Large scale structure formation

with the

Schrödinger method

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Outline

1. Structure formation
2. Analytical description of cold dark matter

   a. dust model

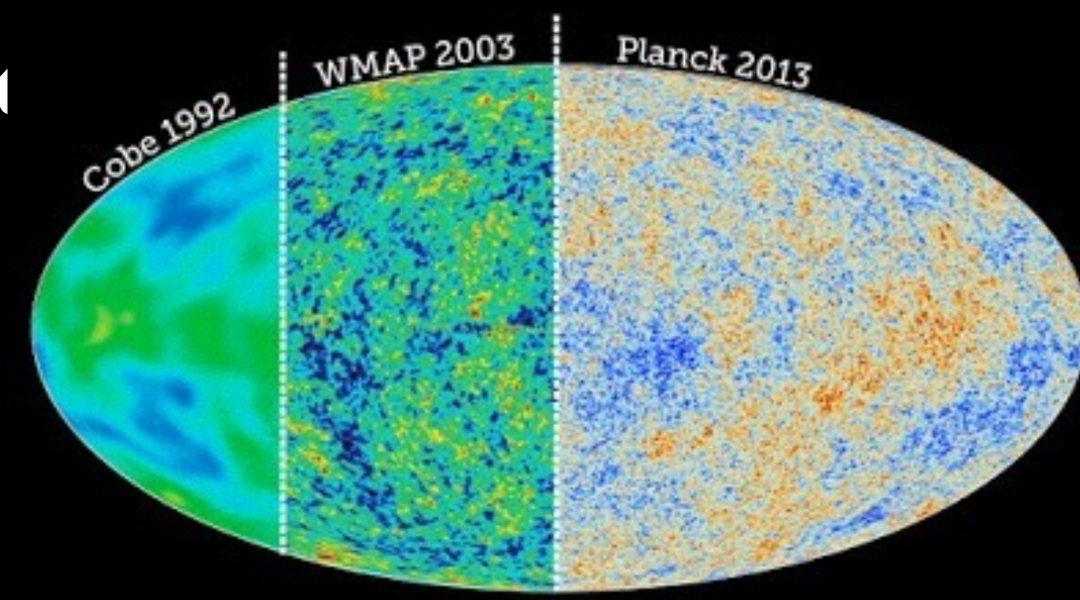
   b. Schrödinger method

   c. coarse-grained dust model

3. Summary

Cosmological Structure Formation

-13.8 billion years: nearly uniform initial stat



Inflation

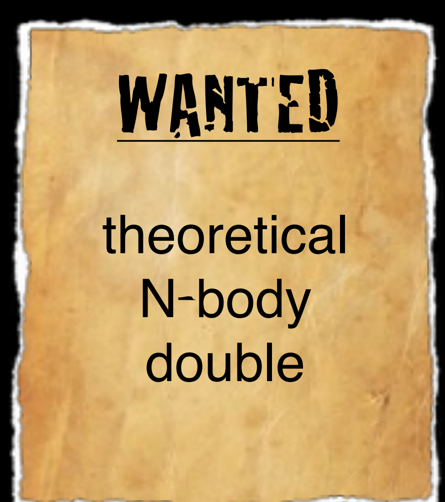
- established `boring` initial conditions
  - quantum fluctuations get amplified
  - primordial plasma cools → recombination → CMB

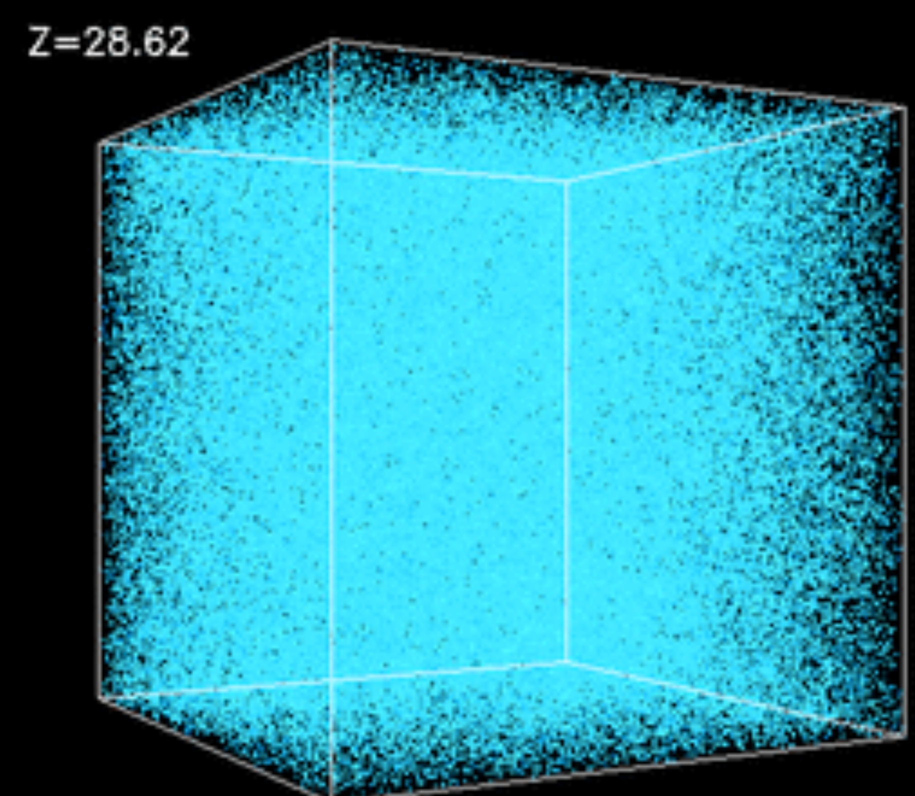
Structure formation

- hierarchical
- tiny over-densities act as seeds
  - congregation via gravitational instability
  - collapse into bound structures

Large scale structure: Dark Matter

- linear regime
  - ✓ analytically understood
- nonlinear stage
  - ?! N-body simulations inevitable





today: rich structures in cosmic web

Kravtsov & Klypin (simulations @NCSA)

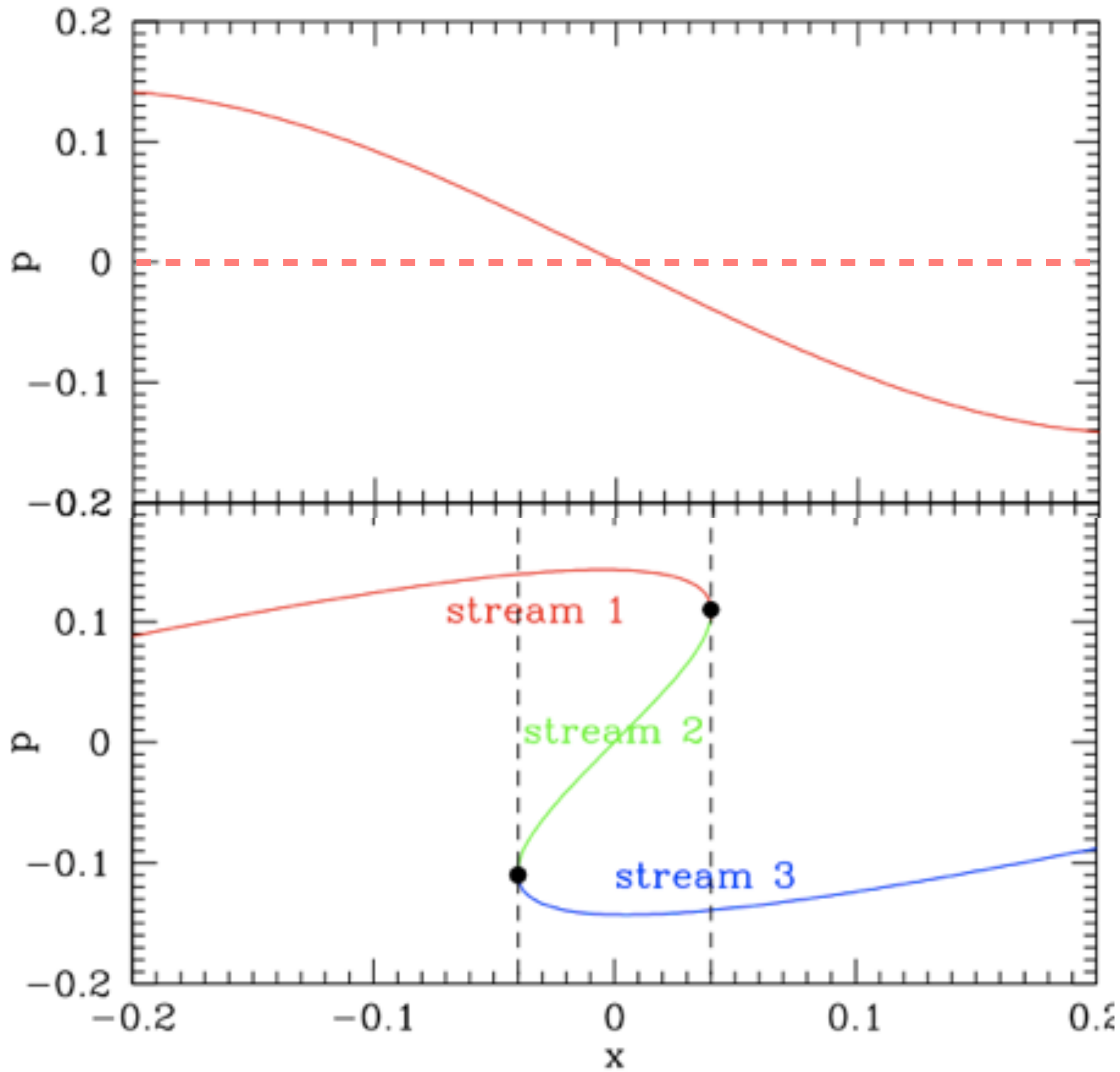
Describing Cold Dark Matter with the Schrödinger method

Describing Cold Dark Matter

phase space distribution function f(t,x,p)

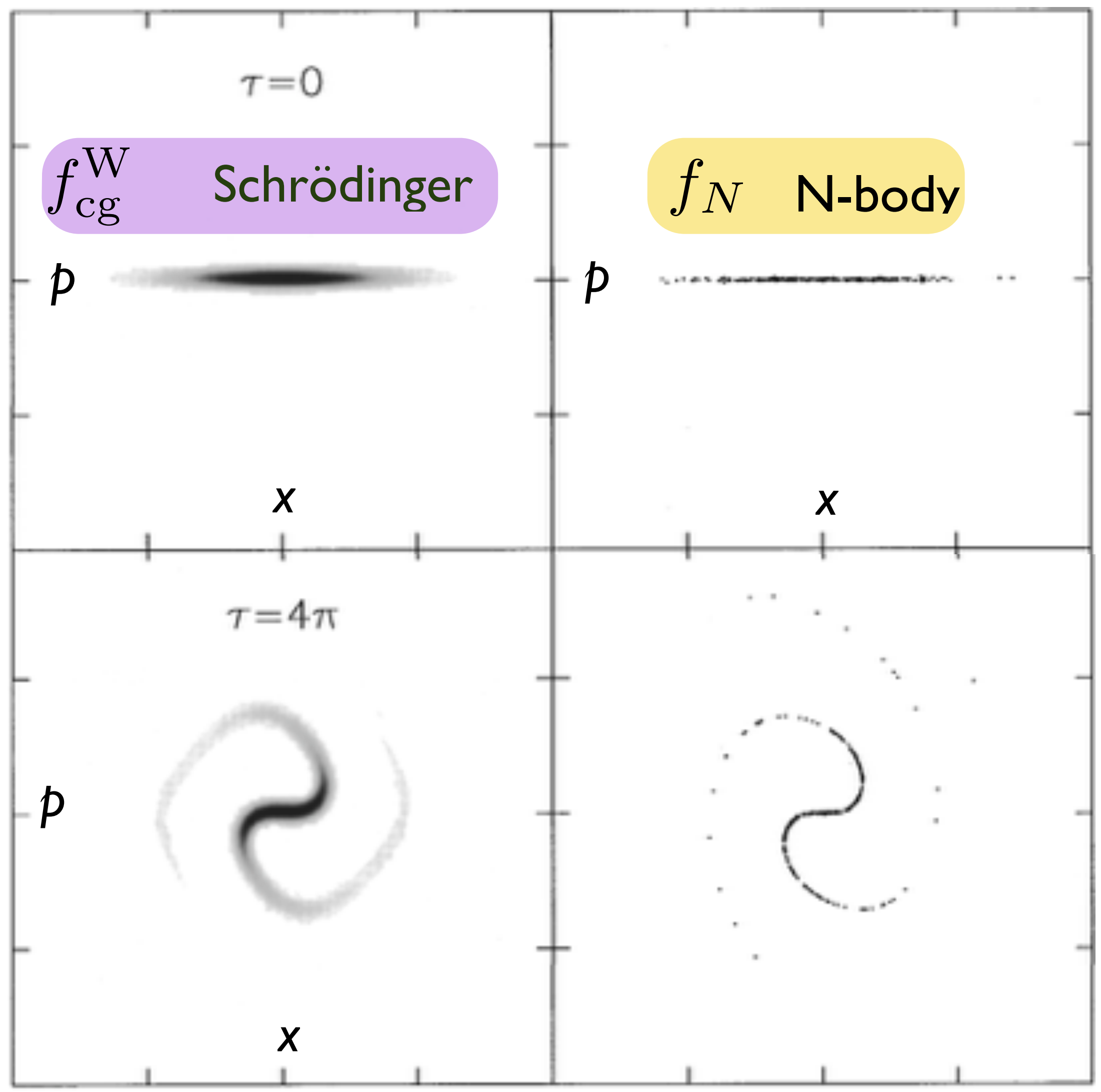
- describes number density & distribution of momenta *p*

theoretical expectation



Pueblas & Scoccimarro (2009, PRD 80)

numerical realization



Schrödinger method: Widrow & Kaiser (1993, ApJ 416)
Widrow (1997, PRD 55)

Describing Cold Dark Matter

N-body picture

- N non-relativistic particles
- only gravitational interaction

$$f_N(\boldsymbol{x}, \boldsymbol{p}, \tau) = \sum_{i=1}^{N} \delta_{\mathrm{D}}(\boldsymbol{x} - \boldsymbol{x}_i(\tau))\delta_{\mathrm{D}}(\boldsymbol{p} - \boldsymbol{p}_i(\tau))$$

Vlasov - Poisson equation

$$\partial_t f_N(\boldsymbol{x}, \boldsymbol{p}, \tau) = -\frac{\boldsymbol{p}}{am}\boldsymbol{\nabla}_x f_N + am\boldsymbol{\nabla}_x V \boldsymbol{\nabla}_p f_N$$

$$\Delta V(\boldsymbol{x}_i) = \frac{4\pi G m}{a}\left(\sum_{j\neq i}^{N} \delta_{\mathrm{D}}(\boldsymbol{x}_i - \boldsymbol{x}_j) - \langle n_N \rangle\right)$$

Describing Cold Dark Matter

phase space distribution function f(t,x,p)

- **N-body**: non-relativistic, only gravitationally
- **continuous**: ensemble average, no collisions

$$f_N = \sum_i \delta_D(\boldsymbol{x} - \boldsymbol{x}_i)\delta_D(\boldsymbol{p} - \boldsymbol{p}_i)$$

$$f$$

Vlasov - Poisson equation

$$\partial_\tau f(\boldsymbol{x}, \boldsymbol{p}, \tau) = -\frac{\boldsymbol{p}}{am}\boldsymbol{\nabla}_x f + am\boldsymbol{\nabla}_x V \boldsymbol{\nabla}_p f$$

gravitational potential

$$\Delta V(\boldsymbol{x}, \tau) = \frac{4\pi G m}{a}\left(n(\boldsymbol{x}, \tau) - \langle n \rangle\right)$$

3+3+1 variables

partial

nonlinear

integro

number density

$$n = \int \mathrm{d}^3 p \ \ f$$

Solving is hard!

have to choose a special ansatz for f(x,p)

reduce some information content

Describing Cold Dark Matter

continuous distribution function f(t,x,p)

- ensemble average
- dropping collision terms ~1/N

f_N → f

Vlasov - Poisson equation

$$\partial_\tau f(\boldsymbol{x},\boldsymbol{p},\tau) = -\frac{\boldsymbol{p}}{am}\boldsymbol{\nabla}_x f + am\boldsymbol{\nabla}_x V \boldsymbol{\nabla}_p f$$

gravitational potential

$$\Delta V(\boldsymbol{x},\tau) = \frac{4\pi G m}{a}\left(n(\boldsymbol{x},\tau) - \langle n \rangle\right)$$

Hierarchy of Moments $M^{(n)}(\boldsymbol{x}) = \int d^3p \; p_{i_1}\ldots p_{i_n} f$

- density n(x): $M^{(0)} = n(\boldsymbol{x})$, velocity **v**(x): $M^{(1)} = n\boldsymbol{v}(\boldsymbol{x})$
- velocity dispersion $\boldsymbol{\sigma}$(x): $M^{(2)} = n\left(\boldsymbol{v}\boldsymbol{v} + \boldsymbol{\sigma}\right)(\boldsymbol{x}), \ldots$

cumulant

$$\partial_t M^{(n)} = -\frac{1}{a^2 m}\nabla \cdot M^{(n+1)} - m\nabla V \cdot M^{(n-1)}$$

infinite coupled hierarchy

Describing Cold Dark Matter

phase space distribution function f(t,x,p)

- **N-body**: non-relativistic, only gravitationally
- **continuous**: ensemble average, no collisions

f_N → f

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gravitational potential

$$\Delta V(\boldsymbol{x},\tau) = \frac{4\pi G m}{a}\left(n(\boldsymbol{x},\tau) - \langle n\rangle\right)$$

Hierarchy of Cumulants

- density n(x): $C^{(0)} = \ln n(\boldsymbol{x})$, velocity **v**(x): $C^{(1)} = \boldsymbol{v}(\boldsymbol{x})$
- velocity dispersion $\boldsymbol{\sigma}$(x): $C^{(2)} = \boldsymbol{\sigma}(\boldsymbol{x})$, …

$$\partial_t C^{(n)} = -\frac{1}{a^2 m}\left[\nabla\cdot C^{(n+1)} + \sum_{|S|=0}^{n} C^{(n+1-|S|)}\cdot\nabla C^{(|S|)}\right] - \delta_{n1}\cdot m\nabla V$$

consistent **truncation** $C^{(\geq n)} \equiv 0 \quad \exists n : |S| \leq 1 \wedge |S| \geq n \quad \forall S \;\Rightarrow\; n \overset{!}{=} 2$

Dust model

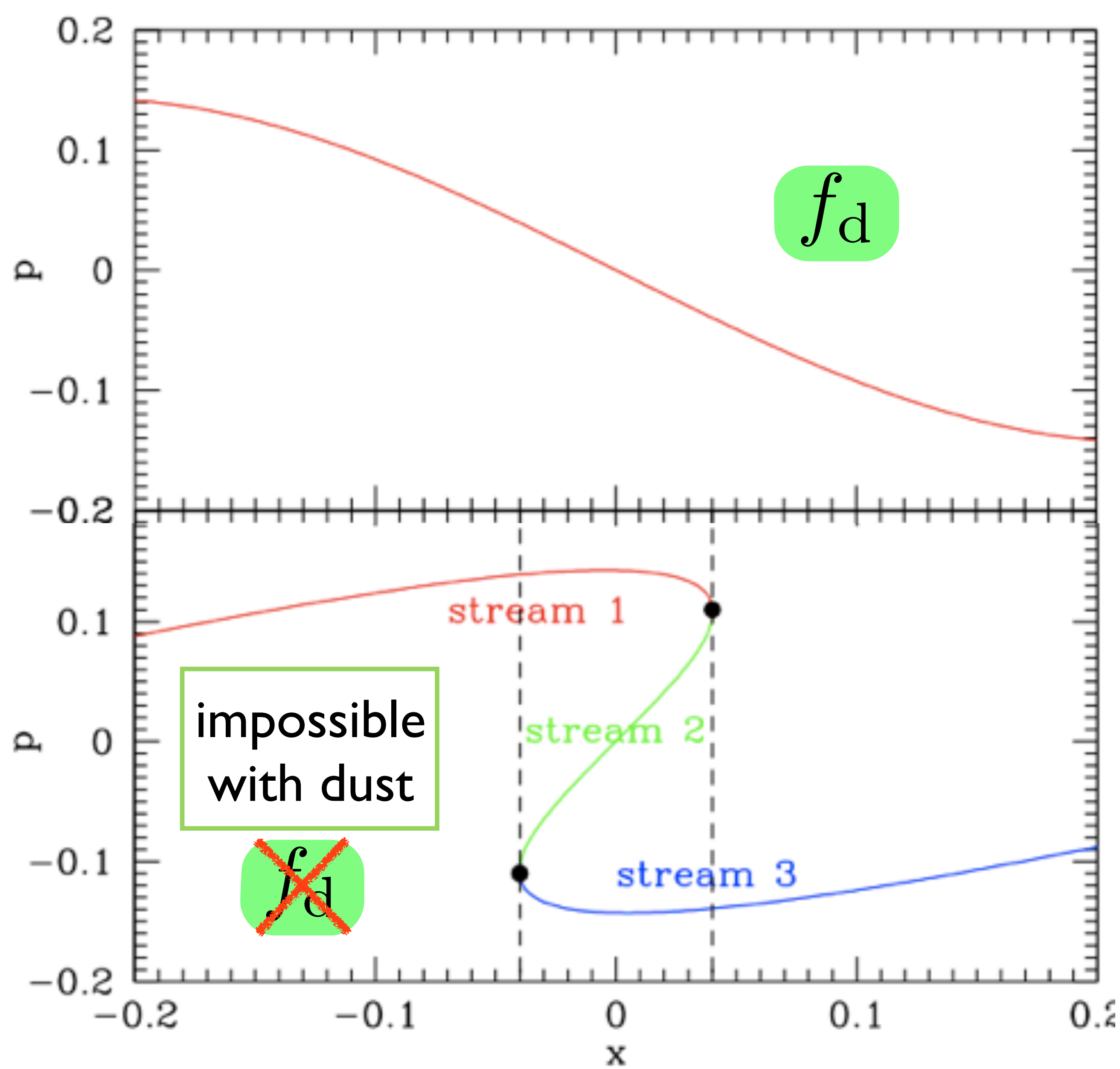
dust model

- only consistent truncation of hierarchy
- pressureless fluid: density and velocity

$$f_{\mathrm{d}}(\boldsymbol{x},\boldsymbol{p},\tau) = n(\boldsymbol{x},\tau)\delta_D^{(3)}(\boldsymbol{p} - \boldsymbol{\nabla}\phi(\boldsymbol{x},\tau))$$

Continuity $\partial_\tau n = -\frac{1}{am}\boldsymbol{\nabla}(n\boldsymbol{\nabla}\phi)$

Euler $\partial_\tau \phi = -\frac{1}{2am}(\boldsymbol{\nabla}\phi)^2 - amV$



- limited to **single-stream**
- no velocity dispersion, …

- shell-crossing singularities
- **no virialization**

Schrödinger method

Schrödinger method

self-gravitating field $$\psi = \sqrt{n}\,\exp\left(\frac{i}{\hbar}\phi\right)$$

Schrödinger - Poisson equation

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2a^2m}\Delta + mV\right]\psi$$

$$\Delta V = \frac{4\pi G\rho_0}{a}(|\psi|^2 - 1)$$

Schrödinger method

Schrödinger method

Wigner function, constructed from self-gravitating field $\psi = \sqrt{n}\exp\left(\frac{i}{\hbar}\phi\right)$

$$f_W(\boldsymbol{x},\boldsymbol{p}) = \int \frac{d^3\tilde{x}}{(2\pi\hbar)^3}\exp\left[2\frac{i}{\hbar}\boldsymbol{p'}\cdot\tilde{\boldsymbol{x}}\right]\psi(\boldsymbol{x'}-\tilde{\boldsymbol{x}})\bar{\psi}(\boldsymbol{x'}+\tilde{\boldsymbol{x}})$$

Schrödinger - Poisson equation

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$$\Delta V = \frac{4\pi G\rho_0}{a}(|\psi|^2-1)$$

Wigner-Vlasov equation

$$\partial_t f_W = \left[\frac{\boldsymbol{p}^2}{2a^2m} + mV\right]\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}(\overleftarrow{\boldsymbol{\nabla}}_x\overrightarrow{\boldsymbol{\nabla}}_p - \overleftarrow{\boldsymbol{\nabla}}_p\overrightarrow{\boldsymbol{\nabla}}_x)\right)f_W$$

Vlasov equation

$$\partial_t f = \left[\frac{\boldsymbol{p}^2}{2a^2m} + mV\right]\left(\overleftarrow{\boldsymbol{\nabla}}_x\overrightarrow{\boldsymbol{\nabla}}_p - \overleftarrow{\boldsymbol{\nabla}}_p\overrightarrow{\boldsymbol{\nabla}}_x\right)f$$

Problems

- Wigner distribution function not manifestly positive
- time evolution not in good correspondence to Vlasov

Schrödinger method

Schrödinger method

- Coarse-grained Wigner function, constructed from self-gravitating field

$$\psi = \sqrt{n}\exp\left(\tfrac{i}{\hbar}\phi\right)$$

$$f_{cg}^{W}(\boldsymbol{x},\boldsymbol{p}) = \int \frac{d^3x'd^3p'}{(\pi\sigma_x\sigma_p)^3}\exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{x'})^2}{2\sigma_x^2}-\frac{(\boldsymbol{p}-\boldsymbol{p'})^2}{2\sigma_p^2}\right]\int\frac{d^3\tilde{x}}{(2\pi\hbar)^3}\exp\left[2\frac{i}{\hbar}\boldsymbol{p'}\cdot\tilde{\boldsymbol{x}}\right]\psi(\boldsymbol{x'}-\tilde{\boldsymbol{x}})\bar{\psi}(\boldsymbol{x'}+\tilde{\boldsymbol{x}})$$

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Vlasov equation

$$\partial_t f = \left[\frac{\boldsymbol{p}^2}{2a^2m}+mV\right]\left(\overleftarrow{\nabla}_x\overrightarrow{\nabla}_p-\overleftarrow{\nabla}_p\overrightarrow{\nabla}_x\right)f$$

Problems

- Wigner distribution function not manifestly positive
- time evolution not in good correspondence to Vlasov
- **solution: add a coarse-graining** $\sigma_x\sigma_p \gtrsim \hbar/2$

Schrödinger method

Schrödinger method

- Coarse-grained Wigner function, constructed from self-gravitating field

$$\psi = \sqrt{n}\exp\left(\tfrac{i}{\hbar}\phi\right)$$

$$f_{cg}^{W}(\boldsymbol{x},\boldsymbol{p}) = \int \frac{d^3x'd^3p'}{(\pi\sigma_x\sigma_p)^3}\exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{x'})^2}{2\sigma_x^2}-\frac{(\boldsymbol{p}-\boldsymbol{p'})^2}{2\sigma_p^2}\right]\int\frac{d^3\tilde{x}}{(2\pi\hbar)^3}\exp\left[2\frac{i}{\hbar}\boldsymbol{p'}\cdot\tilde{\boldsymbol{x}}\right]\psi(\boldsymbol{x'}-\tilde{\boldsymbol{x}})\bar{\psi}(\boldsymbol{x'}+\tilde{\boldsymbol{x}})$$

Schrödinger - Poisson equation

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2a^2m}\Delta + mV\right]\psi$$

$$\Delta V = \frac{4\pi G\rho_0}{a}(|\psi|^2-1)$$

degrees of freedom

- 2: amplitude n & phase ϕ

parameters

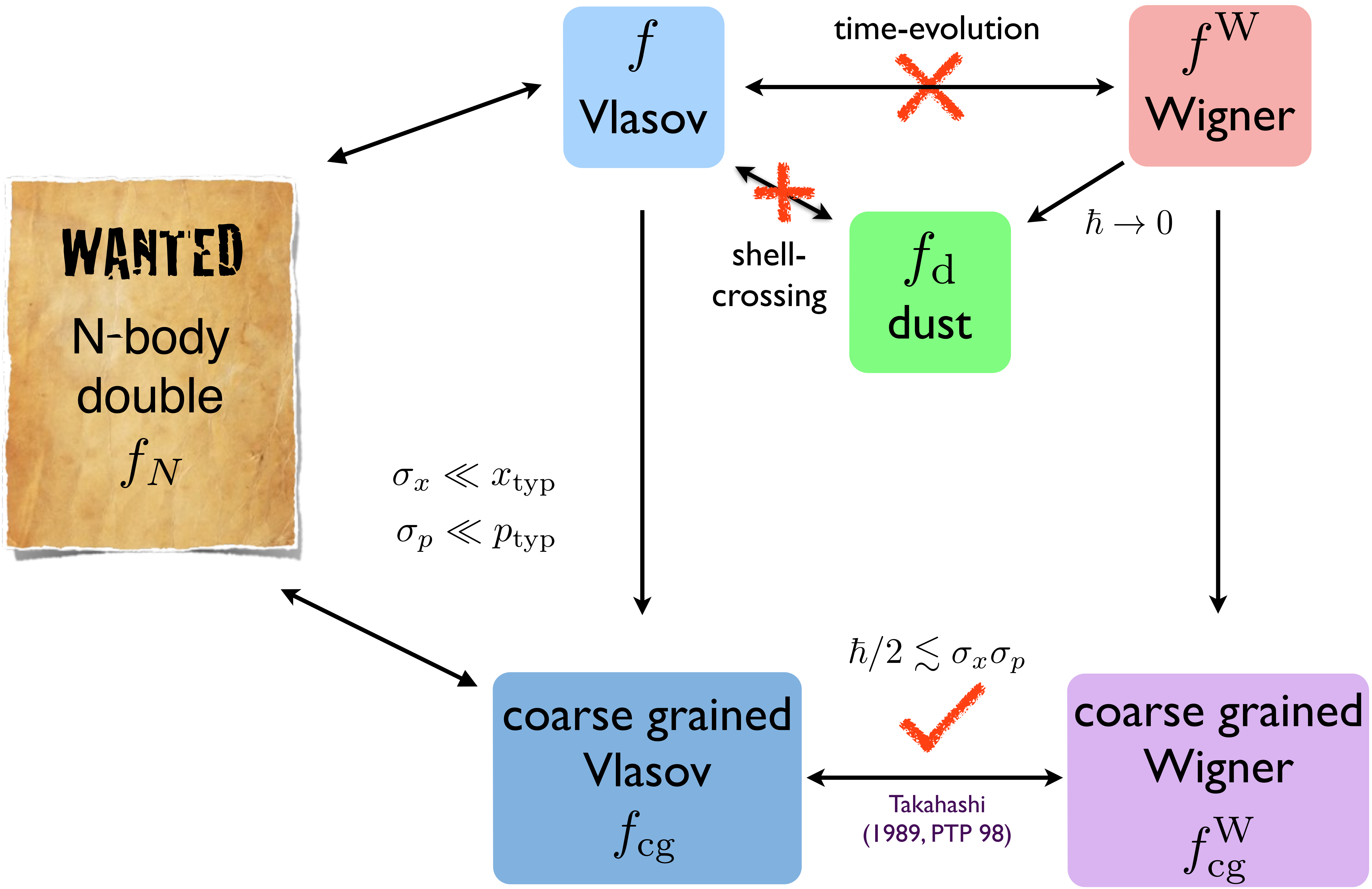
- coarse-graining σ_x, σ_p
  - fundamental resolution $\sigma_x\sigma_p \gtrsim \hbar/2$
- Schrödinger scale $\hbar$
  - degree of restriction
  - dust as special case

Continuity $\partial_\tau n = -\frac{1}{am}\boldsymbol{\nabla}(n\boldsymbol{\nabla}\phi)$

Euler $\partial_\tau\phi = -\frac{1}{2am}(\boldsymbol{\nabla}\phi)^2 - amV + \frac{\hbar^2}{2am}\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right)$

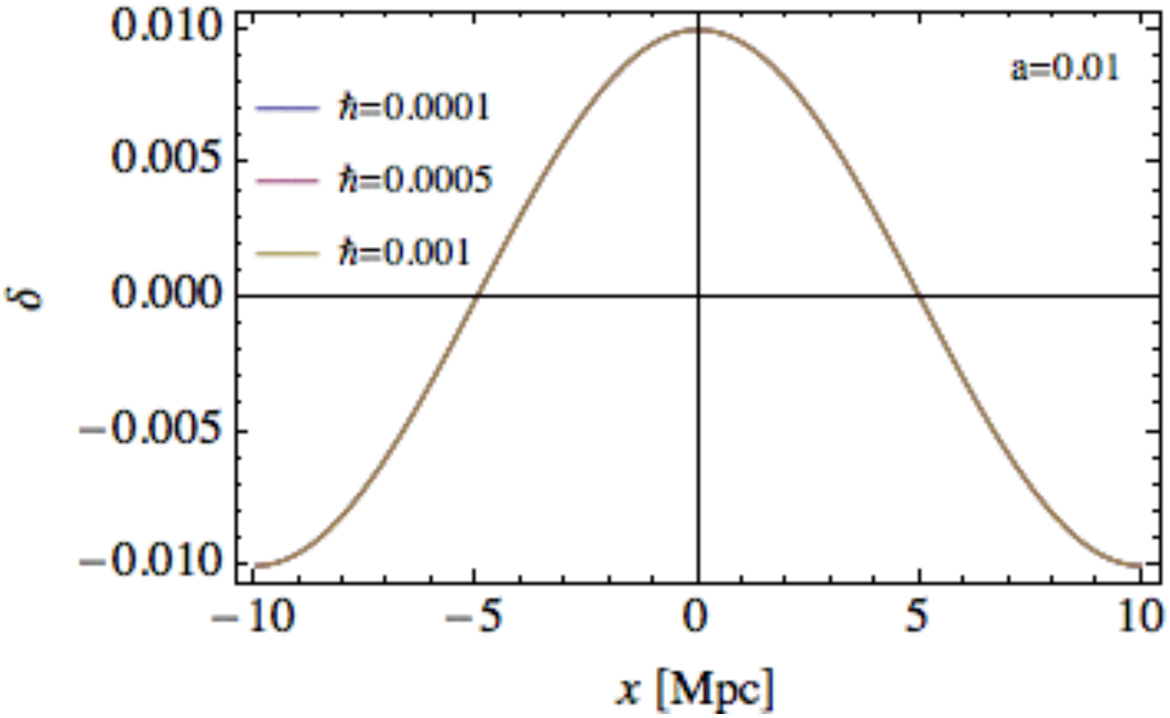
quantum potential

Schrödinger method at a glance

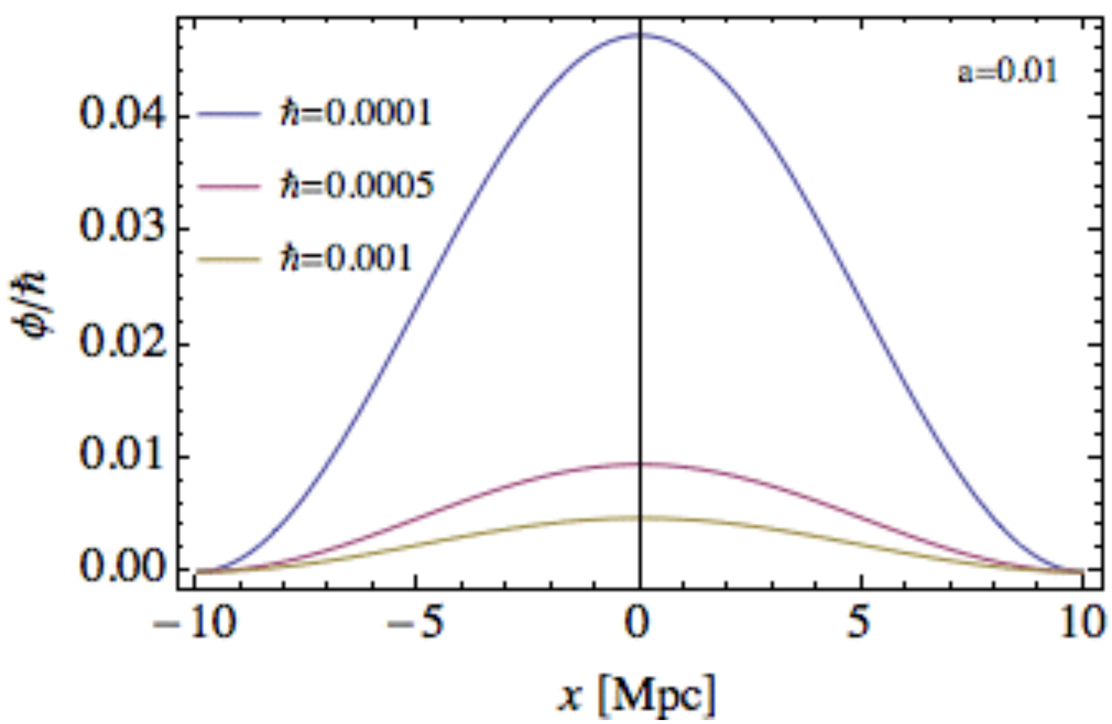


Features of Schrödinger Method

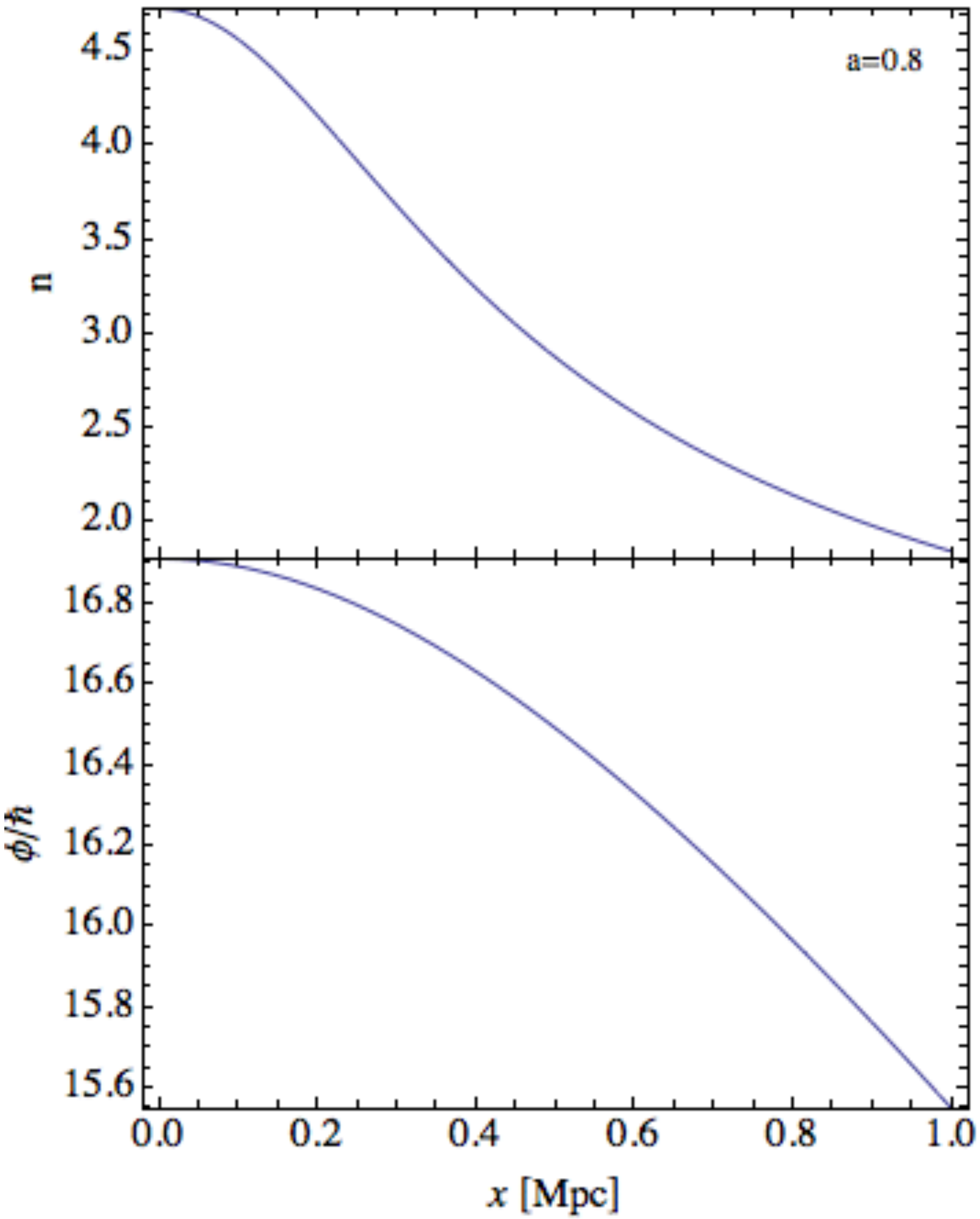
✅ **prevention of shell-crossing singularities**



⁉️ **occurrence of phase jumps**



‼️ $\psi = \sqrt{n} e^{i\phi/\hbar}$ **free of pathologies**



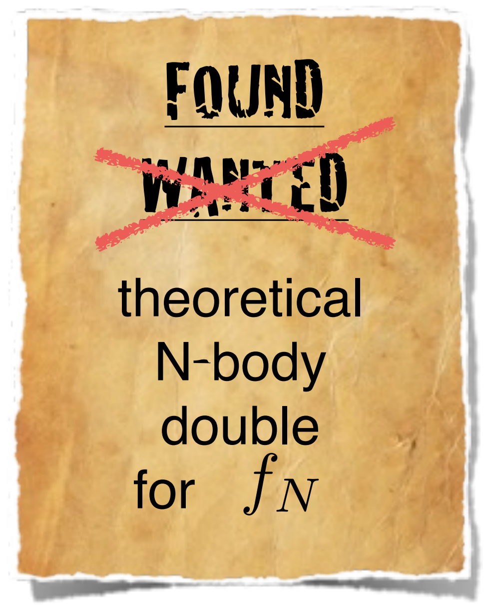
Features of Schrödinger Method

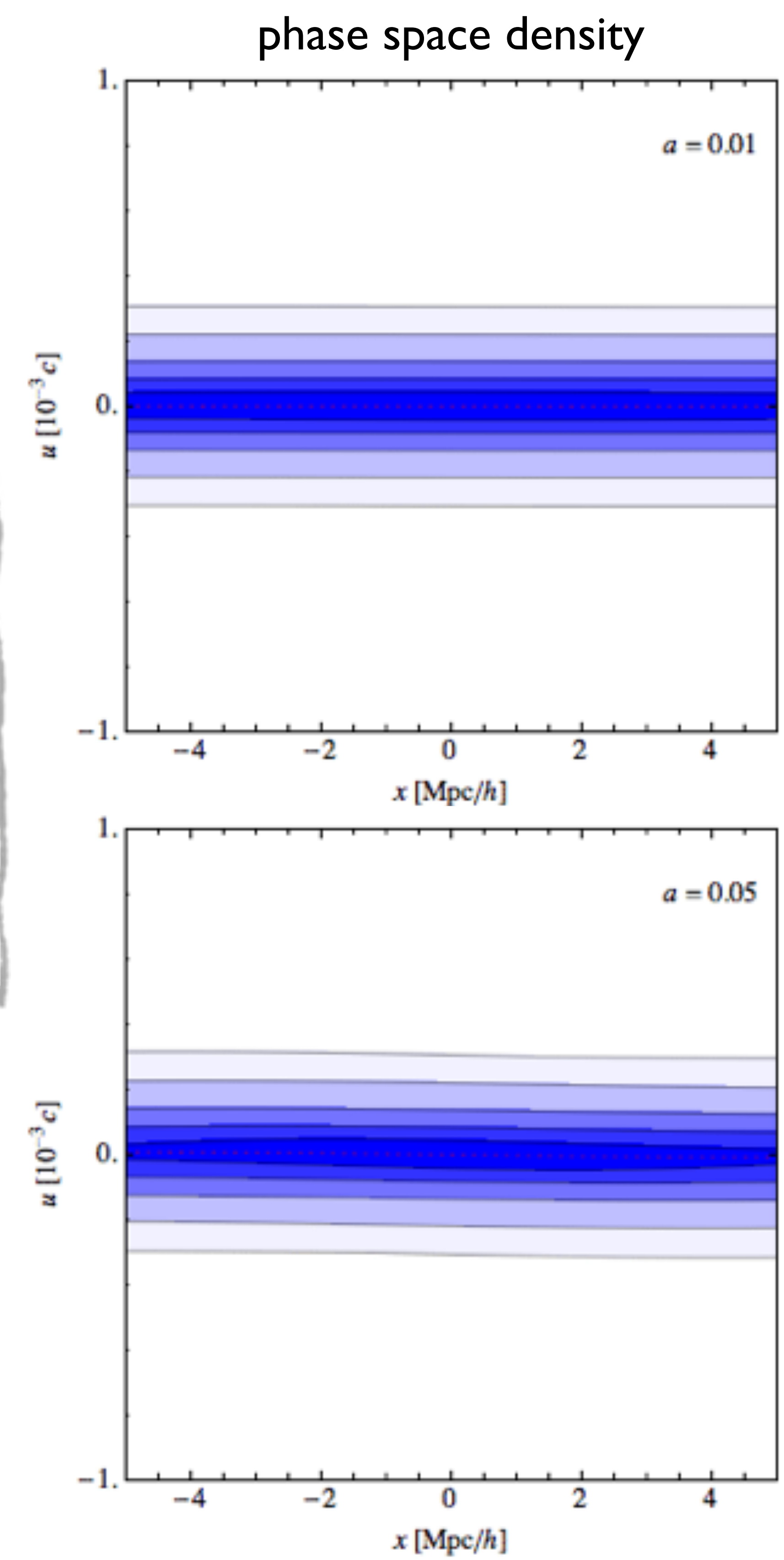
Multi-streaming

❌ dust model: fails at shell-crossing

✅ Schrödinger method: can go **beyond shell-crossing**

blue S contours: Schrödinger method
red dotted Z line: Zeldovich solution (dust model)





Virialization

❌ even in extended models: no **virialization**

✅ Schrödinger method: **bound structures** like halos

Features of Schrödinger Method

Schrödinger method

- Coarse-grained Wigner function, constructed from self-gravitating field

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special p-dependence

Cumulants

- lowest two: macroscopic density & velocity

$$\bar{n}(\boldsymbol{x}) = \exp\left[\tfrac{1}{2}\sigma_x^2\Delta\right]n(\boldsymbol{x}) \qquad \bar{\boldsymbol{v}}(\boldsymbol{x}) = \frac{1}{am\bar{n}(\boldsymbol{x})}\exp\left[\tfrac{1}{2}\sigma_x^2\Delta\right](n\boldsymbol{\nabla}\phi)(\boldsymbol{x})$$

- higher cumulants given self-consistently
  evolution equations fulfilled automatically

$$C^{(0)} = \ln n, \quad C^{(1)} = \boldsymbol{\nabla}\phi$$

$$C^{(n+2)} = -\frac{\hbar^2}{4}\boldsymbol{\nabla}\boldsymbol{\nabla}C^{(n)}$$

add coarse-graining on top of that

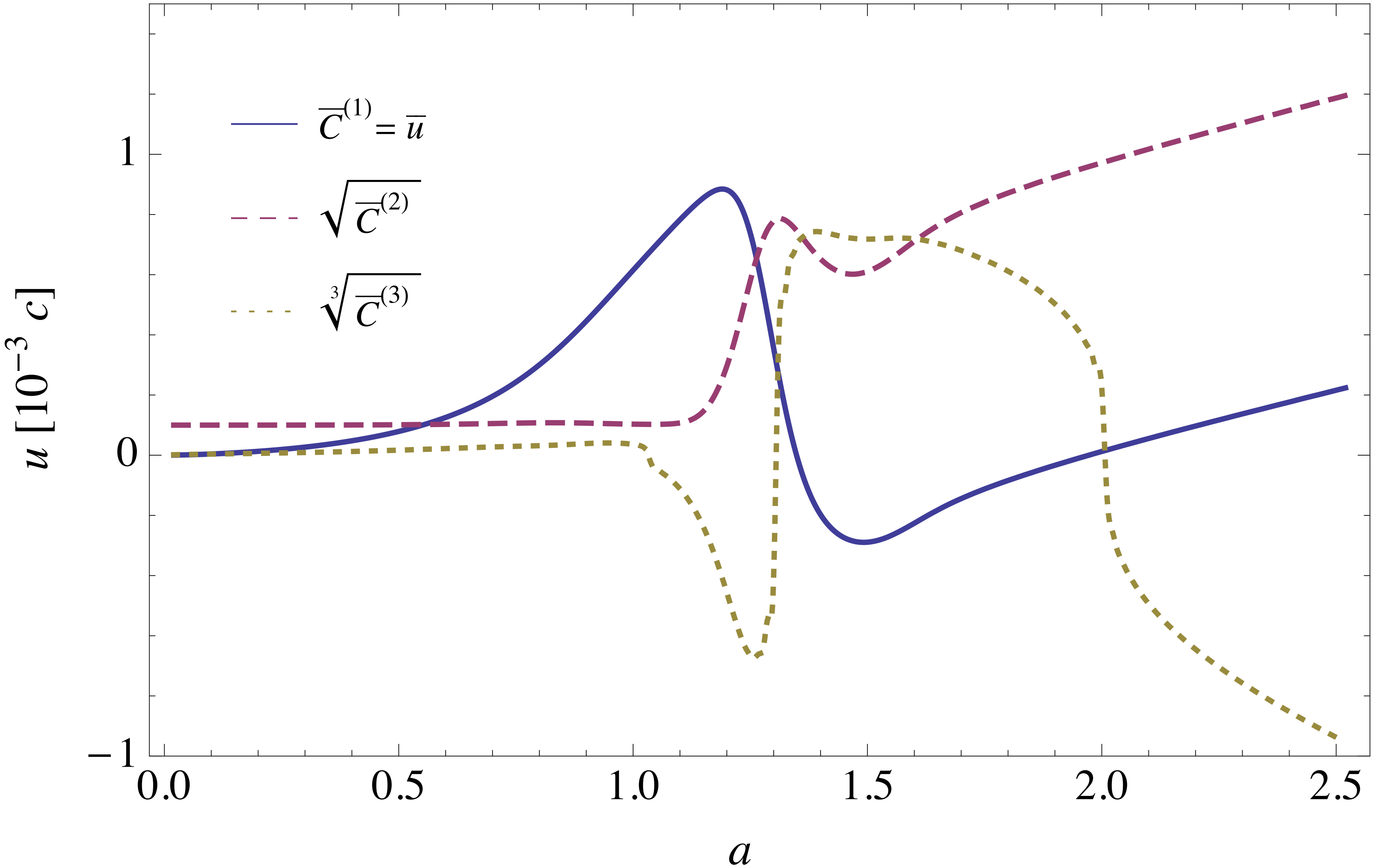
closure of hierarchy

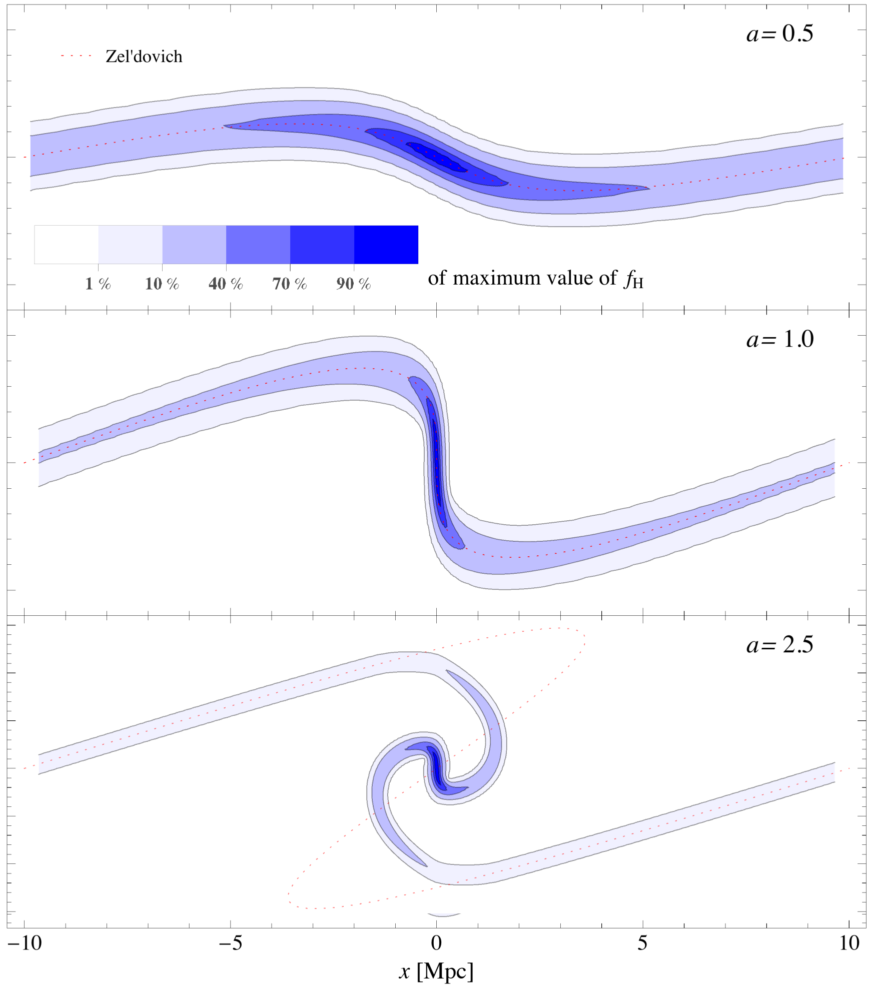
CU, Kopp & Haugg (2014, PRD 90, 023517)

Features of Schrödinger Method

Multi-streaming

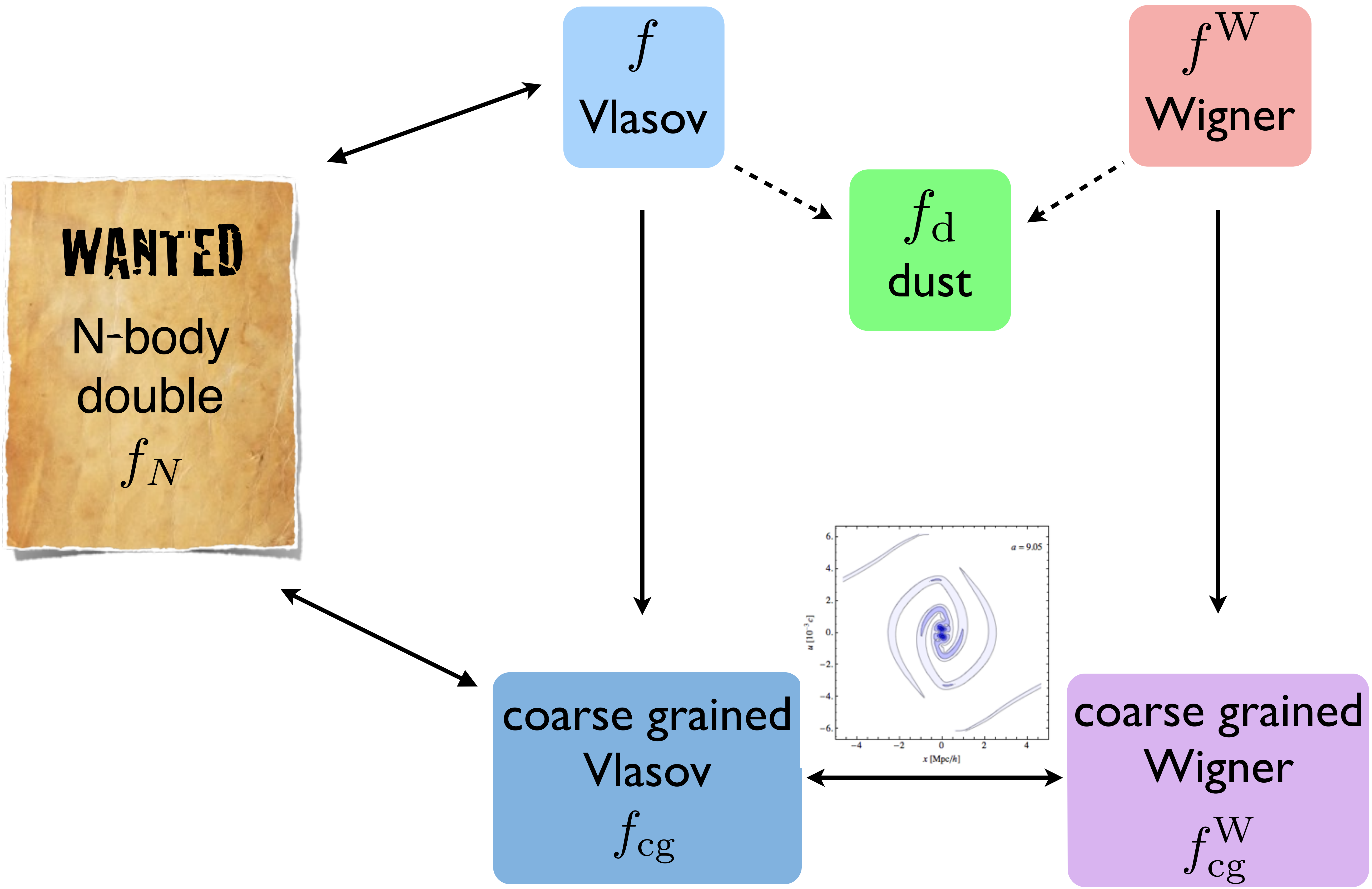
- higher cumulants encode **multi-streaming** effects
- during shell-crossing: **higher moments sourced dynamically**





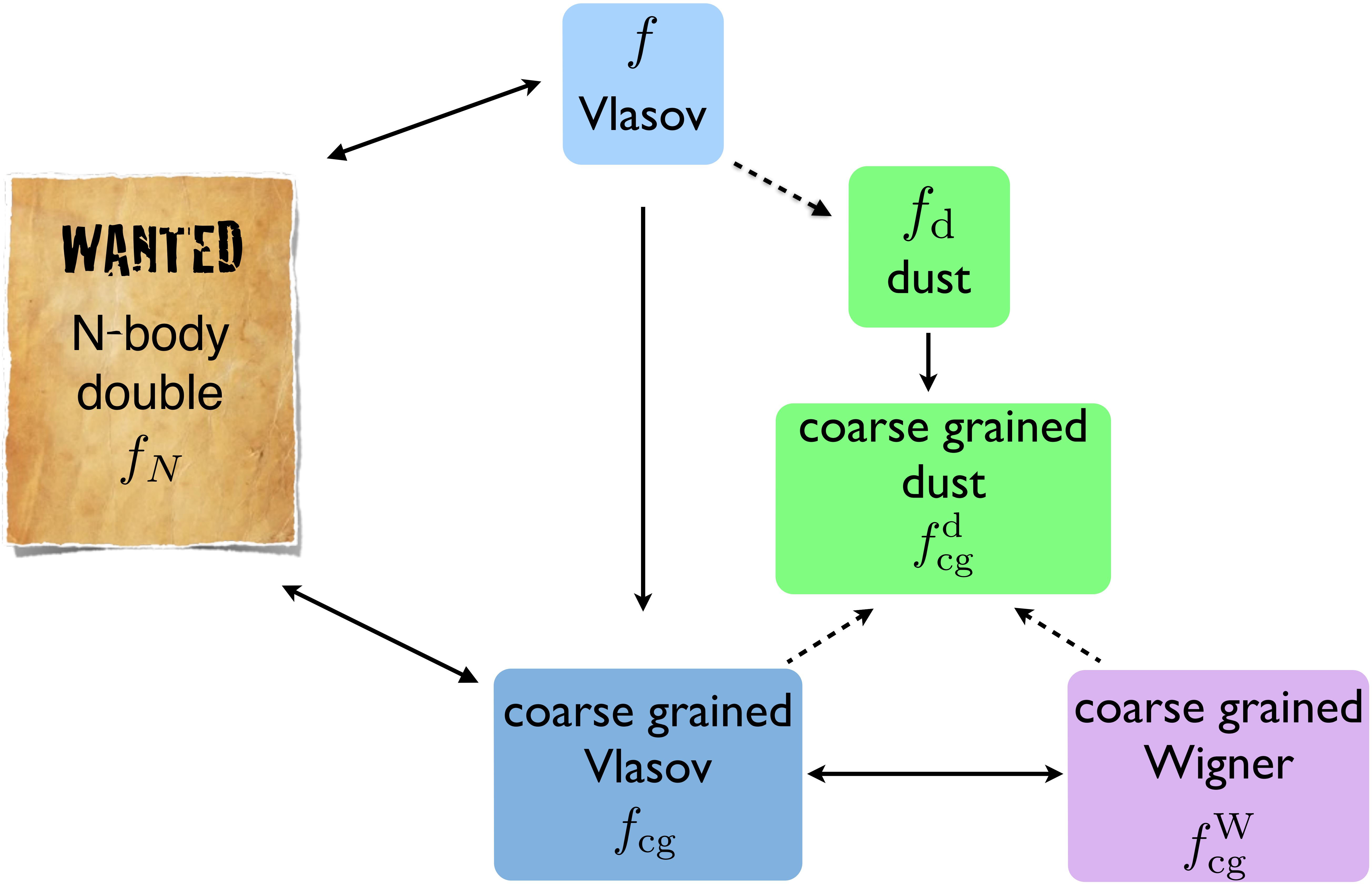
Schrödinger method: cumulants at x = −0.5 Mpc:
all equally important after shell crossing

Schrödinger method at a glance



Dark Matter power spectrum with the Coarse-grained dust model

Coarse-grained perturbation theory



Eulerian Perturbation Theory

Dust model

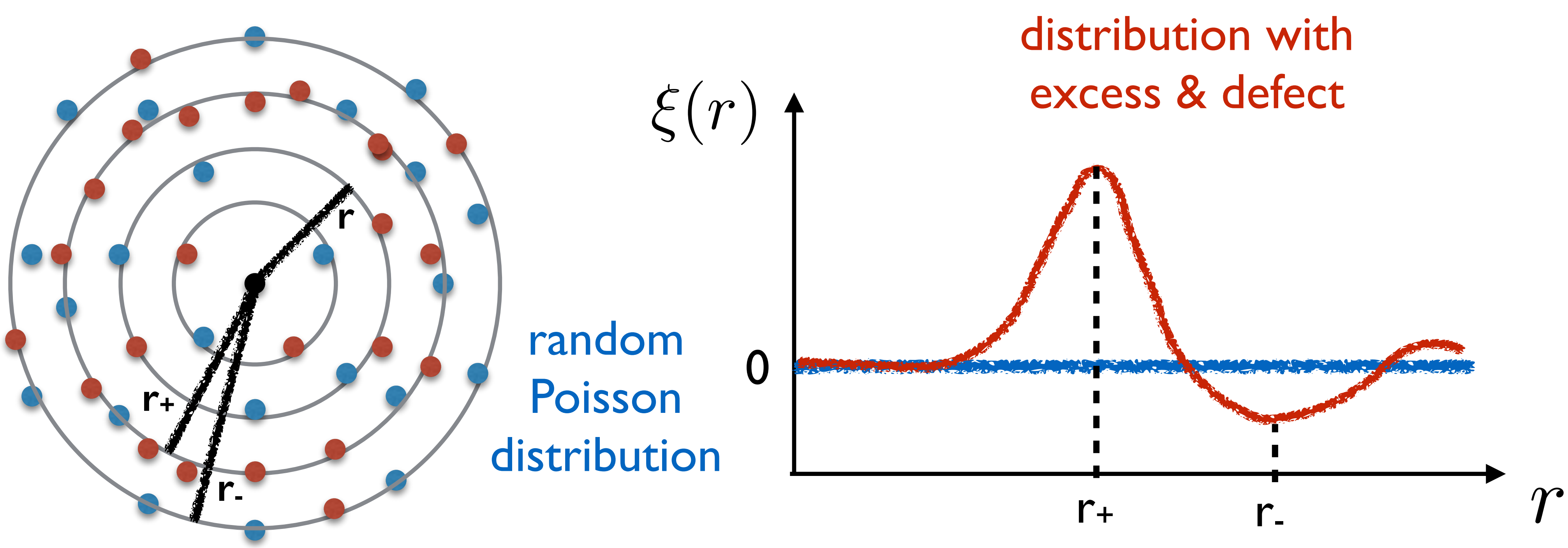
- express fluid equations in terms of $\delta = n - 1$ and $\theta = \boldsymbol{\nabla} \cdot \boldsymbol{v} \propto \Delta\phi$ (no vorticity)
- perturbative expansion: separation ansatz (fastest growing mode)

$$\delta(\tau, \boldsymbol{k}) = \sum_{n=1}^{\infty} a^n(\tau)\ \delta_n(\boldsymbol{k}) \qquad \theta(\tau, \boldsymbol{k}) = \mathcal{H}(\tau) \sum_{n=1}^{\infty} a^n(\tau)\ \theta_n(\boldsymbol{k})$$

Correlation function

- 2-point correlation: excess probability of finding 2 objects separated by **r**

$dP = n[1 + \xi(r)]dV$ homogeneity & isotropy: $\xi(\boldsymbol{r}) = \xi(r)$



Eulerian Perturbation Theory

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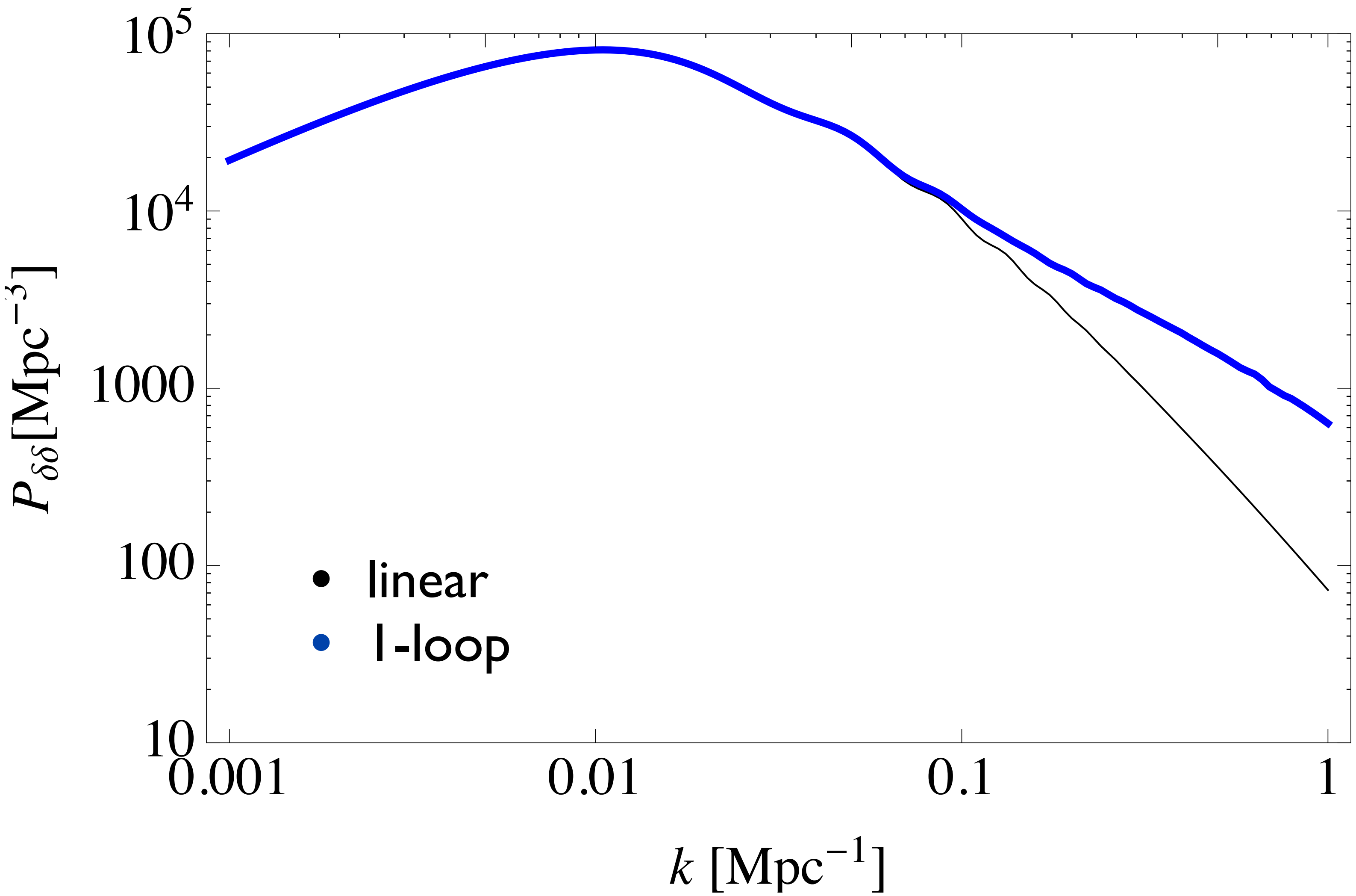
Density power spectrum



correlation function
=
FT of power spectrum

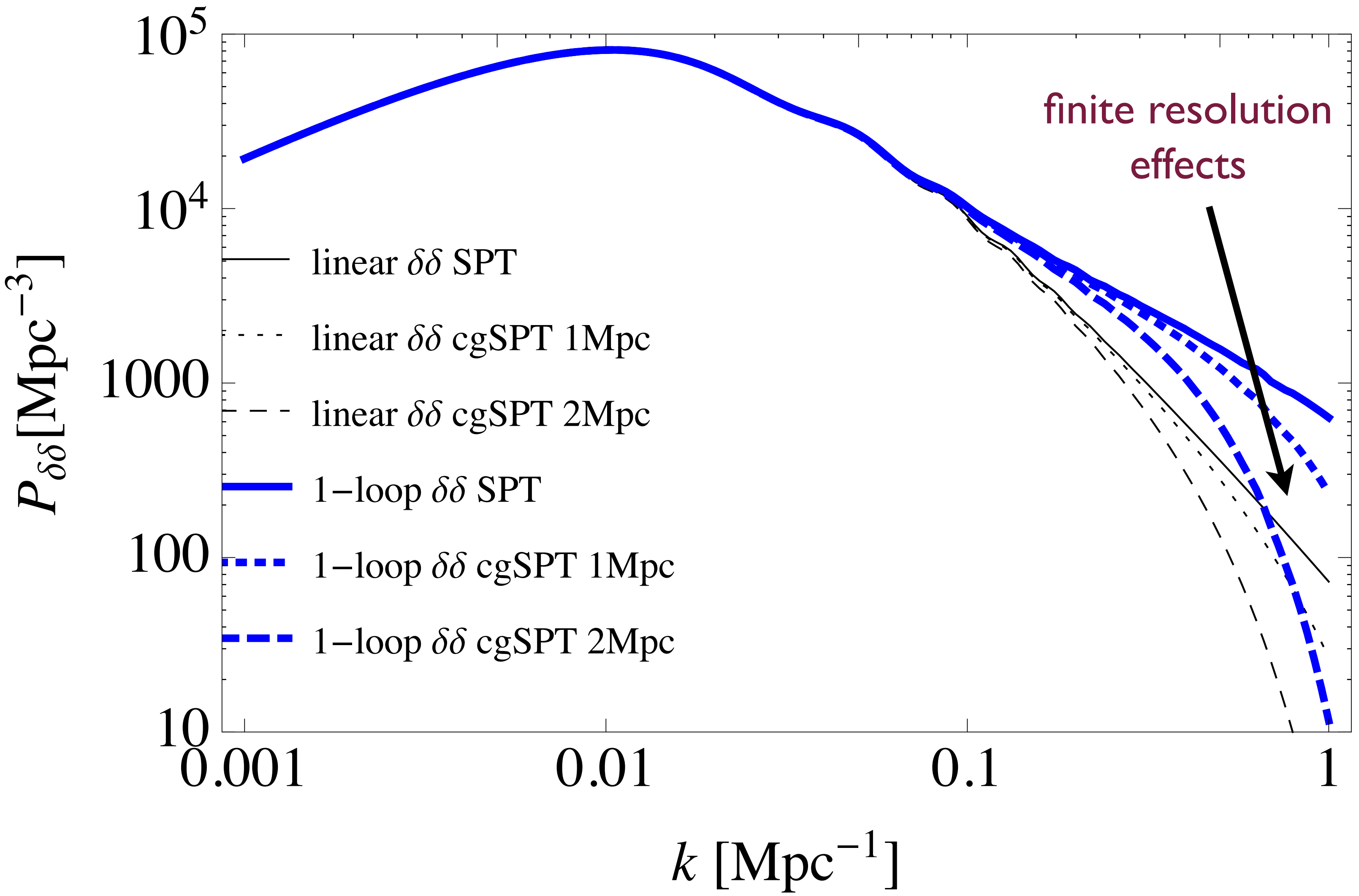
$$\xi(r) = \frac{1}{2\pi^2} \int dk\ k^2 P(k) \frac{\sin(kr)}{kr}$$

Eulerian Perturbation Theory

Coarse grained dust model

- consider only σ_x correction in Schrödinger method
- in 1st order: smoothing of input power spectrum

Density power spectrum



trivial effect:

density power spectrum gets smoothed

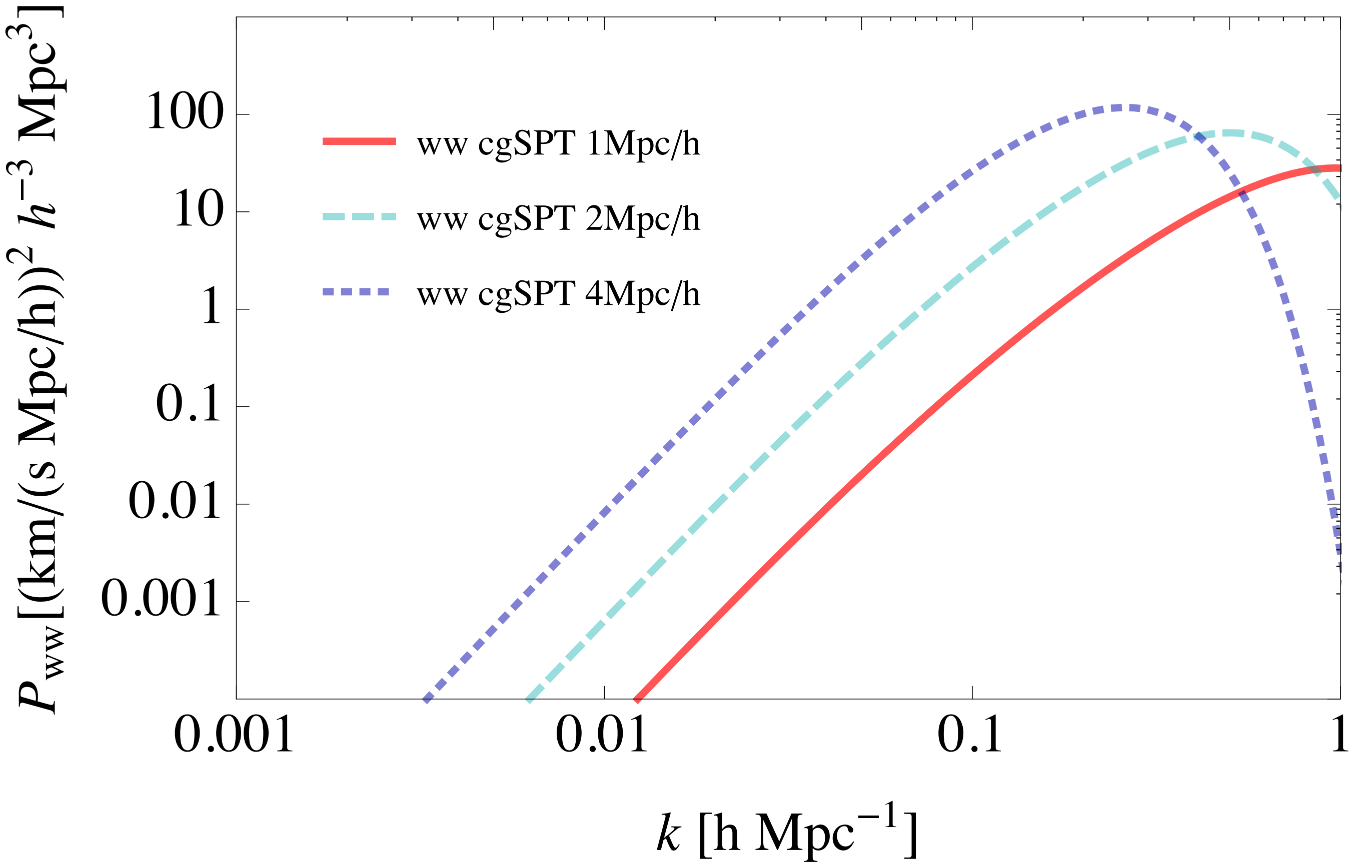
Eulerian Perturbation Theory

Coarse grained dust model

- similar to dust, but mass-weighted velocity $\bar{\boldsymbol{v}} := \frac{\overline{n\boldsymbol{v}}}{\bar{n}}$
- large scale vorticity $\bar{\boldsymbol{w}} := \boldsymbol{\nabla} \times \bar{\boldsymbol{v}} \neq 0$!

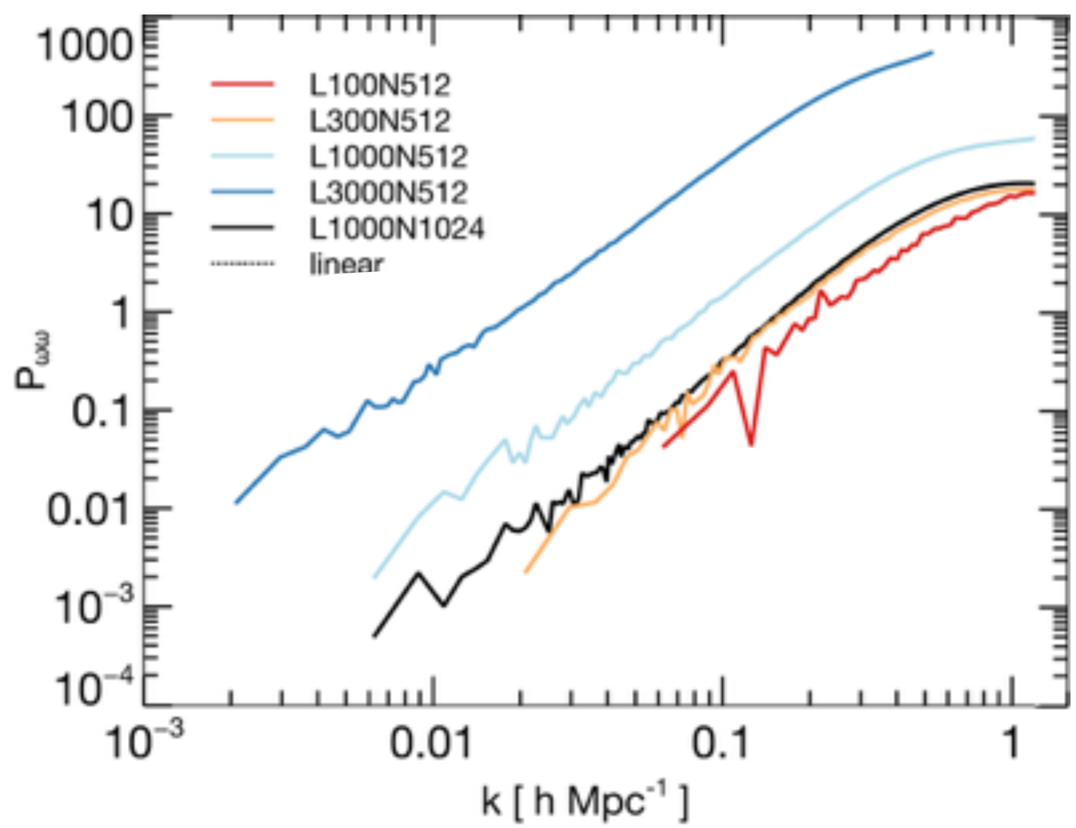
Vorticity power spectrum $P_{ww}(k)$

CU & Kopp (2015, PRD 91, 084010, arXiv: 1407.4810)



corresponding N-body data

Hahn, Angulo & Abel (2014, arXiv:1404.2280)

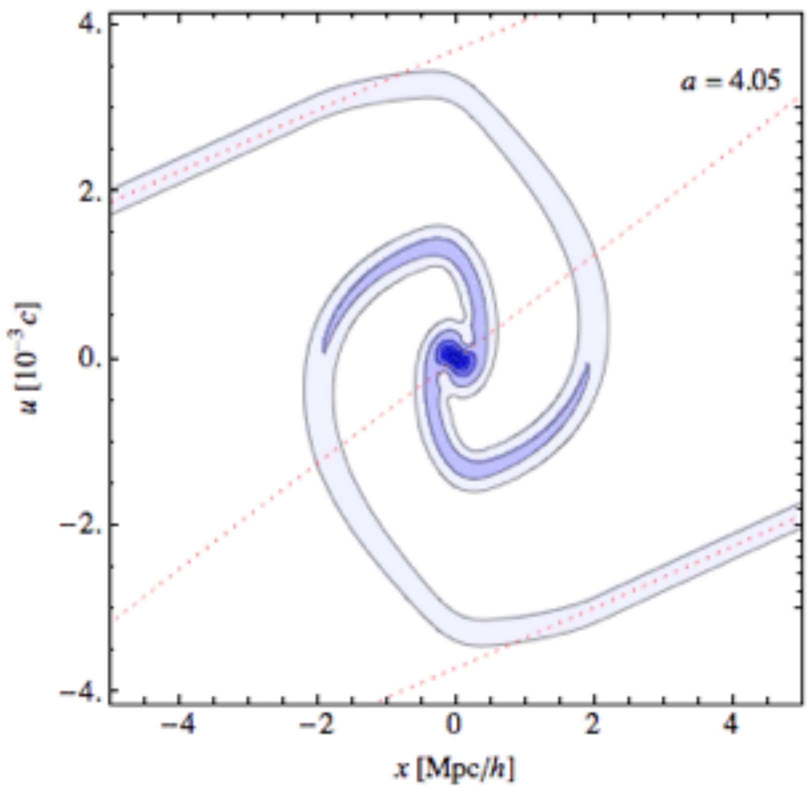


Conclusion & Prospects

Schrödinger method

- models CDM using a self-gravitating scalar field
- analytical tool to access nonlinear stage of structure formation
  - describes multi-streaming & allows for virialization

CU, Kopp, Haugg (2014, PRD 90, 023517, arXiv: 1403.5567)



Coarse-grained dust model

- mass-weighted velocity via Gaussian smoothing
  - vorticity compatible with N-body

CU & Kopp (2015, PRD 91, 084010, arXiv: 1407.4810)

Prospects

- Schrödinger method in terms of a filtering of the distribution function
  - approximate hierarchy closures instead of exact truncation
  - comparison of moments 1D Schrödinger vs. 1D Vlasov-Poisson
  - disentangle limitations of pressureless fluid & perturbation theory
- consider the effect of varying phase-space resolution $\hbar$
- understand universal density profiles of halos (NFW)

Thank You for Your Attention

Questions?