A Statistical Mechanics approach to Geophysical Turbulence

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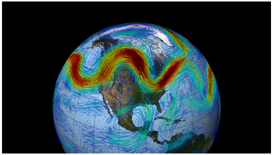
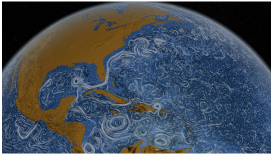
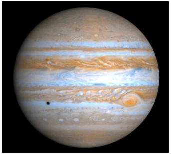
Motivation: geophysical flows

Geophysical flows self-organize into large-scale coherent structures, which play a major part in weather/climate.







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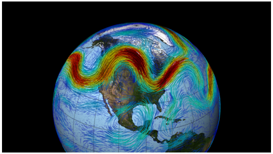
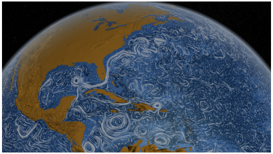
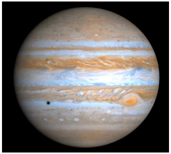
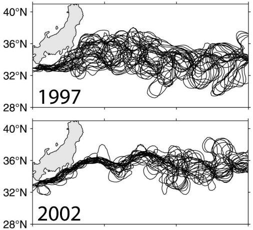


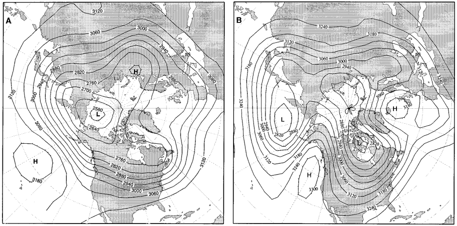




They fluctuate and undergo abrupt transitions.



Kuroshio path[1]



Zonal/blocked Jet Stream transition[2]

[1] B Qiu and S. M. Chen (2005).. *J. Phys. Oceanogr.*

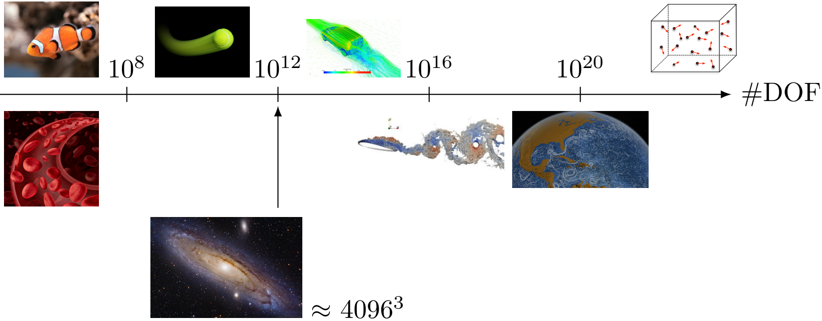
[2] E. R. Weeks et al. (1997).. *Science*

Turbulent flows and degrees of freedom

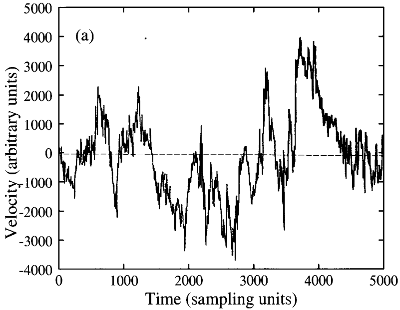
Navier-Stokes equations:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \Delta \mathbf{u}, \qquad \mathrm{Re} = UL/\nu$$

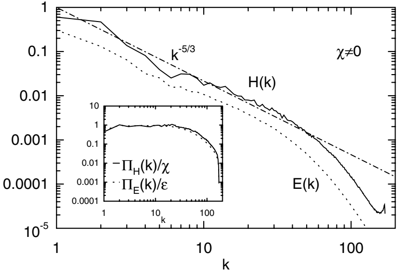
- Chaotic nature: nonlinear term couples wide range of scales.
- Number of degrees of freedom $\sim \mathrm{Re}^{9/4}$.



Predictable and unpredictable observables



Velocity measurement; ONERA wind tunnel (Y. Gagne, E. Hopfinger)



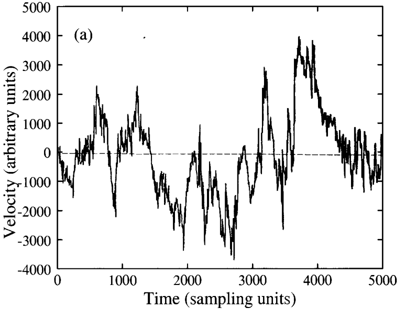
Universal kinetic energy spectrum in a DNS[3].

Can we find a probability distribution describing the system?

Very difficult task!

[3] Q. Chen et al. (2003).. *Phys. Rev. Lett.*

Predictable and unpredictable observables



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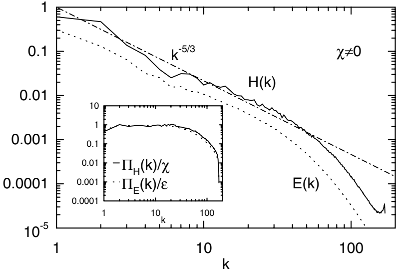

Universal kinetic energy spectrum in a DNS[3].

Kolmogorov theory Probability distribution respects symmetries (homogeneity, isotropy, scale invariance)
Not true: e.g. scale invariance is spontaneously broken (*intermittency*).

Geophysical flows break the symmetries of classical turbulence, which allows for new theoretical approaches.

[3] Q. Chen et al. (2003).. *Phys. Rev. Lett.*

Dynamical Models: 2D and Geophysical Turbulence

Model for incompressible turbulent flows (*Navier-Stokes equation*):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \Delta \mathbf{u} + \mathbf{F},$$
$$\nabla \cdot \mathbf{u} = 0.$$

When $\nu = F = 0$ (no forcing and no dissipation), we have the *Euler equations*.

In terms of *vorticity* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$,

- **For a 2D domain**, $\boldsymbol{\omega} = \omega \mathbf{n}$, vorticity is conserved along trajectories (*Lagrangian invariant*):
$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0.$$
- **For a 3D domain**, it is not:
$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}.$$

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Radically different behavior:

3D HIT

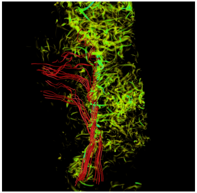
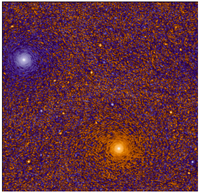


Fig. A. Pouquet (NCAR)

Direct cascade
Intermittency
...

2D Turbulence



Inverse cascade
Conformal invariance
...

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Invariants:

3D HIT

Energy $E = \frac{1}{2} \int \mathbf{u}^2(\mathbf{r}) d\mathbf{r}$

2D Turbulence

Energy $E = \frac{1}{2} \int \omega(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r} \quad (\omega = -\Delta \psi)$

Casimir invariants $\int s(\omega(\mathbf{r})) d\mathbf{r}$

E.g. *enstrophy* $\int \omega^2(\mathbf{r}) d\mathbf{r}$

Dynamical Models: 2D and Geophysical Turbulence

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- **For a 2D domain**, $\boldsymbol{\omega} = \omega \mathbf{n}$, vorticity is conserved along trajectories (*Lagrangian invariant*):
$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0.$$
- **Geophysical flows** are 3D, but subjected to strong rotation and density stratification.
Large scales well described by advection of *potential vorticity* (*quasi-geostrophic*):
$$\partial_t q + \mathbf{u} \cdot \nabla q = 0.$$
E.g. $q = \omega + \partial_z (f_0^2 / N^2 \partial_z \psi) + \beta y$.

Main Questions and Theoretical Tools

Generic questions:

- Can we predict self-organization of geophysical flows into large scale coherent structures?
- Characterize the attractors of geophysical turbulence
- Study fluctuations around the mean state
- What aspects of transitions in turbulent flows are predictable?

Because of strong nonlinearity/huge number of degrees of freedom, classical fluid mechanics+direct numerical simulations do not suffice.

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Because of strong nonlinearity/huge number of degrees of freedom, classical fluid mechanics+direct numerical simulations do not suffice.

Main theoretical tool: Large Deviation Theory

It is a tool to study asymptotic probabilities:

$$\text{Prob}(A[x] = a) \sim e^{-I(a)/\varepsilon} \text{ when } \varepsilon \to 0.$$

The small parameter ε can be

- The inverse of the number of degrees of freedom $\varepsilon = 1/N$.
- The amplitude of a noise term (Freidlin-Wentzell theory)
- The inverse of an observation time $\varepsilon = 1/T$ (Donsker-Varadhan)

Outline

1 Introduction

2 Equilibrium Theory

3 Perturbative approach

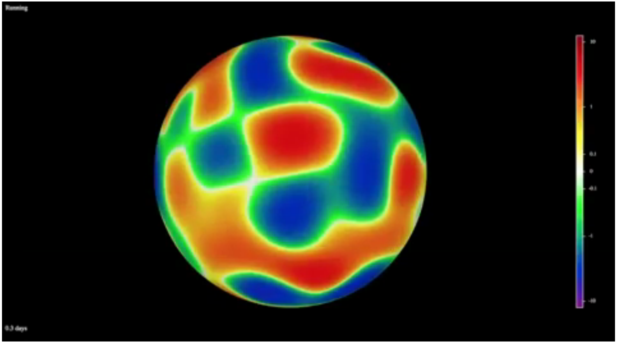
4 Large deviations and transitions

5 Conclusion

Advection of the vorticity field in 2D

$$\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0.$$

Small-scale vorticity is mixed by the flow while large-scale coherent structures form.



Direct Numerical Simulation: Vorticity Contours. Courtesy Brad Marston (Brown University).

The microcanonical measure

Formally, we define the *microcanonical measure* as

$$\mu_{E,(\Gamma_n)_{n\in\mathbb{N}}}(d\omega) = \frac{1}{\Omega(E,(\Gamma_n)_{n\in\mathbb{N}})}\delta(\mathcal{E}[\omega]-E)\prod_{k=1}^{+\infty}\delta(\mathcal{G}_k[\omega]-\Gamma_k)\prod_{i=1}^{+\infty}d\omega_i.$$

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- Invariant measure of the Euler equations[4].
- Difficult to manipulate: e.g.

$$\mathbb{E}[\omega]=0.$$

  Spontaneous symmetry breaking.

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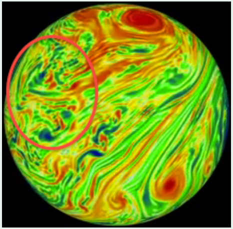
Can we compute macrostates?

Mean-field theory (Miller-Robert-Sommeria)

Two levels of description[4]:

- Microstates: fine-grained vorticity field $\omega(\mathbf{x})$.



[4] R. Robert and J. Sommeria (1991).. *J. Fluid Mech.* J. Miller (1990).. *Phys. Rev. Lett.*

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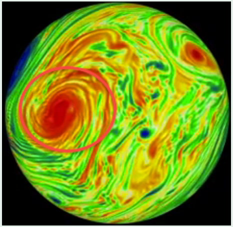
Mean-field theory (Miller-Robert-Sommeria)

Two levels of description[4]:

- Microstates: fine-grained vorticity field $\omega(\mathbf{x})$.
- Macrostates: fine-grained vorticity probability distribution $\rho(\sigma,\mathbf{x})$, $\int \rho(\sigma,\mathbf{x})d\sigma = 1$.
  Mean coarse-grained vorticity: $\overline{\omega}(\mathbf{x}) = \int \sigma\rho(\sigma,\mathbf{x})d\sigma$.

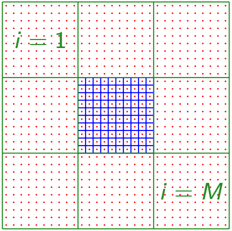


We want to compute the most probable macrostates ρ

[4] R. Robert and J. Sommeria (1991).. *J. Fluid Mech.* J. Miller (1990).. *Phys. Rev. Lett.*

The mean-field approach: counting the microstates[5]

Let us consider a square lattice with N sites, and a "coarse-grained" lattice of M boxes containing $n = N/M$ sites each.



Finite number of vorticity levels $\mathfrak{S} = \{\sigma_1, \dots, \sigma_K\}$.

- *Microstates:*

$$\hat{\omega} = (\omega_{i\alpha})_{\substack{1 \leq i \leq M \\ 1 \leq \alpha \leq n}} \in \mathfrak{S}^N.$$

- *Macrostates:*

$$P = (p_{ik})_{\substack{1 \leq i \leq M \\ 1 \leq k \leq K}} \in [0,1]^{MK}, \sum_{k=1}^{K} p_{ik} = 1.$$

- *Coarse-grained vorticity field:*

$$\overline{\omega}_i \equiv \frac{1}{n} \sum_{\alpha=1}^{n} \omega_{i\alpha} = \sum_{k=1}^{K} \sigma_k p_{ik}.$$

Number of microstates which realize a given macrostate:

$$W(P) = \prod_{i=1}^{M} \frac{n!}{\prod_{k=1}^{K} (np_{ik})!}$$

[5] C. Herbert (2015).. In: *Stochastic Equations for Complex Systems: Theoretical and Computational Topics*. Ed. by S. Heinz and H. Bessaih. Springer

The mean-field approach: large deviation of the macrostate probability[6]

Conservation constraints:

- *Vorticity distribution* (i.e. Casimir invariants) depends only on P
- *Energy* depends only on P in the limit $N \to +\infty$.

Probability of a given macrostate P with energy E:

$$\mathrm{Prob}(P) = \frac{W(P)}{\Omega_N(E,\gamma)},$$

$$\frac{1}{N}\ln \mathrm{Prob}(P) = \underbrace{-\frac{1}{M}\sum_{i=1}^{M}\sum_{k=1}^{K} p_{ik}\ln p_{ik}}_{\text{entropy } \mathscr{S}_{M,K}[P]} - S(E,\gamma) + o(1).$$

This is a *large deviation property*.

[6] C. Herbert (2015).. In: *Stochastic Equations for Complex Systems: Theoretical and Computational Topics*. Ed. by S. Heinz and H. Bessaih. Springer

The mean-field approach: variational problem[7]

Equilibrium states = most probable macrostates. They must minimize the large deviation rate function, while satisfying the global constraints.

Microcanonical variational problem

$$S(E,\gamma) = \max_{\rho}\{\mathscr{S}[\rho] \mid \mathscr{E}[\rho] = E, \forall\sigma \in \mathbb{R}, \mathscr{D}_\sigma[\rho] = \gamma(\sigma)\}.$$

Critical points:

$$\rho(\sigma, \mathbf{r}) = \frac{e^{-\beta\sigma\overline{\psi}(\mathbf{r}) - \alpha(\sigma)}}{\mathcal{Z}_{\beta,\alpha}(\overline{\psi}(\mathbf{r}))} \qquad \text{(Gibbs states)},$$

with

$$\overline{\omega} \equiv -\Delta\overline{\psi}, \qquad \mathcal{Z}_{\beta,\alpha}(u) \equiv \int_{\mathbb{R}} e^{-\beta\sigma u - \alpha(\sigma)} d\sigma.$$

Mean-field equation:

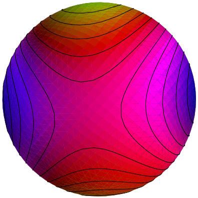
$$\overline{\omega}(\mathbf{r}) = F_{\beta,\alpha}(\overline{\psi}(\mathbf{r})), \qquad \text{with } F_{\beta,\alpha}(u) \equiv -\frac{1}{\beta}\frac{d\ln\mathcal{Z}_{\beta,\alpha}(u)}{du}.$$

[7] C. Herbert (2015).. In: *Stochastic Equations for Complex Systems: Theoretical and Computational Topics*. Ed. by S. Heinz and H. Bessaih. Springer

Exemple: Equilibrium flows on the sphere

Stable equilibrium states[8]

- Solid body rotations: $\psi = \Omega_* \cos\theta$
- Dipoles: $\psi = \Omega_* \cos\theta + \sqrt{3(E - E^*(L))} \sin\theta \cos(\phi - \phi_0)$
- *Quadrupoles:*
  $\psi_\infty = \psi_{20}(3\cos^2\theta - 1) + \psi_{21}\sin(2\theta)\sin(\phi - \phi_1) + \psi_{22}\sin^2\theta \sin(2(\phi - \phi_2))$



Theoretical Equilibrium: Quadrupole

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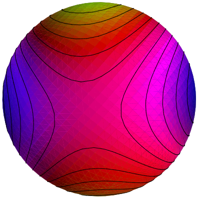

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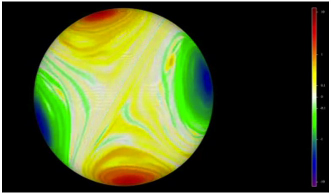

DNS Final State[9]

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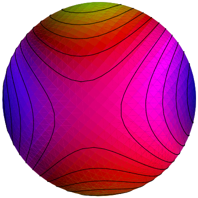

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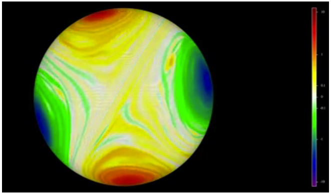

DNS Final State[9]

Generalization to more realistic geophysical flows[10]

[8] C. Herbert et al. (2012).. *J. Stat. Mech.* C. Herbert (2013).. *J. Stat. Phys.*

[9] W. Qi and J. B. Marston (2014).. *J. Stat. Mech.*

[10] F. Bouchet and A. Venaille (2012).. *Phys. Rep.* C. Herbert (2014).. *Phys. Rev. E*; V. Lucarini et al. (2014).. *Rev. Geophys.* A. Renaud et al. (2016).. *J. Stat. Phys.*

Summary

Achievements

- The Microcanonical measure can be built without UV divergences.
- Mean-field theory is exact: in the microcanonical ensemble, vorticity at two different points behaves as statistically independent random variables.
- Macrostates statisfy a large deviation property. Equilibrium states can be computed as solutions of a variational problem.
- They are in qualitative agreement with stationary state of numerical simulations.
- *(Interesting thermodynamical properties (long-range interactions): non-equivalence of ensembles, negative temperatures, etc)*

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Limitations

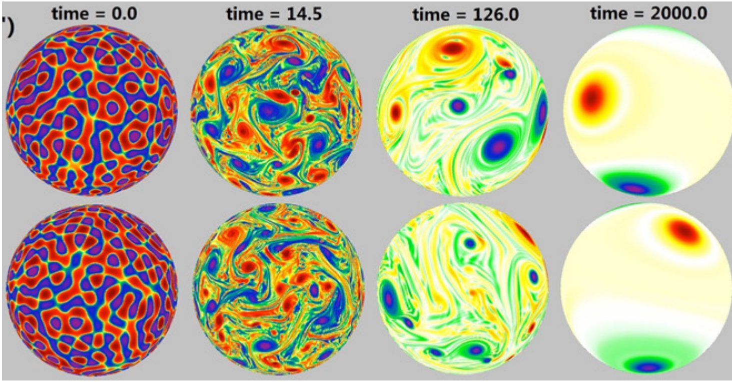
- Non-ergodicity

The effect of rotation (DNS results[11])

When the Rossby waves are sufficiently slow, the system relaxes towards its equilibrium state.

“North Pole”

“South Pole”



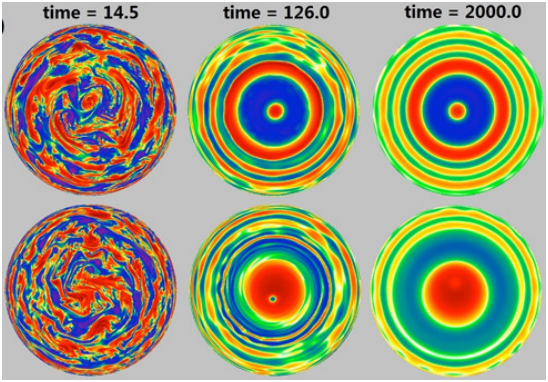
[11] W. Qi and J. B. Marston (2014).. *J. Stat. Mech.*

The effect of rotation (DNS results[11])

For faster rotation rates, Rossby waves arrest the cascade at the Rhines scale and lead to the emergence of zonal flows.

"North Pole"

"South Pole"



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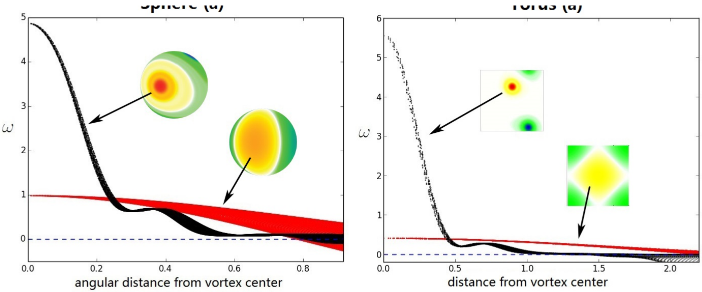
- Quantitative predictions are difficult. The set of MRS equilibria is huge.

Vortex profile and higher-order invariants

Comparing the equilibrium states with numerical simulations[12]:



Perturbative expansion leads to core sharpening, but it is difficult to make quantitative predictions.
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[12] W. Qi and J. B. Marston (2014).. *J. Stat. Mech.*

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- Forcing and dissipation not taken into account

Outline

1. Introduction
2. Equilibrium Theory
3. Perturbative approach
4. Large deviations and transitions
5. Conclusion

The closure problem for Homogeneous Isotropic Turbulence

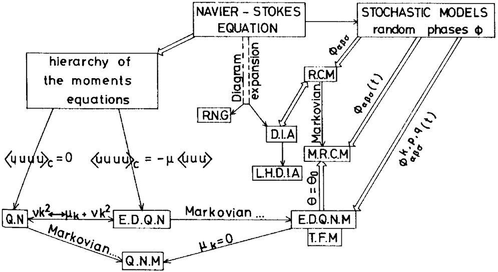
Incompressible Navier-Stokes equation in Fourier space:

$$(\partial_t + \nu k^2)\hat{u}_i(\mathbf{k}) = \sum_{\mathbf{p},\mathbf{q}} \mathcal{P}_i^{jl}(\mathbf{k})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\hat{u}_j(\mathbf{p})\hat{u}_l(\mathbf{q}), \qquad k^i \hat{u}_i(\mathbf{k}) = 0.$$

Formally,

$$[\partial_t + \nu(k^2+p^2)]\langle \hat{u}(\mathbf{k})\hat{u}(\mathbf{p})\rangle = \langle \hat{u}\star\hat{u}\star\hat{u}\rangle,$$
$$[\partial_t + \nu(k^2+p^2+q^2)]\langle \hat{u}(\mathbf{k})\hat{u}(\mathbf{p})\hat{u}(\mathbf{q})\rangle = \langle \hat{u}\star\hat{u}\star\hat{u}\star\hat{u}\rangle,$$
$$\dots$$

Closing the hierarchy requires arbitrary hypothesis (e.g. Gaussianity, etc)



Analytical closure theories[13]

[13] M. Lesieur (2008). *Turbulence in Fluids*. 4th edition. Springer-Verlag, New York.

The closure problem for Homogeneous Isotropic Turbulence

Reynolds decomposition:

$$u_i = \bar{u}_i + u_i',$$

where $\bar{\cdot}$ is a projection operator.
The Navier-Stokes equations become:

$$\partial_t \bar{u}_i + \bar{u}_j \partial^j \bar{u}_i = -\partial_i \bar{P} + \nu \partial_j \partial^j \bar{u}_i - \partial^j \overline{u_i' u_j'},$$
$$\partial_t u_i' + \bar{u}_j \partial^j u_i' + u_j' \partial^j \bar{u}_i = -\partial_i P' + \nu \partial_j \partial^j u_i' - \partial^j u_i' u_j' + \partial^j \overline{u_i' u_j'}.$$

Modeling approaches:

- *Large Eddy Simulations*: spatial filtering

$$\bar{u}_i(\mathbf{x}, t) = \int G(\mathbf{x} - \mathbf{y}) u_i(\mathbf{y}, t) d\mathbf{y}$$

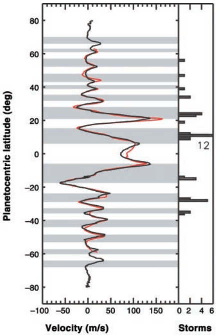
- *Reynolds Average Navier-Stokes*: time filtering

These are *phenomenological models*.
The major difficulty is to compute the Reynolds stress tensor $-\partial^j \overline{u_i' u_j'}$.

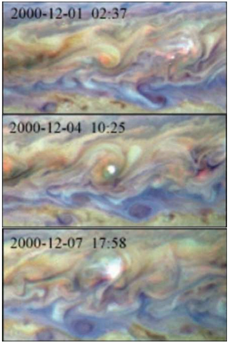
Timescale separation in geophysical flows

In some flows, there is a *natural timescale separation*, usually associated to a broken symmetry of the Navier-Stokes equations.

E.g. Jupiter[14]:



Zonal wind measured by Voyager 2 (1979, red) and Cassini (2000, black).



Cassini

[14] C. C. Porco et al. (2003).. *Science*.

Adiabatic elimination of fast variables[15] (*stochastic averaging*)

Slow-fast SDE:

$$dX_t = f(X_t, Y_t)dt + \sqrt{2\epsilon}dW_t,$$
$$dY_t = \alpha^{-1}g(X_t, Y_t)dt + \sqrt{\alpha^{-1}}h(X_t, Y_t)dW_t.$$

- Joint PDF $P(x, y; t)$; Fokker-Planck equation $\partial_t P = (\alpha^{-1}L_0 + L_1)P$.
- Stationary distribution for fast modes at fixed x and projection operator:

$$L_0 P^x_\infty(y) = 0, \quad \mathcal{P}\phi = P^x_\infty(y)\int dy \phi(x, y).$$

- Write $P_s = \mathcal{P}P$, $P_f = (1 - \mathcal{P})P$. We have $\partial_t P_s = \mathcal{P}(\alpha^{-1}L_0 + L_1)P = \mathcal{P}L_1 P$.
- At lowest order, $\partial_t P_s = \mathcal{P}L_1 P_s + O(\alpha)$ and $P_s(x, y) = P^x_\infty(y)Q(x)$ with

$$\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial x}\left[\mathbb{E}^x_\infty[f]Q(x)\right] + \epsilon\frac{\partial^2}{\partial x^2}Q + O(\alpha).$$

Finally, after adiabatic reduction:

$$dX_t = \mathbb{E}^{X_t}_\infty[f]dt + \sqrt{2\epsilon}dW_t.$$

[15] e.g. C. W. Gardiner (2009). *Handbook of Stochastic Methods for physics, chemistry, and the natural sciences*. 4th edition. Springer, Berlin.

Adiabatic elimination of fast variables: zonal jets

Reynolds decomposition for the zonal jets

$\omega = \bar{\omega} + \omega'$, with $\bar{\cdot}$ the projection on the (slow) zonal modes.
Formally,

$$\partial_t \bar{\omega} + L_{\bar{\omega}}[\bar{\omega}] = -\partial^i \overline{u_i' \omega'} + \bar{\eta},$$
$$\partial_t \omega' + L'_{\bar{\omega}}[\omega'] = -u_i' \partial^i \omega' + \partial^i \overline{u_i' \omega'} + \eta'.$$

Adiabatic reduction at lowest order[16]**:**

$$\partial_t \bar{\omega} + L_{\bar{\omega}}[\bar{\omega}] = -\partial^i \mathbb{E}_{\bar{\omega}}[\overline{u_i' \omega'}] + \bar{\eta},$$
$$\partial_t \omega' + L'_{\bar{\omega}}[\omega'] = \eta'.$$

- No UV divergences
- Eddy-eddy interactions do not contribute at leading order.

The fluctating vorticity field is an Ornstein-Uhlenbeck process characterized by the two-point correlation function $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}_{\bar{\omega}}[\omega'(\mathbf{r}_1, t)\omega'(\mathbf{r}_2, t)]$, which satisfies the Lyapunov equation:

$$\partial_t g + L'^{(1)}_{\bar{\omega}} g + L'^{(2)}_{\bar{\omega}} g = C',$$

with $C'(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbb{E}[\eta'(\mathbf{r}_1, t)\eta'(\mathbf{r}_2, t)]$ the correlation matrix of the Gaussian white noise η'.

[16] F. Bouchet et al. (2013).. *J. Stat. Phys.*

Adiabatic elimination of fast variables: zonal jets

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$$\partial_t \omega' + L'_{\bar{\omega}}[\omega'] = \eta'.$$

- No UV divergences
- Eddy-eddy interactions do not contribute at leading order.

Numerical simulations in the quasi-linear framework:

- Stochastic Structural Stability Theory[17]
- Cumulant Expansion "CE2"[18]

[16] F. Bouchet et al. (2013).. *J. Stat. Phys.*

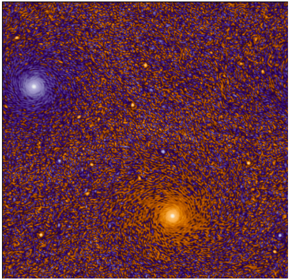
[17] B. F. Farrell and P. J. Ioannou (2003).. *J. Atmos. Sci.*

[18] S. M. Tobias and J. B. Marston (2013).. *Phys. Rev. Lett.* J. B. Marston et al. (2016).. *Phys. Rev. Lett.*

Explicit computations in the vortex condensate[19]

Let us go back to the periodic square box with small-scale random forcing:

$$\mathbf{v} = \mathbf{U} + \mathbf{u}, \text{ with } \mathbf{U} = U\mathbf{e}_\theta, \mathbf{u} = u\mathbf{e}_\theta + v\mathbf{e}_r \text{ and } \langle \mathbf{u} \rangle = 0,$$
$$\omega = \Omega + \omega', \text{ with } \langle \omega' \rangle = 0.$$
$$\partial_t \Omega + \mathbf{U} \cdot \boldsymbol{\nabla} \Omega = -\alpha \Omega - \boldsymbol{\nabla} \cdot \langle \mathbf{u} \omega' \rangle.$$



DNS: 1024^2, $k_f = 100$, hyperviscosity, $\alpha = 1.1 \times 10^{-4}$.

[19] C. Herbert, A. Frishman and G. Falkovich, to appear

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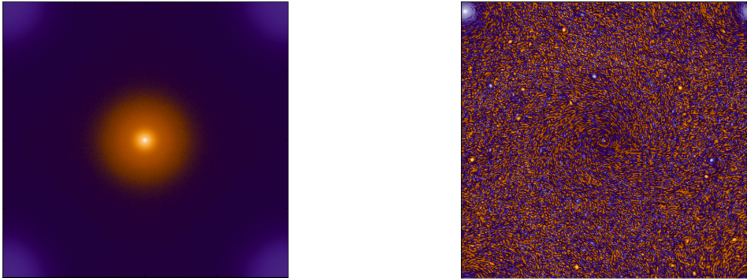


Ω ω'

DNS: 1024^2, $k_f = 100$, hyperviscosity, $\alpha = 1.1 \times 10^{-4}$.

[19] C. Herbert, A. Frishman and G. Falkovich, to appear

Explicit computations in the vortex condensate[20]

Let us go back to the periodic square box with small-scale random forcing:

$$\mathbf{v} = \mathbf{U} + \mathbf{u}, \text{ with } \mathbf{U} = U\mathbf{e}_\theta, \mathbf{u} = u\mathbf{e}_\theta + v\mathbf{e}_r \text{ and } \langle \mathbf{u} \rangle = 0,$$
$$\omega = \Omega + \omega', \text{ with } \langle \omega' \rangle = 0.$$
$$\partial_t \Omega + \mathbf{U} \cdot \nabla \Omega = -\alpha \Omega - \nabla \cdot \langle \mathbf{u} \omega' \rangle.$$

Timescale separation

Perturbative expansion of the equations of motion in $\delta = \alpha L^{2/3}/\varepsilon^{1/3} \ll 1$ leads at first order to (Momentum and energy balance)[19]:

$$r^{-1}\partial_r(r^2\langle uv \rangle) = -\alpha r U,$$
$$r^{-1}\partial_r(rU\langle uv \rangle) + \alpha U^2 = \varepsilon.$$

Solution:

$$U = \sqrt{3\varepsilon/\alpha}, \qquad \langle uv \rangle = -r\sqrt{\alpha\varepsilon/3}.$$

Therefore $\Omega(r) = \sqrt{3\varepsilon/\alpha}\, r^{-1}$.

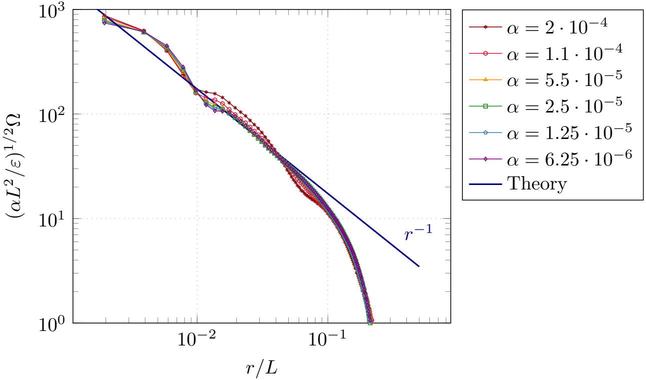
Global energy balance neglecting small-scale dissipation yields $U_{\text{rms}} = \sqrt{\varepsilon/\alpha}$.

[19] J. Laurie et al. (2014).. *Phys. Rev. Lett.*

[20] C. Herbert, A. Frishman and G. Falkovich, to appear

Explicit computation for the mean vorticity profile (DNS)[21]

Theoretical prediction: $\Omega(r) = \sqrt{3\varepsilon/\alpha}\, r^{-1}$

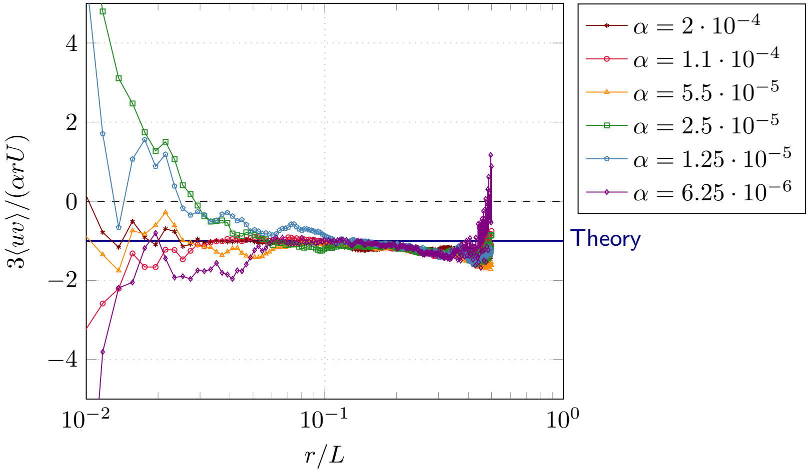


Our DNS (512^2 and 1024^2) support the α-scaling on a wide range of α, and seem compatible with the r-scaling.

[21] C. Herbert, A. Frishman and G. Falkovich, to appear

Explicit computation for the Reynolds tensor (DNS)[22]

$\langle uv \rangle / U^2 = O(\delta^{3/2})$ and not sign definite.



DNS: 512^2, $k_F = 100$, hyperviscosity, $\sim$ 300000 turnover times.

[22] C. Herbert, A. Frishman and G. Falkovich, to appear

Summary & Prospects

Due to the existence of a small parameter, we can close asymptotically the hierarchy of moments for the 2D Navier-Stokes equations, and compute the statistics of the mean-flow (e.g. vortex condensate, jets) and fluctuations.

Salient features of the theory

- Theoretical and Numerical arguments support the timescale separation hypothesis.
- Explicit formula for the mean-flow in the vortex condensate
- Explicit computation of the average Reynolds stress tensor agrees with long time DNS.
- Dominant interactions are non-local between mean-flow and fluctuations.

Prospects

- Slow dynamics of large-scale flow (e.g. zonal jets): attractors, fluctuations,. . .
- Large deviations of the Reynolds tensor

What do we learn about mean-flow-turbulence interactions in general flows?

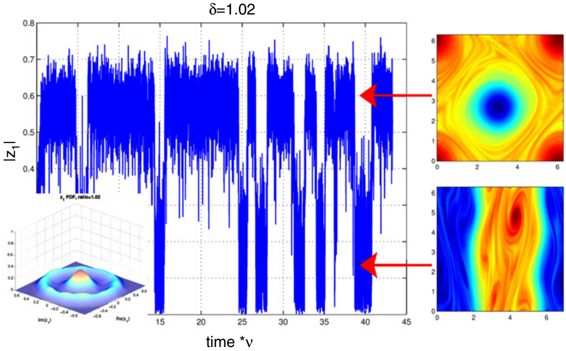
Outline

1. Introduction
2. Equilibrium Theory
3. Perturbative approach
4. Large deviations and transitions
5. Conclusion

Transitions in the stochastic 2D Navier-Stokes equations

Stochastic 2D Navier-Stokes equations on a double periodic domain with aspect ratio close to one[23].

$$z_1 = \int dxdye^{iy}\omega(x,y)$$



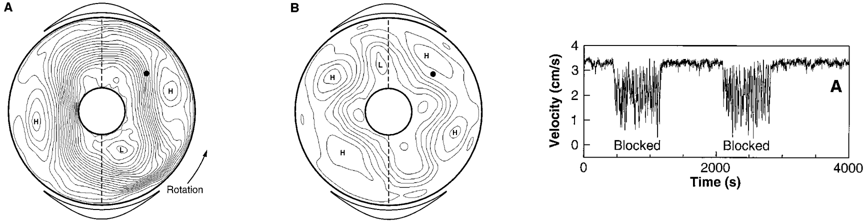
- Unidirectional flows: $|z_1| \approx 0$.
- Dipoles: $|z_1| > 0$.

Both states are close to stationary states of the Euler equations.

[23] F. Bouchet and E. Simonnet (2009).. *Phys. Rev. Lett.*

Zonal-Blocked transitions

Transitions between zonal and blocked states in rotating tank experiments[24]:

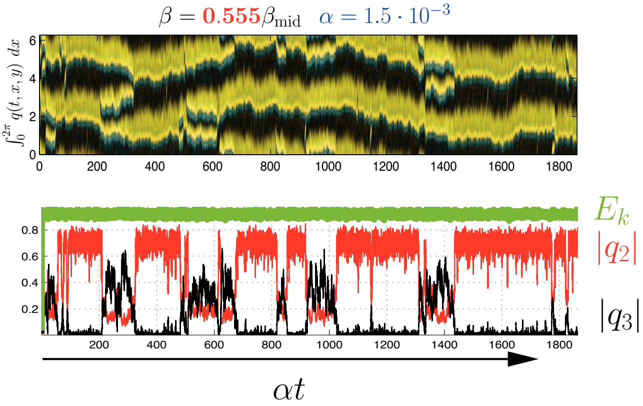


Connecting blocking and bistability is an old idea[25].

[24] E. R. Weeks et al. (1997).. *Science*; Y. D. Tian et al. (2001).. *J. Fluid Mech.*
[25] J. Charney and J DeVore (1979).. *J. Atmos. Sci.*

Rare transitions in jet dynamics

Zonal jets in the stochastic barotropic vorticity equation:

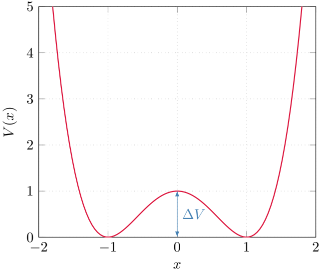


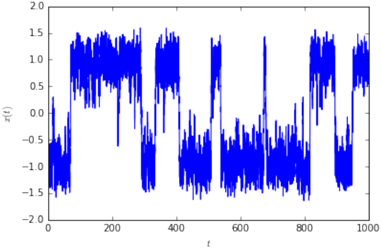
Simulations by Eric Simonnet (INLN).

Theoretical framework for noise induced transitions: the Kramers problem[26]

Overdamped Langevin dynamics:

$$\dot{x} = -V'(x) + \sqrt{2\epsilon}\eta, \quad V(x) = (x^2-1)^2, \quad \mathbb{E}[\eta(t)\eta(t')] = \delta(t-t').$$





[26] H. A. Kramers (1940).. *Physica*.

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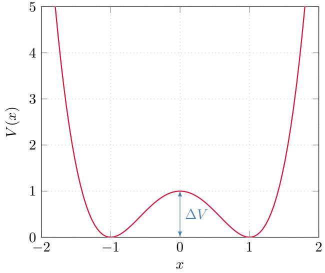

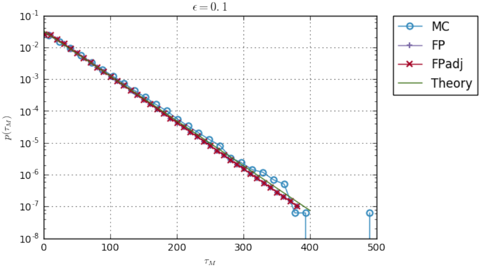

Transition probability

In the weak noise limit, transition times form a Poisson point process with transition rate λ, given by

$$\lambda = \tau^{-1} e^{-\Delta V/\epsilon}$$

This is a large deviation result.

26 H. A. Kramers (1940).. *Physica*.

Theoretical framework for noise induced transitions: the Kramers problem[26]

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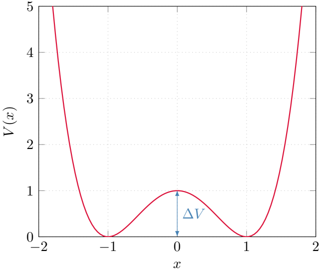


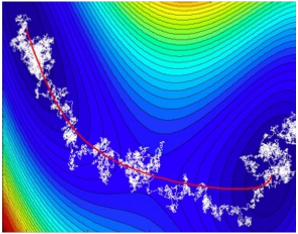


Fig. E. Vanden-Eijnden (Courant)

Instantons

Path integral formalism

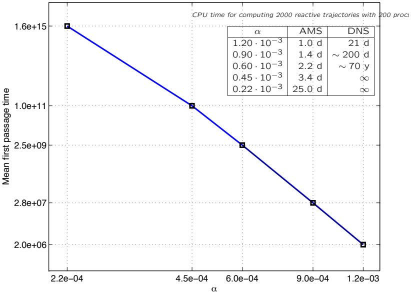
$$\mathbb{E}[\mathcal{O}] = \int \mathcal{D}[x]\mathcal{O}[x]\exp(-\mathcal{A}[x]/\epsilon), \quad \text{Action: } \mathcal{A}[x] = \frac{1}{4}\int dt(\dot{x} + V'(x))^2.$$

Instanton: most probable path: $\min_x\{\mathcal{A}[x] | x(-T) = -1, x(T) = 1\}$.

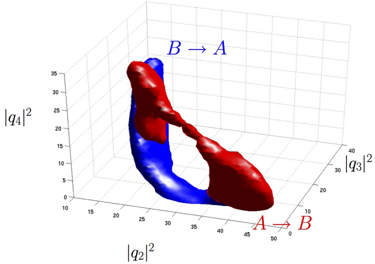
[26] H. A. Kramers (1940).. *Physica*.

Arrhenius law and Instantons in jet transitions

Numerical algorithms to compute large deviations: dynamics biased in a controlled way[27].



Arrhenius law



"Instantons"

Jet transition simulations with rare event algorithm (AMS) by Eric Simonnet (INLN).

27 C Giardina et al. (2011).. *J. Stat. Phys.* F. Cérou and A. Guyader (2007).. *Stoch. Anal. Appl.*

Summary and Prospects

Theoretical and numerical tools have recently been developped to study abrupt transitions in a statistical manner.

Exemples of quantities we can compute

- Probability of transition between attractors
- Most probable path (instanton theory)
- Large deviations of any observable

Recent developments

- More complex dynamics: bifurcations[28], non-gradient dynamics[29]
- Large deviations and return time for time-averaged observables[30]: applications for heat waves, cold spells, etc
- ...

[28] C. Herbert and F. Bouchet, to appear.

[29] F. Bouchet and J. Reygner (2016).. *Ann. Henri Poincaré*.

[30] T. Lestang, F. Ragone, C. Herbert and F. Bouchet, to appear

Summary

Developping statistical theory for 2D and geophysical turbulence

- The mean-field theory allows one to compute statistical equilibrium states, which correspond to observed large-scale structures.
- Time scale separation allows for perturbative closure of hierarchy of moments. Explicit computation for fundamental quantities in turbulence: mean flow and Reynolds tensor.
- Abrupt transitions in turbulent flows can be studied with large deviations theory and rare event algorithms.

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Tobias, S. M. and J. B. Marston (2013).. *Phys. Rev. Lett.* 110, p. 104502.
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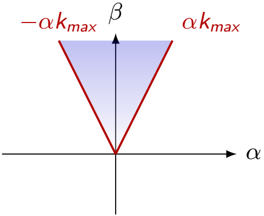
Waleffe, F. (1992).. *Phys. Fluids A* 4, p. 350.
Weeks, E. R. et al. (1997).. *Science* 278, p. 1598.
Xia, H et al. (2009).. *Phys. Fluids* 21.12, p. 125101.

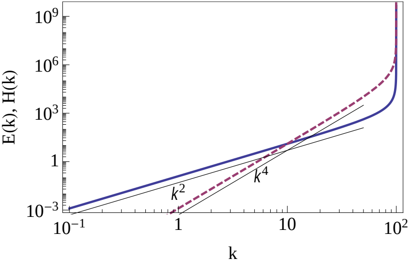
Canonical distribution for Galerkin-truncated 3D flows

Invariants: energy $E = 1/2\int \mathbf{u}^2$ and helicity $H = \int \mathbf{u}\cdot\boldsymbol{\omega}$.
The Liouville theorem holds[31].
Canonical probability density:

$$\rho(\{u_+(\mathbf{k}), u_-(\mathbf{k})\}) = \frac{1}{\mathcal{Z}} e^{-\beta E - \alpha H},$$
$$= \frac{1}{\mathcal{Z}} e^{-\sum_{\mathbf{k}}[(\beta+\alpha k)|u_+(\mathbf{k})|^2 + (\beta-\alpha k)|u_-(\mathbf{k})|^2]}.$$



Partition Function: $\mathcal{Z} = \prod_{\mathbf{k}} \frac{2\pi}{\sqrt{\beta^2 - \alpha^2 k^2}}$. $\beta > |\alpha| k_{\max} > 0$.



$$\langle E\rangle = -\frac{\partial \ln \mathcal{Z}}{\partial \beta},$$
$$= \sum_{\mathbf{k}} \frac{\beta}{\beta^2 - \alpha^2 k^2},$$
$$\langle E(k)\rangle = \frac{4\pi\beta k^2}{\beta^2 - \alpha^2 k^2}.$$

Ultraviolet Divergence[32]

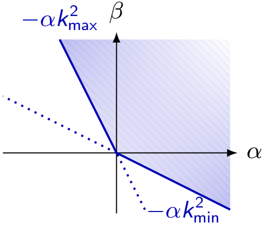
[31] T. D. Lee (1952).. *Q. Appl. Math.*
[32] R. H. Kraichnan (1973).. *J. Fluid Mech.*

Canonical distribution for Galerkin-truncated 2D flows

Invariants: Energy and Enstrophy:

$$\mathcal{E}[\omega] = \frac{1}{2}\int_{\mathcal{D}} \omega\psi = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{B}} \frac{|\omega_{\mathbf{k}}|^2}{k^2},$$

$$\mathcal{G}_2[\omega] = \frac{1}{2}\int_{\mathcal{D}} \omega^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{B}} |\omega_{\mathbf{k}}|^2.$$

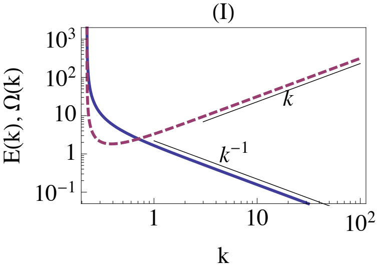


Canonical probability density[33]:

$$\rho(\{\omega_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{B}}) = \frac{1}{\mathcal{Z}} e^{-\beta\mathcal{E}[\omega]-\alpha\mathcal{G}_2[\omega]},$$

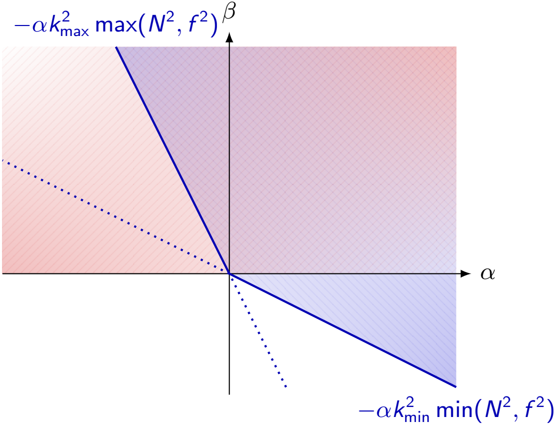
$$= \frac{1}{\mathcal{Z}} e^{-\sum_{\mathbf{k}\in\mathcal{B}}(\beta+\alpha k^2)\frac{|\omega_{\mathbf{k}}|^2}{2k^2}},$$

$$\langle E(k)\rangle = \frac{\pi k}{\beta + \alpha k^2}.$$



Infrared divergence in the $\beta < 0$ regime. Inverse cascade for 2D Turbulence.

[33] R. H. Kraichnan (1967).. *Phys. Fluids*; R. H. Kraichnan (1975).. *J. Fluid Mech.*



Accessible thermodynamic space for rotating-stratified flows, waves (red) and slow manifold (blue).

The helical decomposition for the 3D Euler equation

Euler equations for 3D homogeneous isotropic turbulence:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P,$$
$$\nabla \cdot \mathbf{u} = 0.$$

Helical decomposition in Fourier space[34]: $\nabla \times \mathbf{h}_\pm(\mathbf{k}) = \pm k \mathbf{h}_\pm(\mathbf{k})$,

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} [u_+(\mathbf{k})\mathbf{h}_+(\mathbf{k}) + u_-(\mathbf{k})\mathbf{h}_-(\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\boldsymbol{\omega}(\mathbf{x}) = \nabla \times \mathbf{u} = \sum_{\mathbf{k}} k[u_+(\mathbf{k})\mathbf{h}_+(\mathbf{k}) - u_-(\mathbf{k})\mathbf{h}_-(\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}}$$

Automatically enforces incompressibility: $\mathbf{k} \cdot \mathbf{h}_\pm(\mathbf{k}) = 0$.
Energy and Helicity:

$$E = \frac{1}{2} \int \mathbf{u}(\mathbf{x})^2 d\mathbf{x} = \frac{1}{2} \sum_{\mathbf{k}} [|u_+(\mathbf{k})|^2 + |u_-(\mathbf{k})|^2],$$
$$H = \frac{1}{2} \int \mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x}) d\mathbf{x} = \frac{1}{2} \sum_{\mathbf{k}} k[|u_+(\mathbf{k})|^2 - |u_-(\mathbf{k})|^2].$$

[34] A Craya (1958).. *Publ. Sci. Tech. Ministère de l'Air*; J. R. Herring (1974).. *Phys. Fluids*; F. Waleffe (1992).. *Phys. Fluids A*

Macrostates and global constraints[35]

Coarse-grained vorticity field:

$$\overline{\omega}_i = \frac{1}{n}\sum_{\alpha=1}^{n}\omega_{i\alpha} = \sum_{k=1}^{K}\sigma_k p_{ik}.$$

- The energy does not depend on the microstate but only on the macrostate

$$\mathcal{E}[\hat{\omega}] = \frac{1}{2N^2}\sum_{(i,\alpha)\neq(j,\beta)} G_{i\alpha,j\beta}\omega_{i\alpha}\omega_{j\beta},$$
$$= \frac{1}{2M^2}\sum_{i\neq j} G_{ij}\overline{\omega}_i\overline{\omega}_j + o\left(\frac{1}{n}\right).$$

- For $\hat{\omega} \in \mathfrak{M}(P)$,

$$\nu_k^T[\hat{\omega}] = \sum_{i=1}^{N}\nu_{ik}[\hat{\omega}] = n\sum_{i=1}^{N} p_{ik},$$

Global vorticity distribution constraints:

$$\frac{\nu_k^T[P]}{N} = \gamma_k.$$

[35] C. Herbert (2015).. In: *Stochastic Equations for Complex Systems: Theoretical and Computational Topics*. Ed. by S. Heinz and H. Bessaih. Springer

The mean-field approach: thermodynamic limit

Microstates

$$\hat{\omega} = (\omega_{i\alpha})_{\substack{1\le i\le M\\ 1\le\alpha\le n}} \in \mathfrak{S}^N \underset{n,M,K\to+\infty}{\longrightarrow} \omega(\mathbf{r}) \in L^2(\mathcal{D})$$

Macrostates

$$P = (p_{ik})_{\substack{1\le i\le M\\ 1\le k\le K}} \in [0,1]^{MK} \underset{n,M,K\to+\infty}{\longrightarrow} \rho(\sigma,\mathbf{r})$$

$$\forall i \in [\![1,M]\!], \sum_{k=1}^{K} p_{ik} = 1 \underset{n,M,K\to+\infty}{\longrightarrow} \forall \mathbf{r}\in\mathcal{D}, \int_{\mathbb{R}} \rho(\sigma,\mathbf{r})d\sigma = 1$$

$$\overline{\omega}_i = \frac{1}{n}\sum_{\alpha=1}^{n}\omega_{i\alpha} = \sum_{k=1}^{K}\sigma_k p_{ik} \underset{n,M,K\to+\infty}{\longrightarrow} \overline{\omega}(\mathbf{r}) = \int_{\mathbb{R}} \sigma\rho(\sigma,\mathbf{r})d\sigma$$

$$\mathscr{S}_{M,K}[P] = -\frac{1}{M}\sum_{i=1}^{M}\sum_{k=1}^{K} p_{ik}\ln p_{ik} \underset{n,M,K\to+\infty}{\longrightarrow} \mathscr{S}[\rho] \equiv -\int_{\mathcal{D}} d\mathbf{r}\int_{\mathbb{R}} d\sigma\rho(\sigma,\mathbf{r})\ln\rho(\sigma,\mathbf{r})$$

Constraints

$$\frac{1}{2}\sum_{i,j=1}^{M} G_{ij}\overline{\omega}_i\overline{\omega}_j = E \underset{n,M,K\to+\infty}{\longrightarrow} \mathscr{E}[\rho] \equiv \frac{1}{2}\int_{\mathcal{D}^2} d\mathbf{r}d\mathbf{r}' G(\mathbf{r},\mathbf{r}')\overline{\omega}(\mathbf{r})\overline{\omega}(\mathbf{r}') = E$$

$$\forall k \in [\![1,K]\!], \frac{1}{M}\sum_{i=1}^{M} p_{ik} = \gamma(\sigma_k) \underset{n,M,K\to+\infty}{\longrightarrow} \forall\sigma\in\mathbb{R}, \mathscr{D}_\sigma[\rho] \equiv \int_{\mathcal{D}} \rho(\sigma,\mathbf{r})d\mathbf{r} = \gamma(\sigma)$$

The mean-field equation for the coarse-grained vorticity field

Mean-field equation:

$$\overline{\omega}(\mathbf{r}) = F_{\beta,\alpha}(\overline{\psi}(\mathbf{r})), \qquad \text{with } F_{\beta,\alpha}(u) = -\frac{1}{\beta}\frac{d\ln \mathcal{Z}_{\beta,\alpha}(u)}{du}.$$

In particular, the equilibrium coarse-grained vorticity field is a *stationary solution of the 2D Euler equation*. Further, it is *dynamically stable*.

In general, this equation is difficult to solve:

- Nonlinear partial differential equation.
- Analytic computation of the partition function $\mathcal{Z}_{\beta,\alpha}(u)$ is rarely possible.
- Relate *a posteriori* the Lagrange parameters $\beta, \alpha(\sigma)$ to invariants $E, \gamma(\sigma)$.

Numerical methods: relaxation equations[36], Turkington-Whitaker algorithm[37],...

When the function $F_{\beta,\alpha}$ is linear, the mean-field equation can be solved analytically. When does this happen?

- "Strong mixing" limit[38]: $\beta \to 0$, or "low-energy" limit: $\overline{\psi} \to 0$.
- Energy-enstrophy variational problem
- Subclass of the full MRS equilibrium states[39].

Then analytical computations are possible, by introducing the eigenmodes of the Laplacian on the domain $\mathcal{D}$.

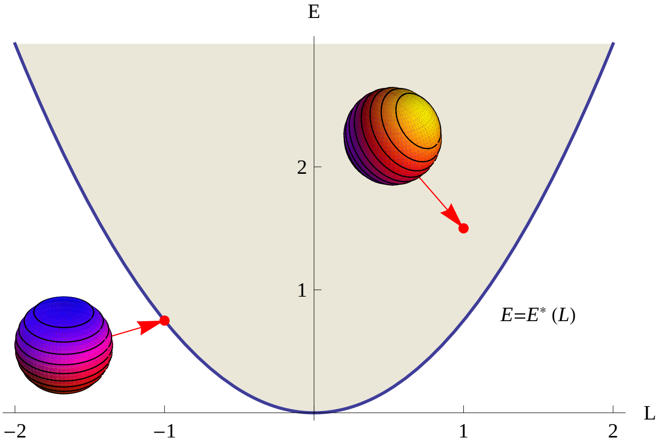
[36] R. Robert and J. Sommeria (1992).. *Phys. Rev. Lett.* P.-H. Chavanis (2009).. *Eur. Phys. J. B*

[37] B. Turkington and N. Whitaker (1996).. *SIAM J. Sci. Comput.*

[38] P.-H. Chavanis and J. Sommeria (1996).. *J. Fluid Mech.*

[39] F. Bouchet (2008).. *Physica D*

Microcanonical Phase Diagram



Second-order phase transition with spontaneous symmetry breaking.[40]

[40] C. Herbert et al. (2012).. *J. Stat. Mech.*