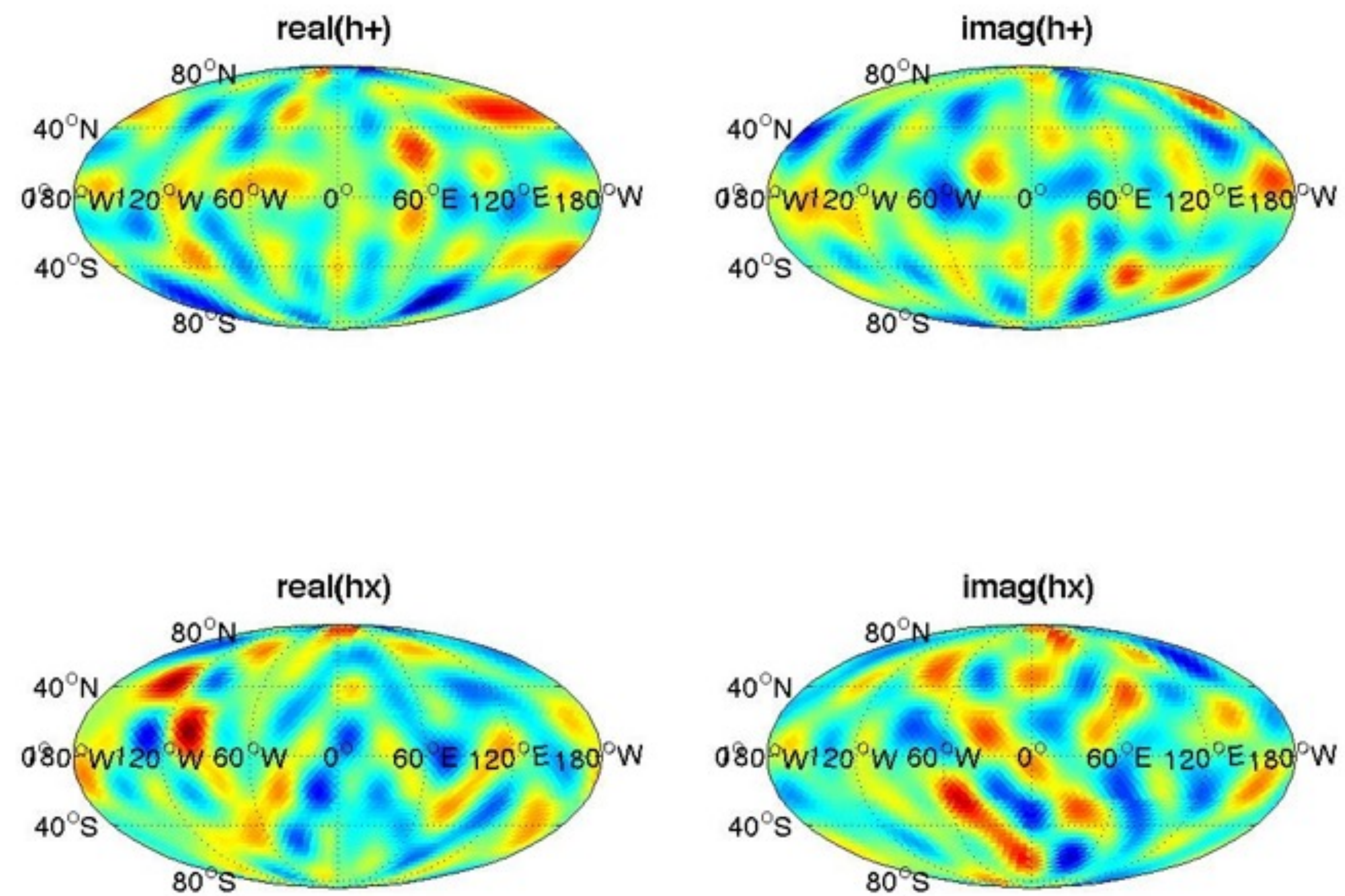
- we expect to measure the low-l modes more precisely. To reach an angular resolution of $l_{\rm max}$ we therefore need an array of

$$N_p \approx (l_{\rm max}+1)^2 - 4$$

- Need $N_p \approx 21$ pulsars to reach $l_{\rm max}$=4 required for an isotropic background; $N_p \approx 100$ to reach single source resolution at $l_{\rm max}$=10.

Background mapping - GR modes

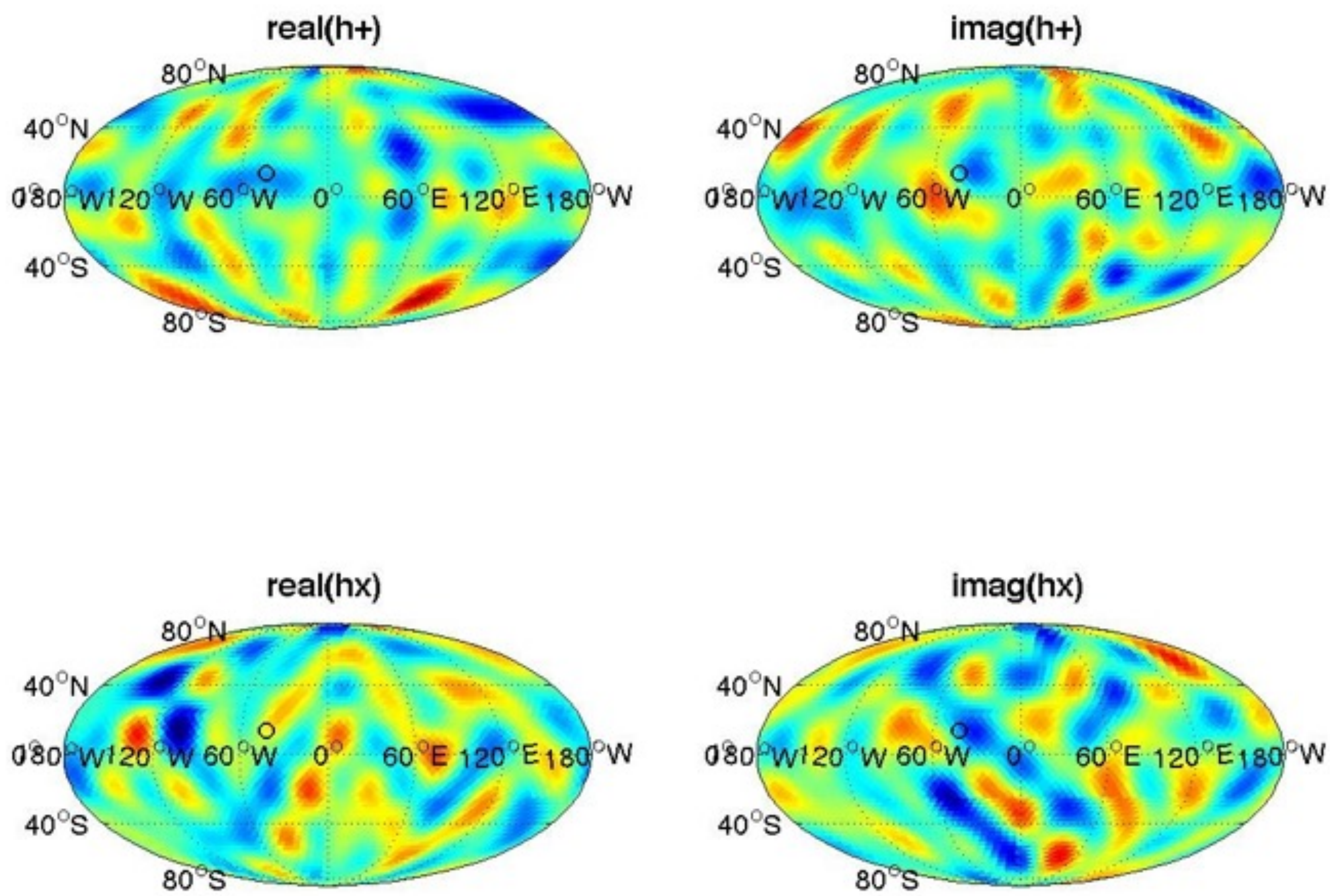
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Injected background

Background mapping - GR modes

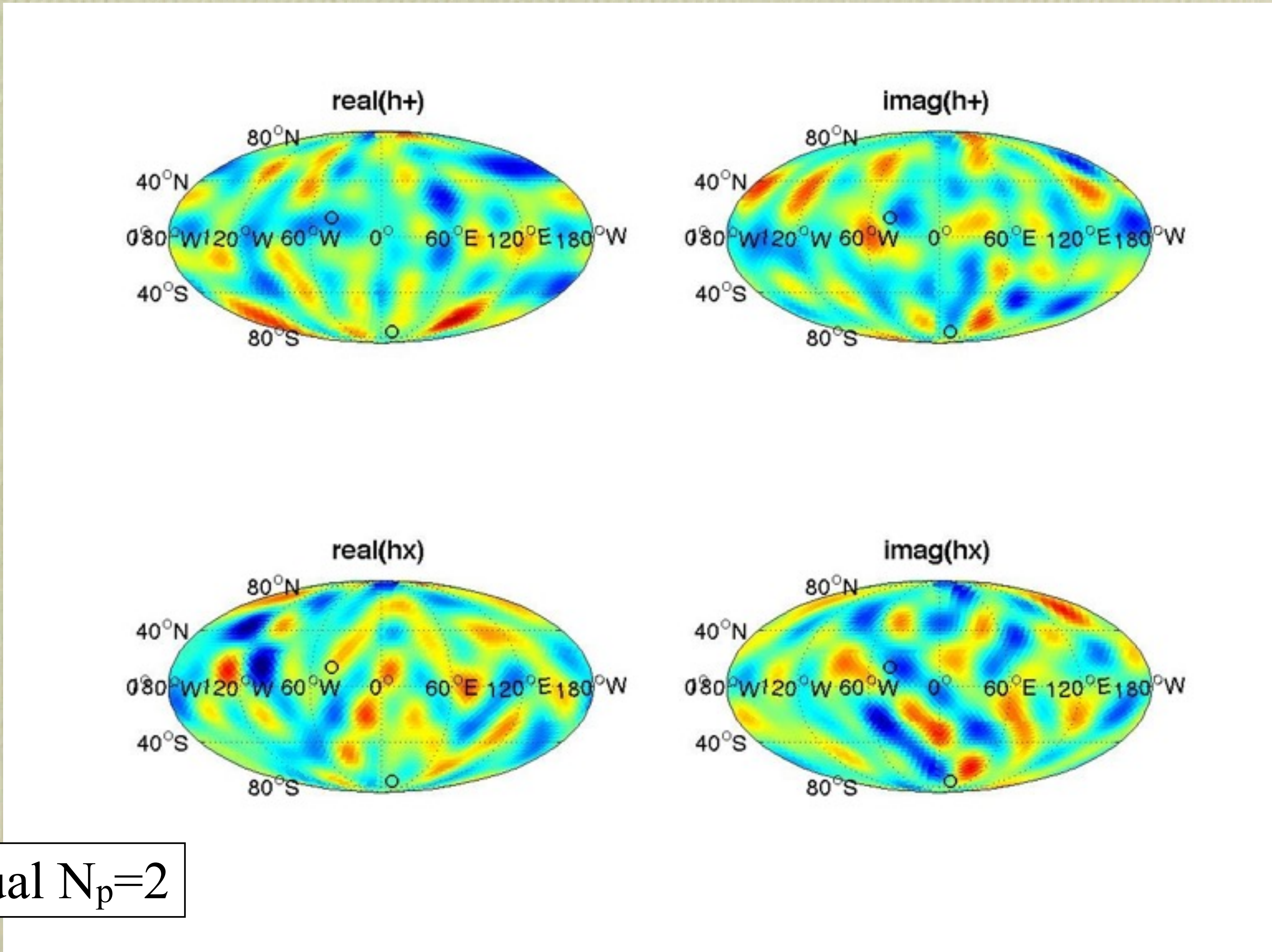
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Residual N_p=1

Background mapping - GR modes

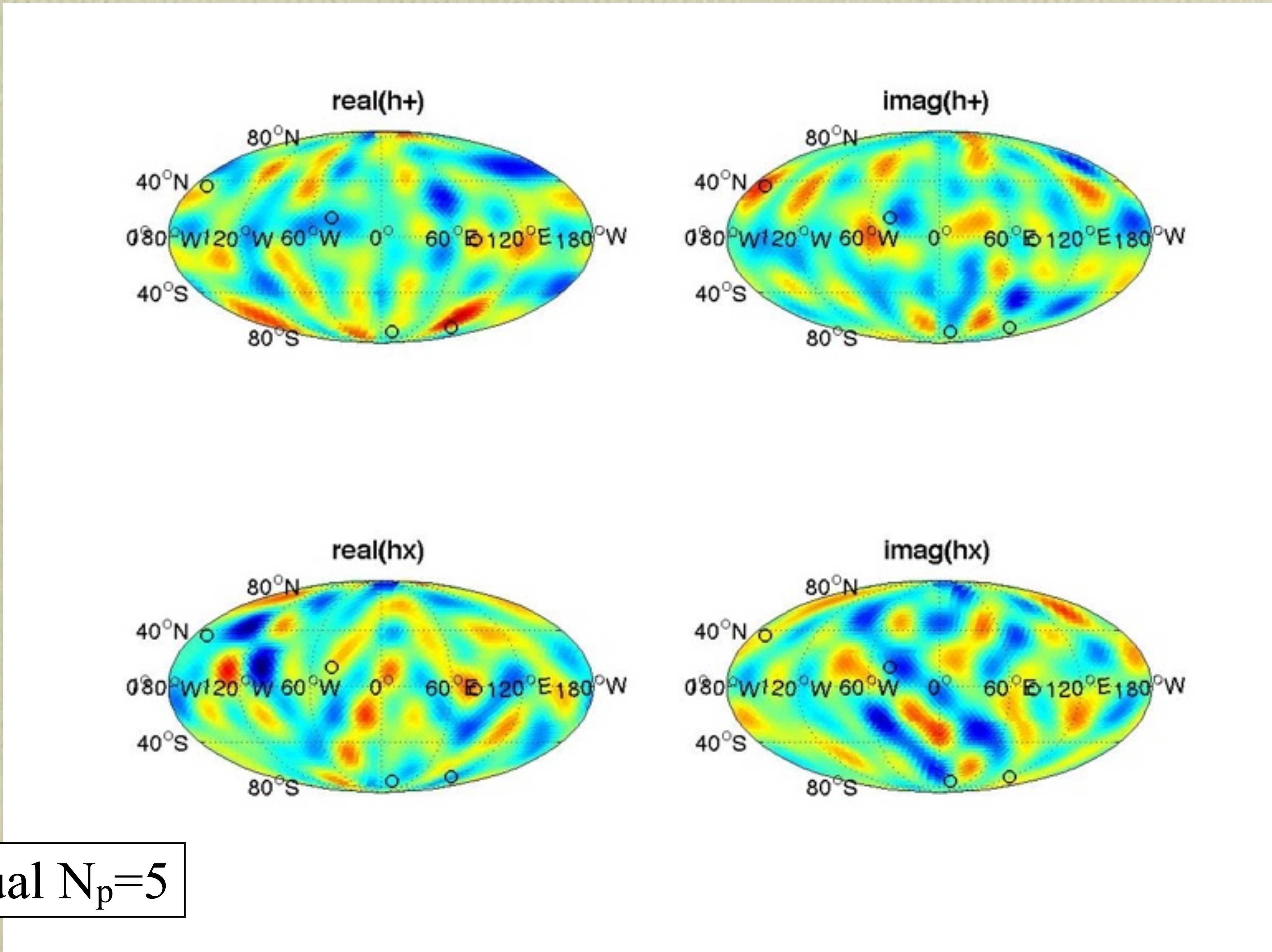
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Residual N_p=2

Background mapping - GR modes

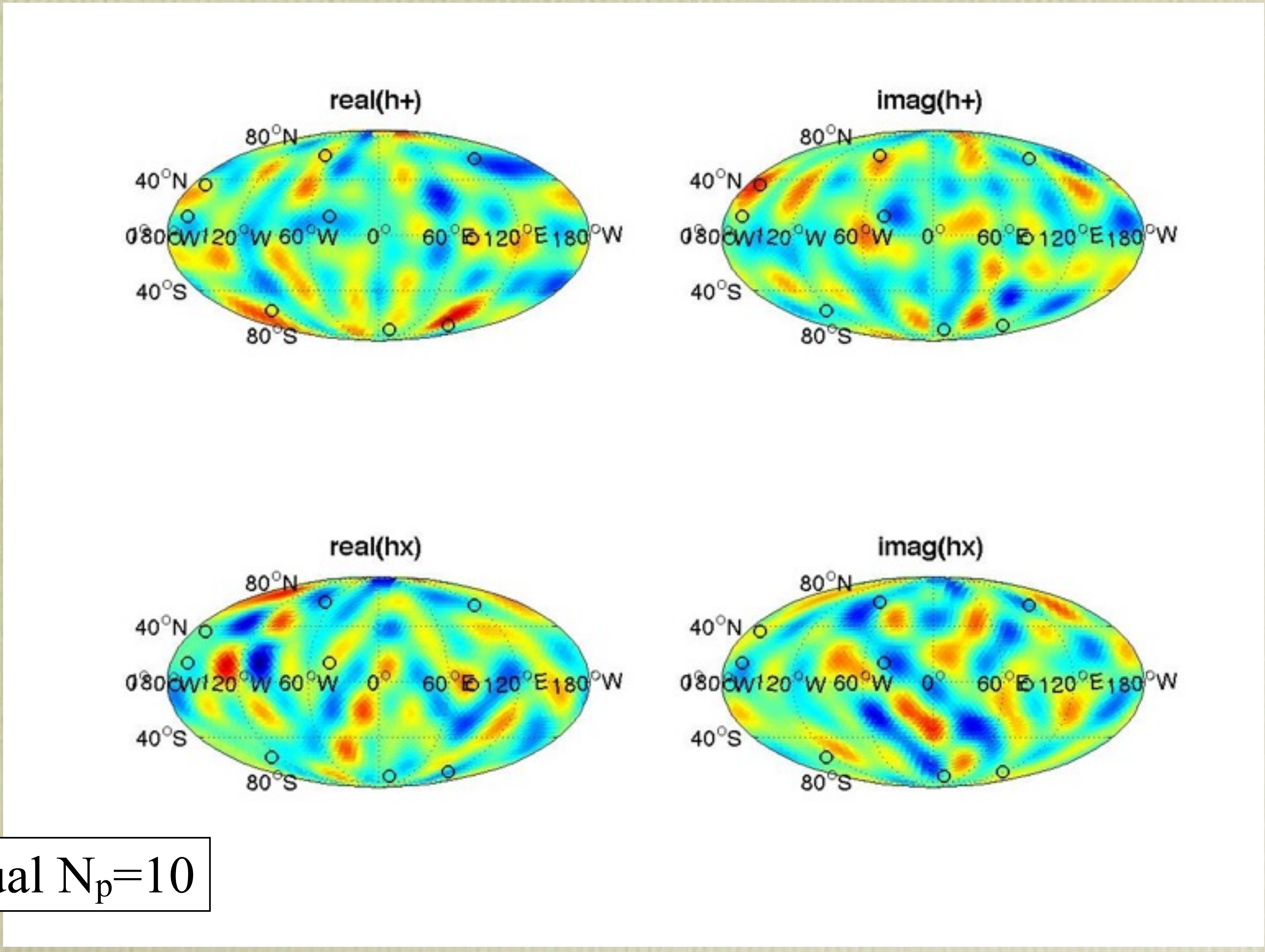
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Residual N_p=5

Background mapping - GR modes

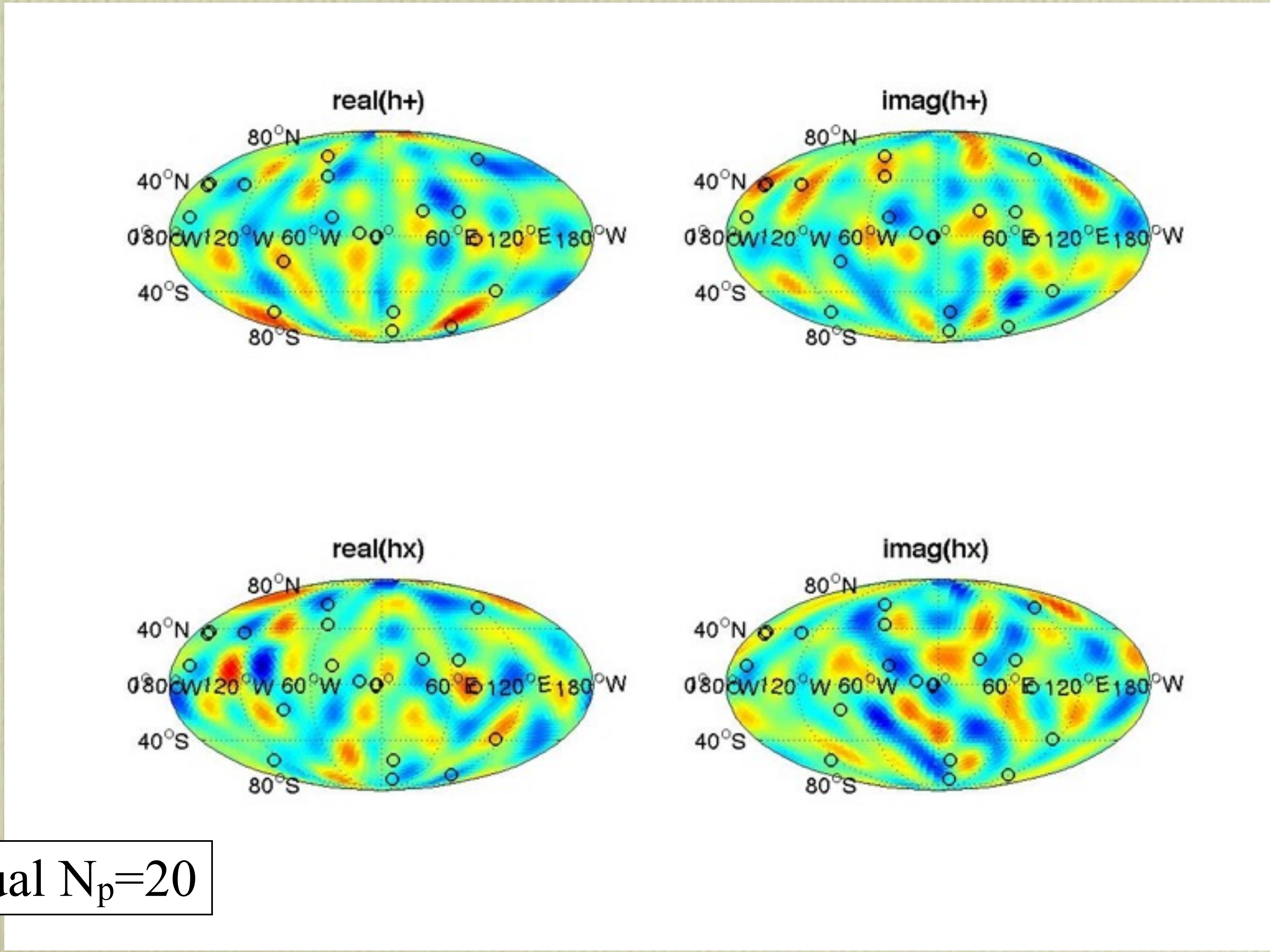
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Residual N_p=10

Background mapping - GR modes

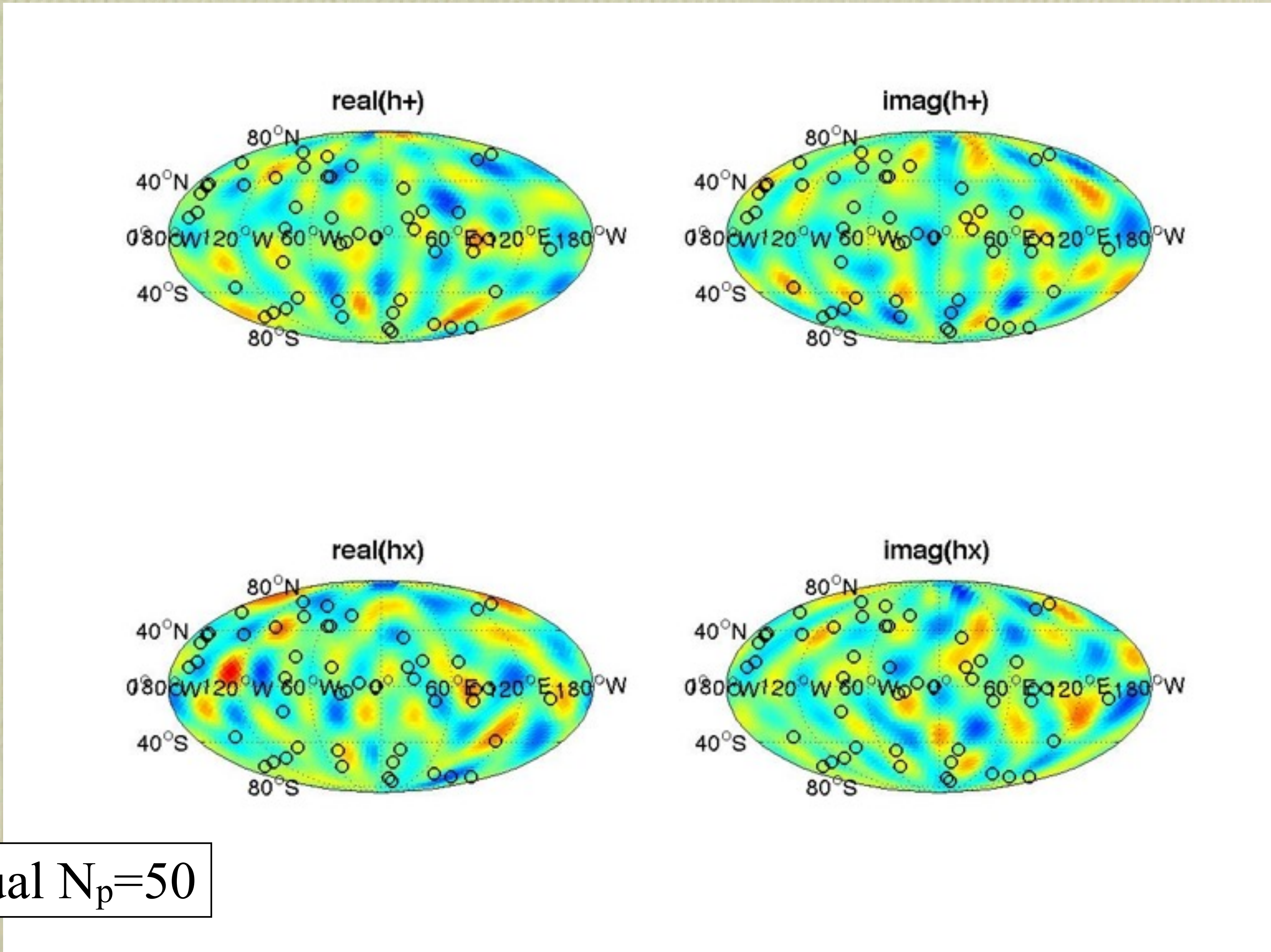
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Residual N_p=20

Background mapping - GR modes

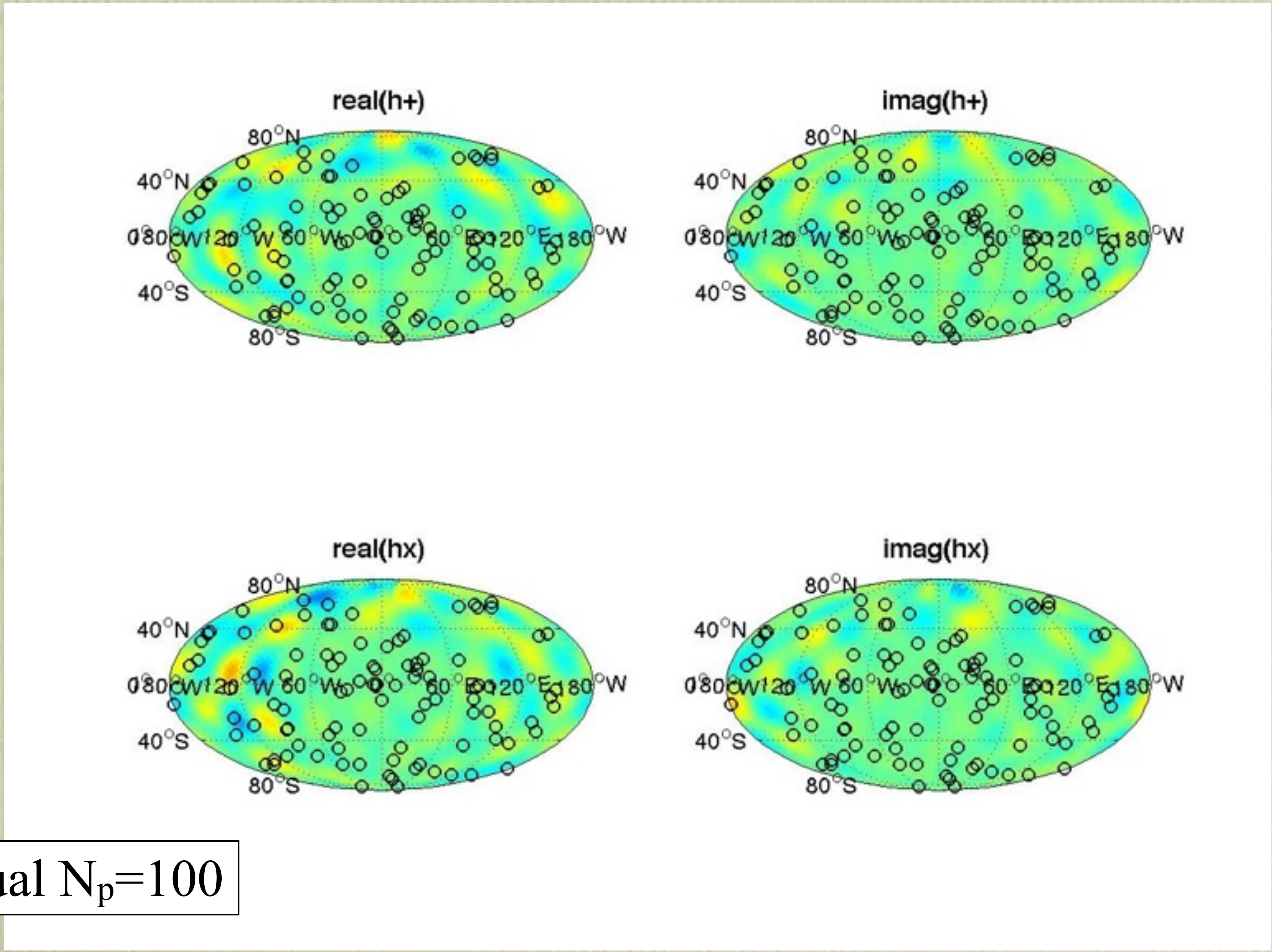
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Residual N_p=50

Background mapping - GR modes

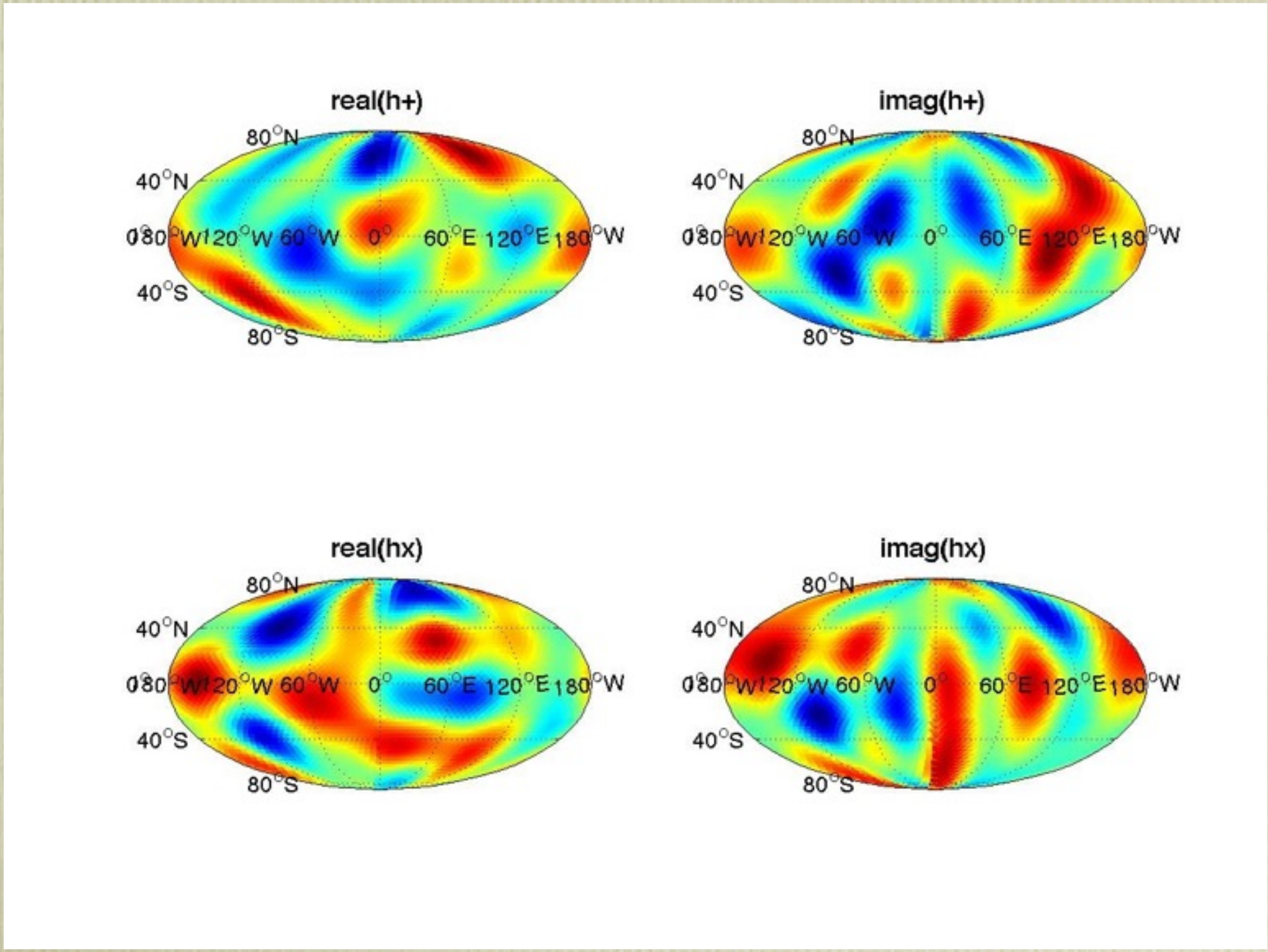
- Gradient piece of background behaves as expected. Adding more pulsars increases resolution of map and reduces residual.



Residual N_p=100

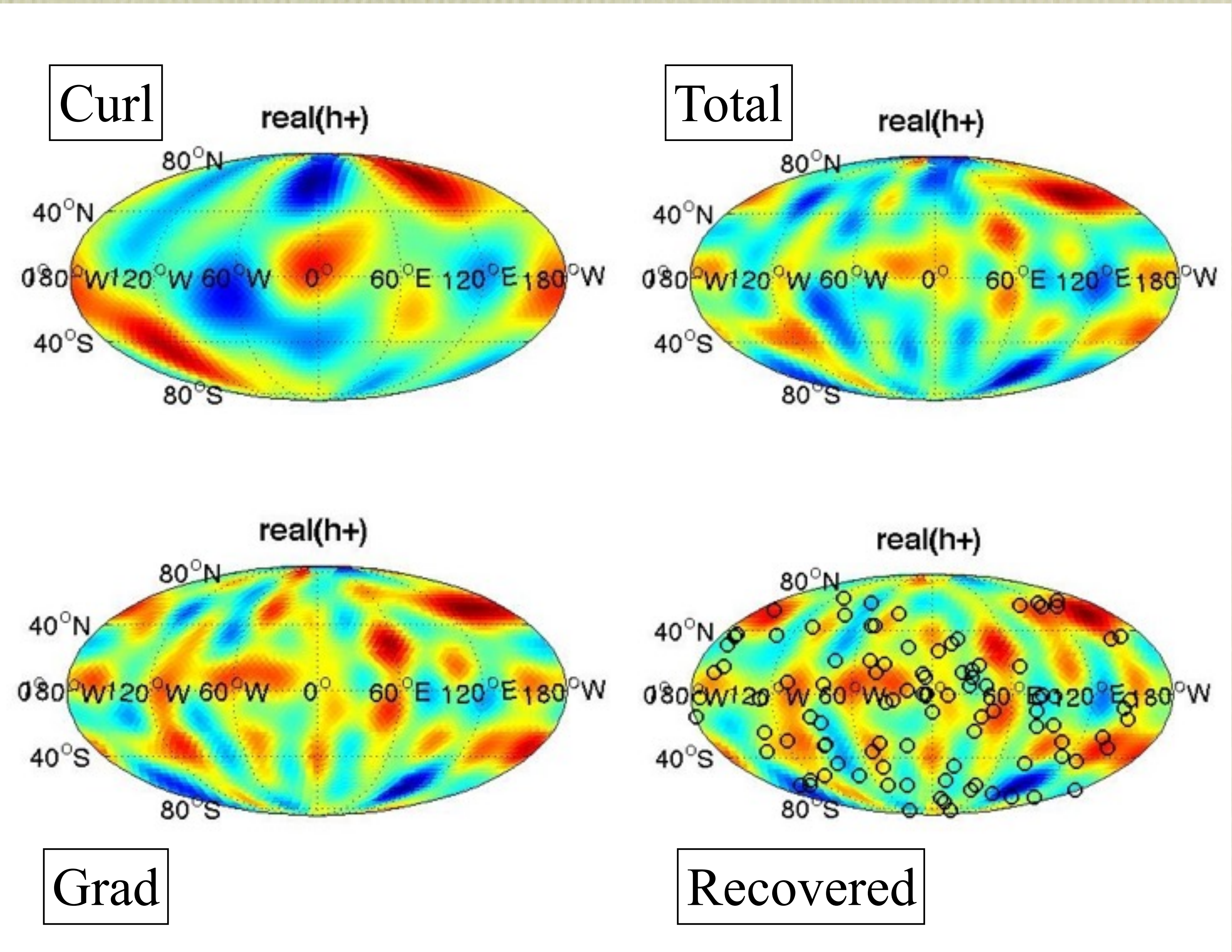
Background mapping - GR modes

- Curl part of background can never be observed.



Background mapping - GR modes

- Total GW background map could still be missing a significant component.

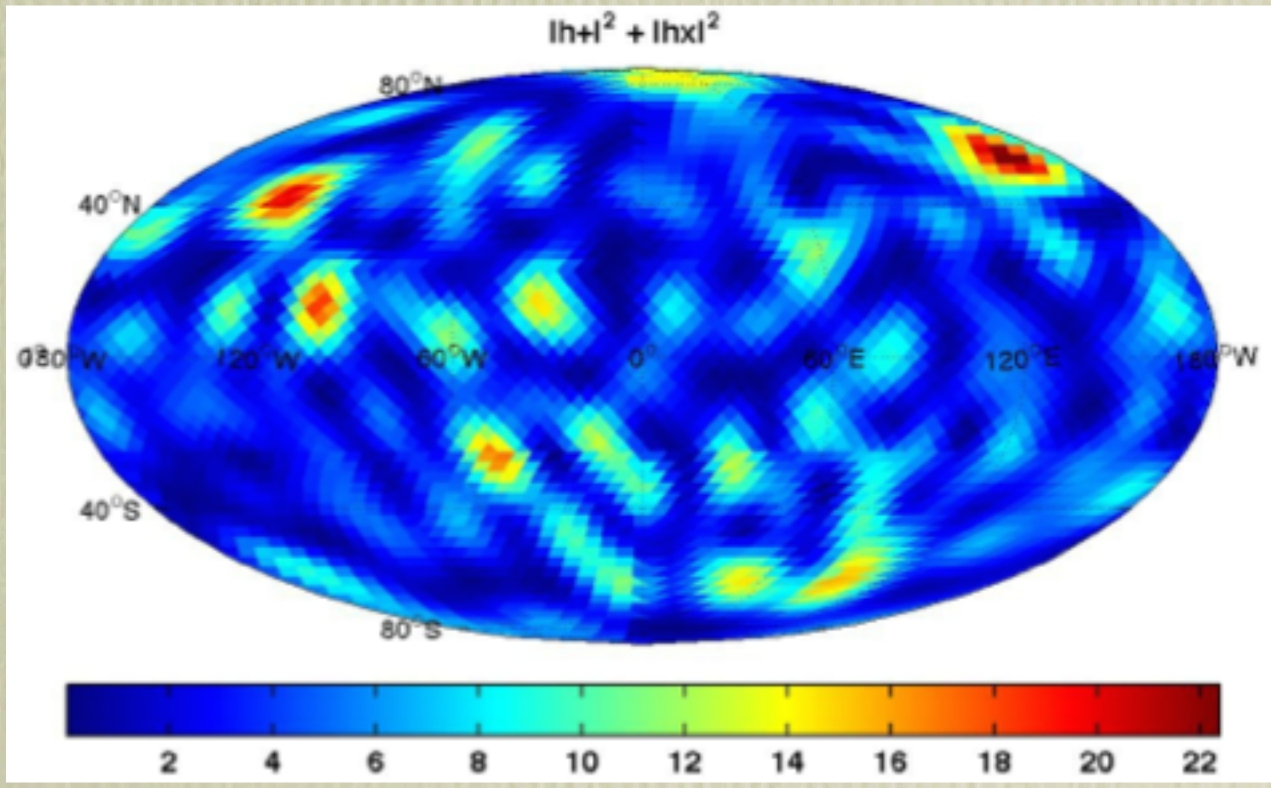


Implications

- Individual modes of the background represent GW emission that is correlated between different points on the sky.

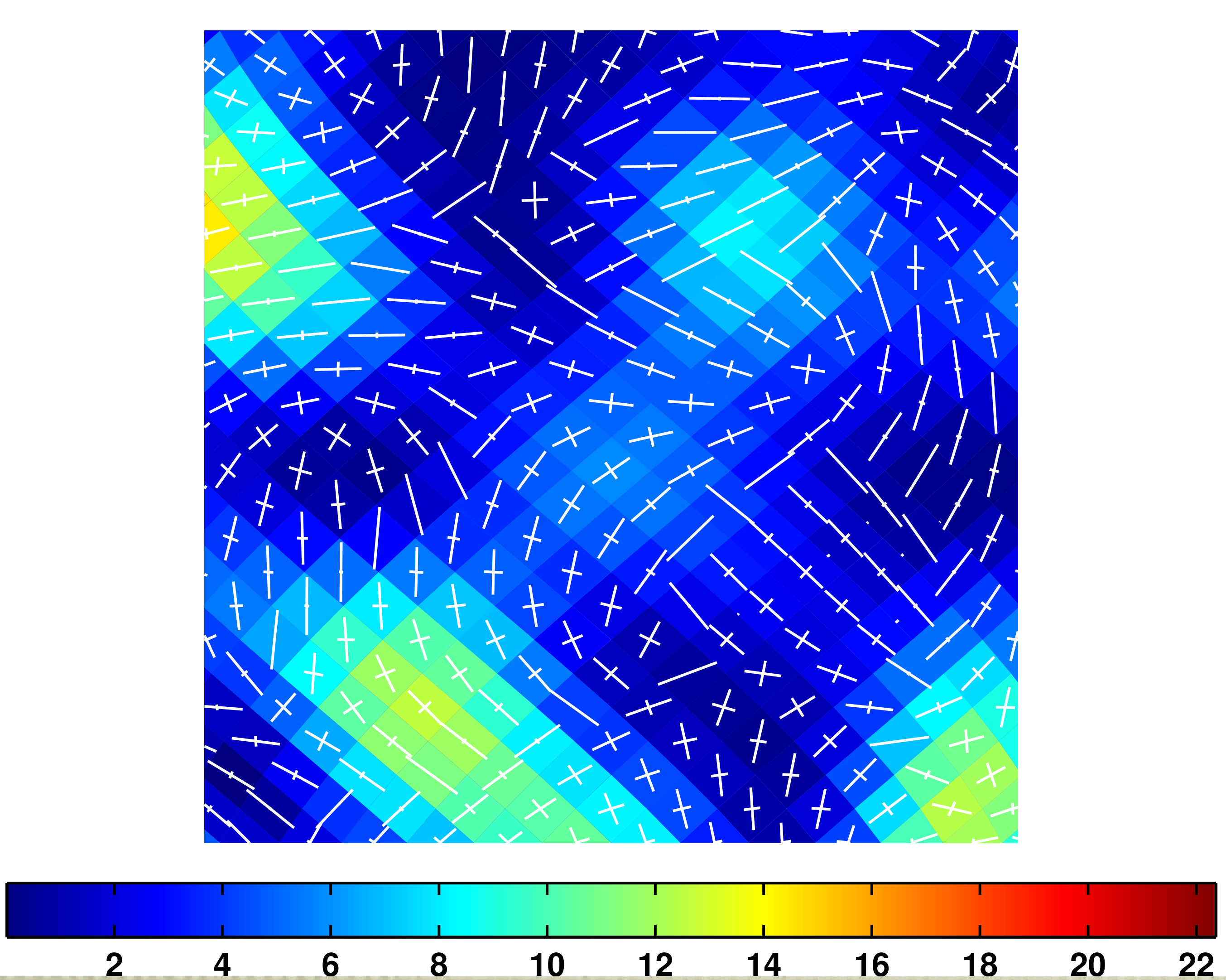
$$\langle h_+(f,\hat{k})\ h_+^*(f',\hat{k}')\rangle_k = \frac{1}{2}\sum_{l=2}^{\infty}\frac{2l+1}{4\pi}(N_l)^2\left[C_l^{GG}(f)G_{l2}^{+}(\cos\theta) + C_l^{CC}(f)G_{l2}^{-}(\cos\theta)\right]\delta(f-f')$$

- No well-established physical mechanism to create such correlations - discovery of a correlated background would be a profound result.
- Mild anisotropy expected in power of GW background - could be consistent with either uncorrelated or correlated background.

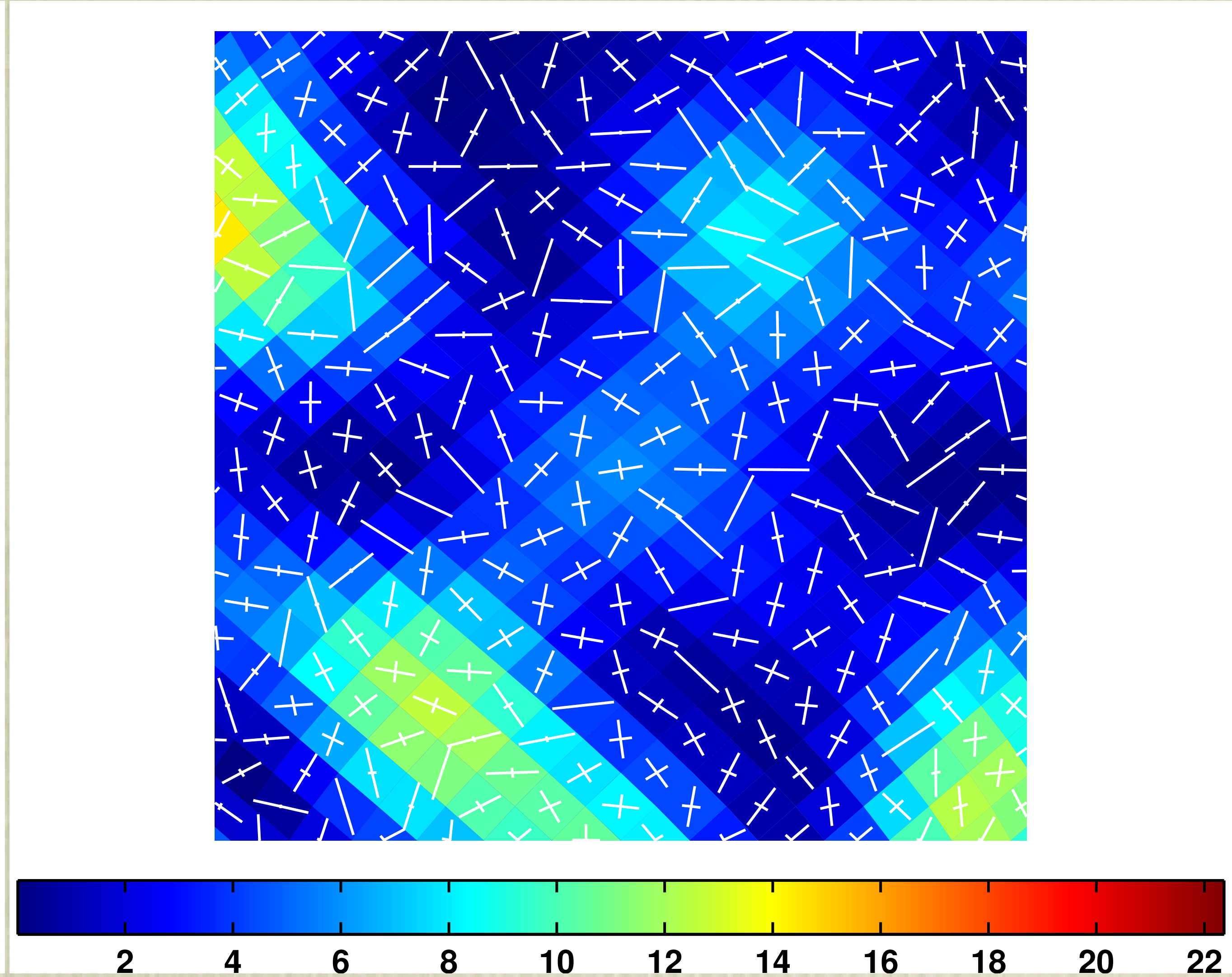


Implications

- Polarization of background can distinguish correlated and uncorrelated origin.



Correlated



Uncorrelated

Background mapping - all modes

- If we allow for alternative polarisations, the number of GR modes we can measure is reduced further.
- The total response of a pulsar in direction $\hat{u}_I$ is

$$R_I(f) = \sum_{lm} \Big(a^B_{(lm)}(f)\mathcal{R}^B_l(y_I) + a^L_{(lm)}(f)\mathcal{R}^L_l(y_I) + a^{V_G}_{(lm)}(f)\mathcal{R}^{V_G}_l(y_I) + a^G_{(lm)}(f)\mathcal{R}^G_l(y_I) \Big) Y_{lm}(\hat{u}_I)$$

- If we have pulsars all over the sky, can decompose "pulsar response" map into spherical harmonic basis. Coefficients are linear combinations of different polarisations.
- No confusion between B and G modes due to range of *l*. Confusion with V_G and L possible unless have pulsars at several distances, i.e., several *y*'s.
- Even with multiple pulsar distances, we expect great confusion between V_G and other modes, due to weaker *y* dependence.

Background mapping - all modes

- Fisher matrix analysis predicts precision with which coefficients will be measured. E.g., analysis with $l_{max} = 2$ and $N_p = 30$.

	(l, m) mode								
	$(0,0)$	$(1,-1)$	$(1,0)$	$(1,1)$	$(2,-2)$	$(2,-1)$	$(2,0)$	$(2,1)$	$(2,2)$
G: transverse-tensor (gradient)	–	–	–	–	0.44	0.38	0.32	0.38	0.44
G: transverse-tensor (gradient)	–	–	–	–	0.49	0.39	0.37	0.39	0.49
B: scalar-transverse (breathing)	0.16	0.53	0.46	0.53	–	–	–	–	–
G: transverse-tensor (gradient)	–	–	–	–	16.2	10.5	11.4	10.5	16.2
B: scalar-transverse (breathing)	4.36	16.1	14.1	16.1	–	–	–	–	–
L: scalar-longitudinal	0.71	0.96	0.84	0.96	1.21	0.78	0.86	0.78	1.21
G: transverse-tensor (gradient)	–	–	–	–	1.4e5	5.4e4	8.0e4	5.4e4	1.4e5
B: scalar-transverse (breathing)	18.4	9.4e4	6.2e4	9.4e4	–	–	–	–	–
L: scalar-longitudinal	3.08	11.5	8.68	11.5	20.9	7.51	11.9	7.52	20.9
V_G: vector-longitudinal (gradient)	–	6.6e4	4.4e4	6.6e4	7.0e4	2.7e4	4.0e4	2.7e4	7.0e4

- Extension to ground-based interferometers.

Extension to interferometers

- Can apply the same formalism to other GW detectors. Consider ground-based interferometers and make static approximation.
- The strain response of a static interferometer in the point detector limit may be approximated by

$$R^A(f,k) = \frac{1}{2} e^A_{ab}(\hat{k}) \left(u_1^a u_1^b - u_2^a u_2^b \right)$$

- Using separate integration frames for each arm, such that the arm is in the *z*-direction, and using the rotation properties of the $a^P_{(lm)}$ coefficients we find

$$R^G_{(lm)}(f) = \frac{4\pi}{5}\sqrt{\frac{1}{3}}\,\delta_{l,2}\left(Y_{2m}(\theta_1,\phi_1) - Y_{2m}(\theta_2,\phi_2)\right)$$
$$R^C_{(lm)}(f) = 0$$

- for a detector with arms pointing in the directions (θ_i, ϕ_i).

Extension to interferometers

- Including transfer function, still have zero curl mode response, but sensitivity to grad modes with $l > 2$ is recovered.
- A moving detector recovers curl mode sensitivity since

$$\Delta(f\hat{k}\cdot\vec{x}/c) \sim 5\times 10^3 \to 5\times 10^5$$

- over a year for $f = 10 \to 1000\text{Hz}$.
- Regard moving detector as superposition of static detectors at different locations.
- Under a translation to a frame with origin at $\vec{x}_0$

$$h_{ab}(f,\hat{k}) \to \bar{h}_{ab}(f,\hat{k}) = h_{ab}(f,\hat{k})\mathrm{e}^{-i2\pi f\hat{k}\cdot\hat{x}_0/c}$$

Extension to interferometers

- Using the identity

$$e^{-i2\pi f\hat{k}\cdot\vec{x}_0/c} = 4\pi\sum_{L=0}^{\infty}(-i)^L j_L(\alpha)\sum_{M=-L}^{L} Y^*_{LM}(\hat{x}_0)Y_{LM}(\hat{k})$$

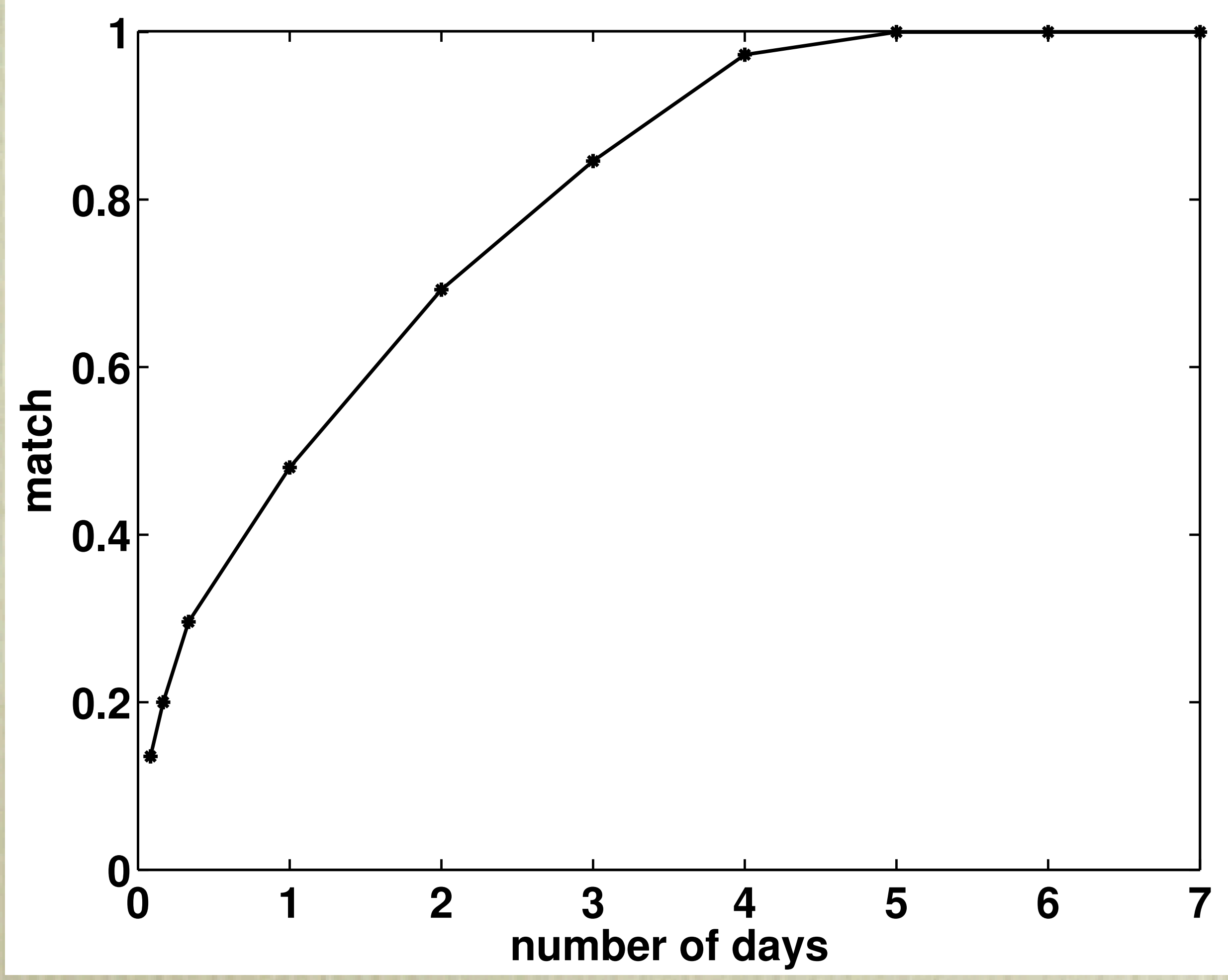
- where $\alpha \equiv 2\pi f|\vec{x}_0|/c$, we can transform the components of the background in the cosmic frame into the frame of the detector at $\hat{x}_0$

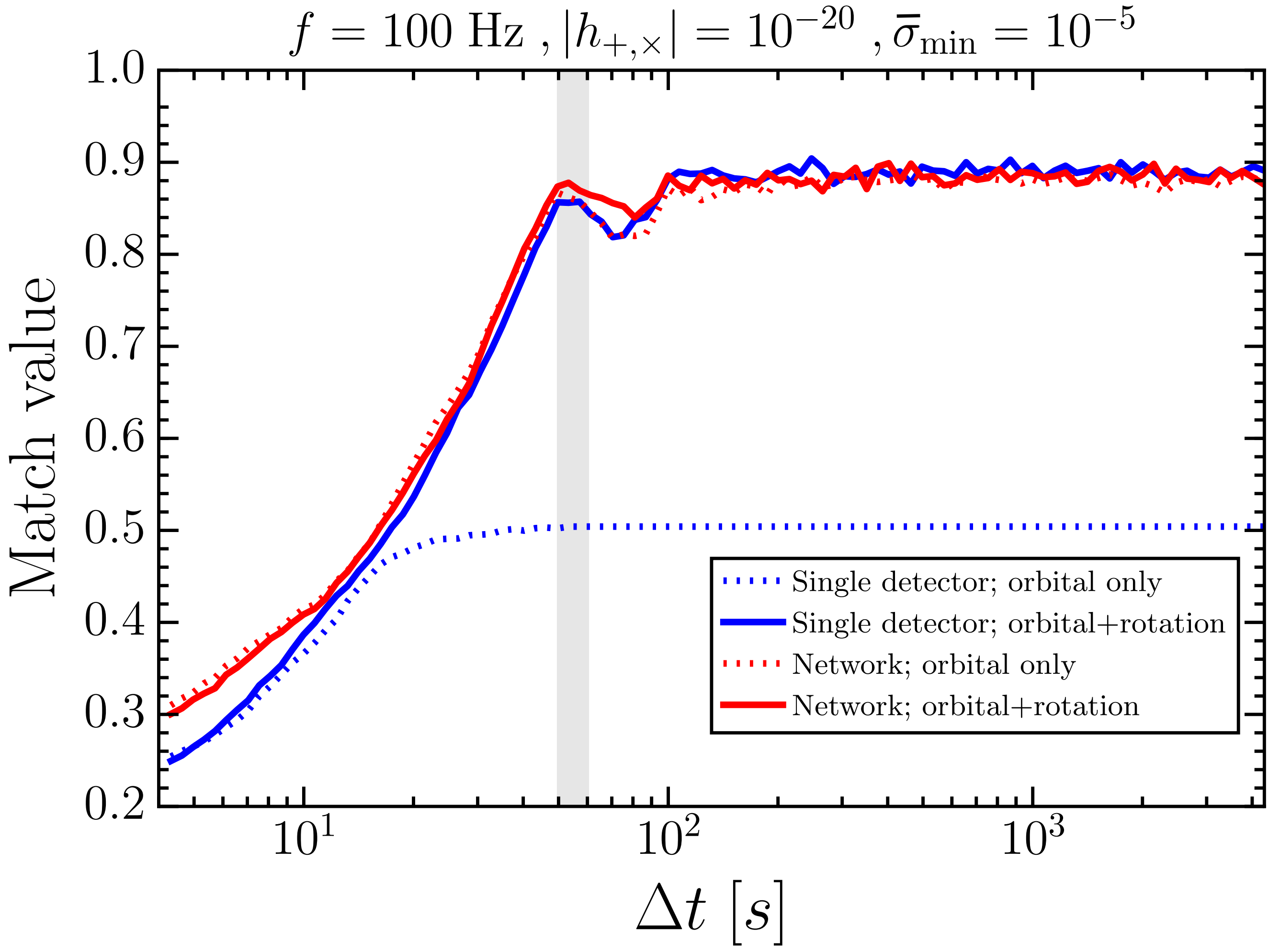
$$R^G_{(lm)}(f) = \sum_{m'=-2}^{2}\sum_{L=l-2}^{l+2}\sum_{M=-L}^{L}\bar{R}^G_{(2m')}4\pi(-i)^L j_L(\alpha)Y^*_{LM}(\hat{x}_0)$$
$$\frac{(-1)^{m'}}{2}\sqrt{\frac{(2\cdot 2+1)(2l+1)(2L+1)}{4\pi}}\begin{pmatrix} 2 & l & L \\ -m' & m & M\end{pmatrix}\begin{pmatrix} 2 & l & L \\ 2 & -2 & 0\end{pmatrix}\left[(-1)^{l+L}+1\right]$$

$$R^C_{(lm)}(f) = \sum_{m'=-2}^{2}\sum_{L=l-2}^{l+2}\sum_{M=-L}^{L}\bar{R}^G_{(2m')}4\pi(-i)^L j_L(\alpha)Y^*_{LM}(\hat{x}_0)$$
$$\frac{(-1)^{m'}}{2i}\sqrt{\frac{(2\cdot 2+1)(2l+1)(2L+1)}{4\pi}}\begin{pmatrix} 2 & l & L \\ -m' & m & M\end{pmatrix}\begin{pmatrix} 2 & l & L \\ 2 & -2 & 0\end{pmatrix}\left[(-1)^{l+L}-1\right]$$

Extension to interferometers

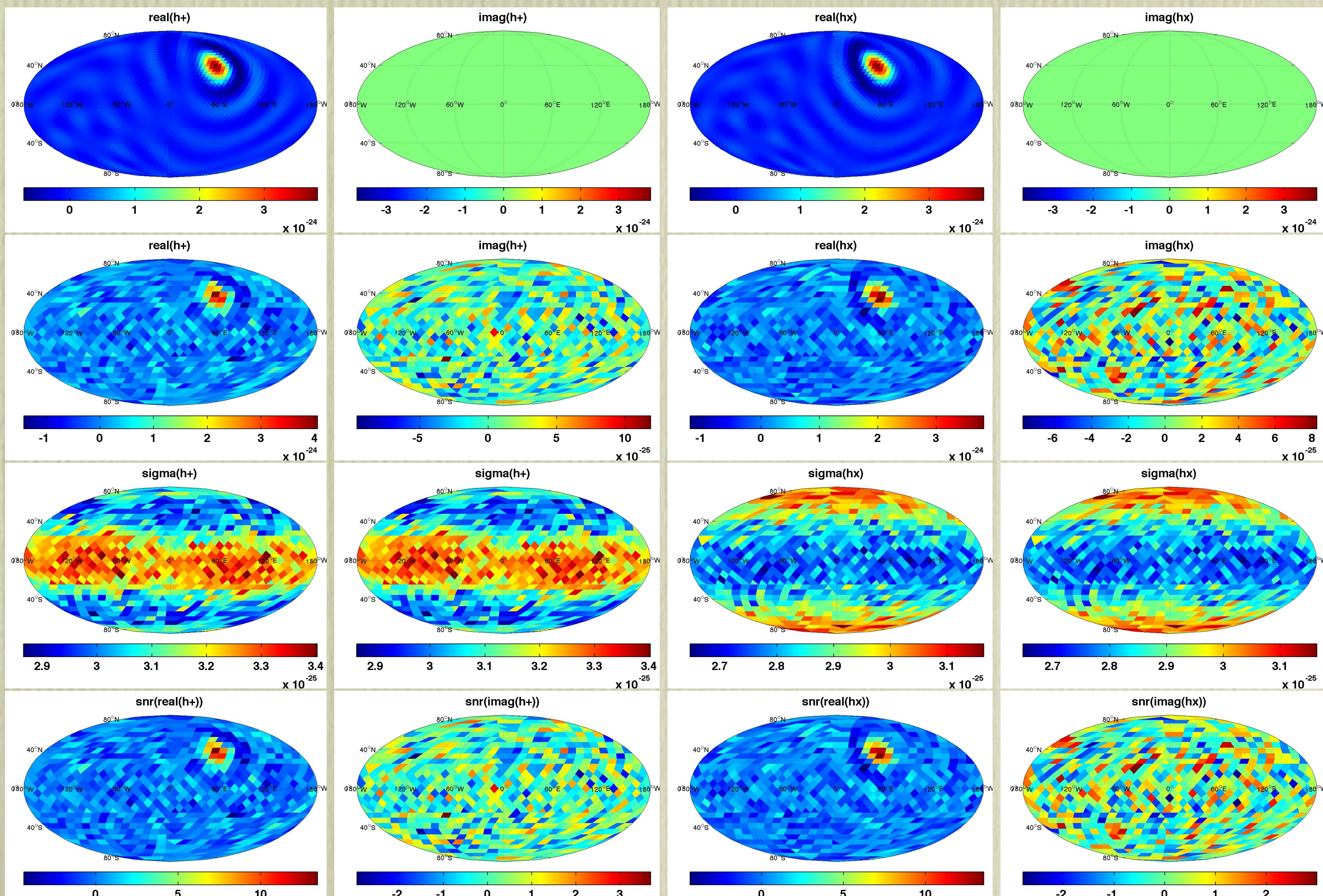
- Recover more and more modes as number of effective independent detectors increases over time.
- Earth rotation crucial to break degeneracies for a single detector network.





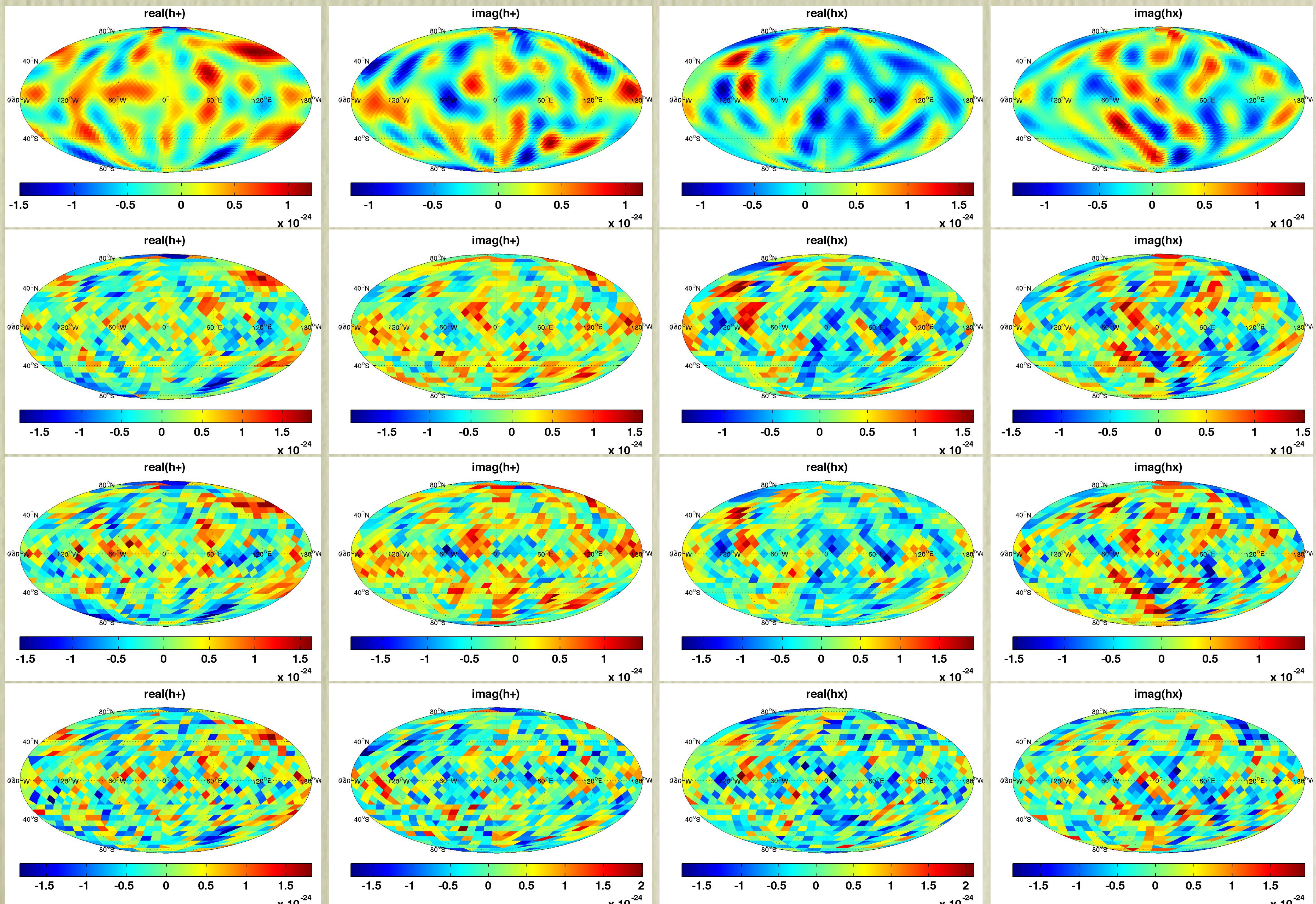
Extension to interferometers

- Recovery of point source.



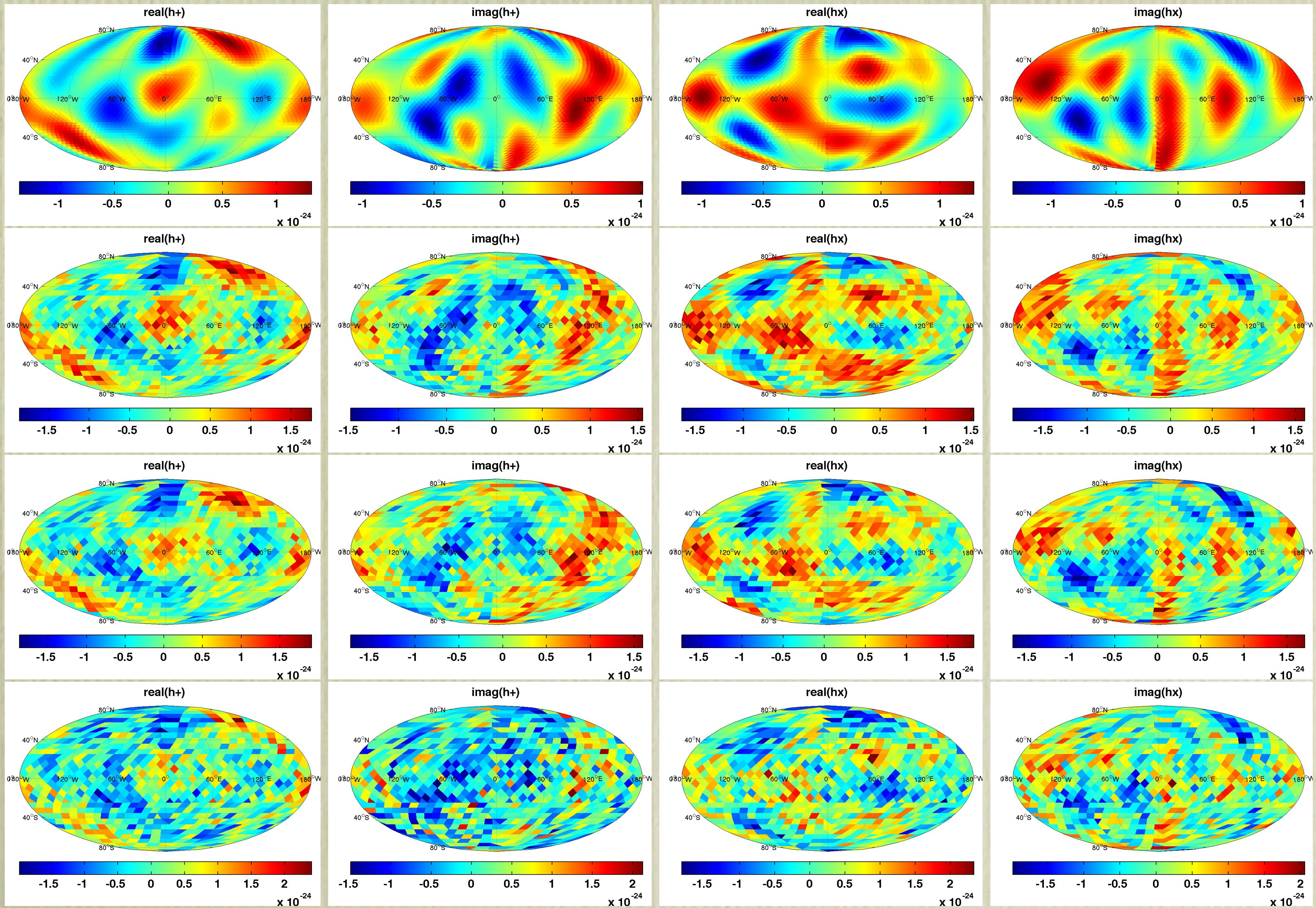
Extension to interferometers

- Recovery of grad mode GR background.



Extension to interferometers

- Recovery of curl mode GR background.



Summary

- The framework used to analyse CMB polarisation can be applied to describe arbitrary gravitational wave backgrounds.
- PTA response to modes of the background takes a simple form - spherical harmonics evaluated at pulsar locations, multiplied by distance-dependent factors.
- PTAs have no sensitivity to the curl components of the background or to modes higher than dipole in scalar-transverse backgrounds.
- Can describe an isotropic uncorrelated background with just three *l*-modes.
- A PTA of N_p pulsars can measure N_p combinations of the grad component of the background. PTAs are blind to the other grad components and the whole curl component.
- A measurement of unexpected values for these components would reveal correlations in the background and profound new physics.