

Non perturbative scalar  
field dynamics in  
de Sitter space

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# Motivations

- ▣ Radiative corrections to inflationary dynamics
- ▣ (Analog) black hole radiation
- ▣ Curvature-induced phase transitions
- ▣ Foundations of QFT in curved space-times



# Scalar fields in de Sitter space (I)

$$ds^2 = -dt^2 + \bar{a}^2(t) d\vec{X}^2$$

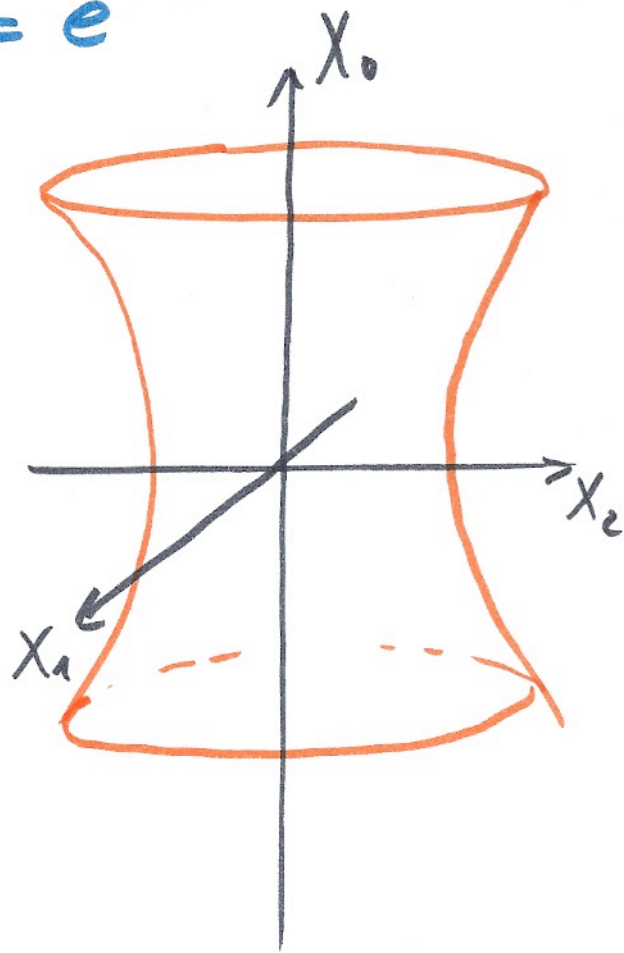
$$\bar{a}(t) = e^{Ht}$$

$$\square d\eta = dt / \bar{a}(t)$$

$$ds^2 = \bar{a}^2(\eta) (-d\eta^2 + d\vec{X}^2)$$

spatially homogeneous  
but nonstationary

$$\square \vec{x} = a(t) \vec{X}$$



$$ds^2 = -(1-x^2)dt^2 - 2\vec{x} \cdot d\vec{x} dt + d\vec{x}^2$$

stationary but inhomogeneous



# Scalar fields in dS space (II)

$$S = \int d^D x \sqrt{-g(x)} \left( \frac{1}{2} \phi \square \phi - \frac{m^2}{2} \phi^2 \right)$$

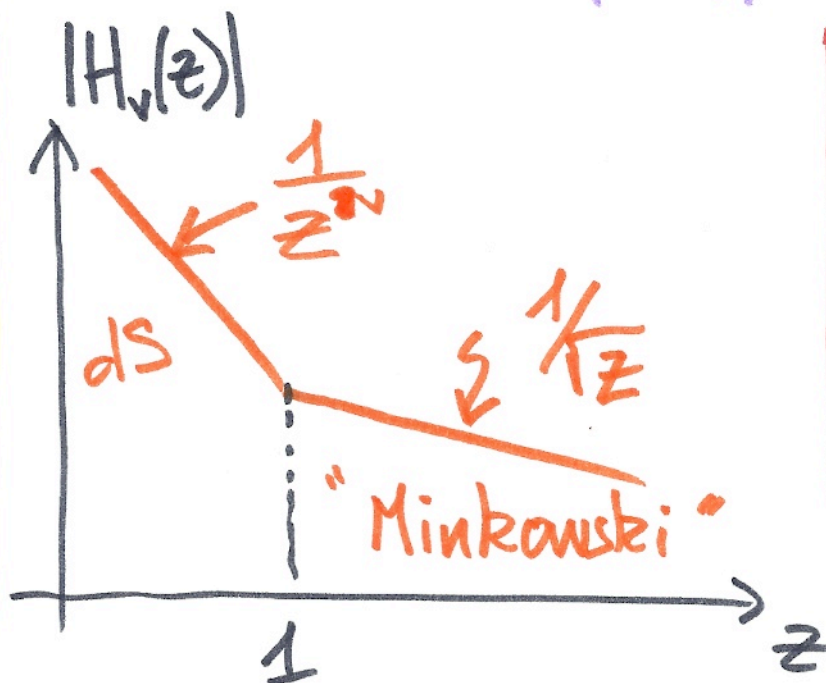
$$D = d + 1; \quad \square = \frac{1}{a^2(\eta)} \left( -\partial_\eta^2 + \frac{d-1}{\eta} \partial_\eta + \vec{\nabla}_x^2 \right)$$

$$(-\square + m^2) \phi = 0$$

$$\phi(\eta, \vec{x}) \sim \int \frac{d^d k}{(2\pi)^d} \left( e^{i \vec{k} \cdot \vec{x}} H_\nu \left( \frac{k}{a(\eta)} \right) a_k + \text{h.c.} \right)$$

$$\nu = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$$

Redshift.



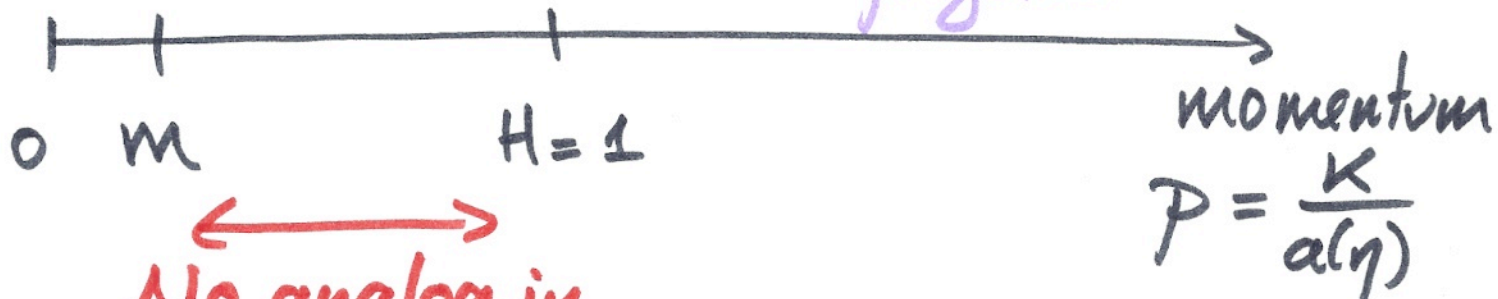
Stationary  
gravitational redshift  
leads to strong  
infrared (IR)  
fluctuations



# Scalar fields in dS (III)

The case of light fields  $m \ll H = 1$

← ... Minkowski physics



Loop corrections :

$$\text{Loop} \sim \frac{H^4}{m^2} : \text{IR divergencies}$$

$$\text{Loop} \sim H^2 \ln\left(\frac{P}{H}\right) : \text{large logs (secular divergencies)}$$



NEED FOR RESUMMATION

# Resummation / nonperturbative methods in de Sitter.

difficulty : nonequilibrium system

- Stochastic approach  
[Starobinsky, Yokoyama ('94)]
  - Euclidean de Sitter  
[Rajaraman ('10); Beneke, Roth ('13)]
  - Dynamical R.G.  
[Burgess et al. ('10)]
  - Wigner-Weisskopf method  
[Boyanovsky ('12)]
- 
- Large -  $N$   
[Riotto, Sloth ('08); Mazzitelli, Pae ('09); J.S. ('11)]
  - Dyson-Schwinger equations  
[Garbrecht, Rigopoulos ('11); Akhmedov, Burda ('12)  
Parentani, J.S. ('12); Gautier, J.S. ('13)]
  - Nonperturbative / Functional R.G.  
[Kaya ('13); Serreau ('14); Guilleux; JS ('15)]

Application :  $\alpha(N)$  theory

$$S = \int d^D x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!N} (\varphi^2)^2 \right)$$



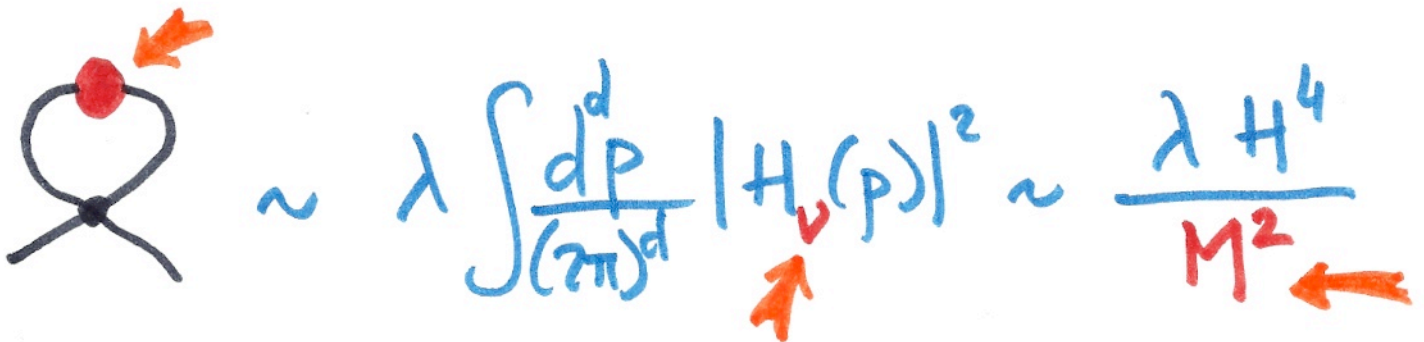
Large  $N$

[Mazzitelli, Pàz (09),  
Riotto, Slotk (08),  
Serreau (11)]

Local IR terms:


$$\text{tadpole} + \text{self-energy} + \dots + \text{complex diagram} \sim N$$

Effective mass resummation


$$\text{tadpole} \sim \lambda \int \frac{d^d p}{(2\pi)^d} |H_\nu(p)|^2 \sim \frac{\lambda H^4}{M^2}$$

GAP EQUATION

$$M^2(\phi) = \frac{m^2 + \lambda \phi^2}{m_a^2(\phi)} + c \frac{\lambda H^4}{M^2(\phi)}$$

$$M^2(\phi) = \frac{m_a^2(\phi)}{2} + \sqrt{\frac{m_a^4(\phi)}{4} + c \lambda H^4}$$

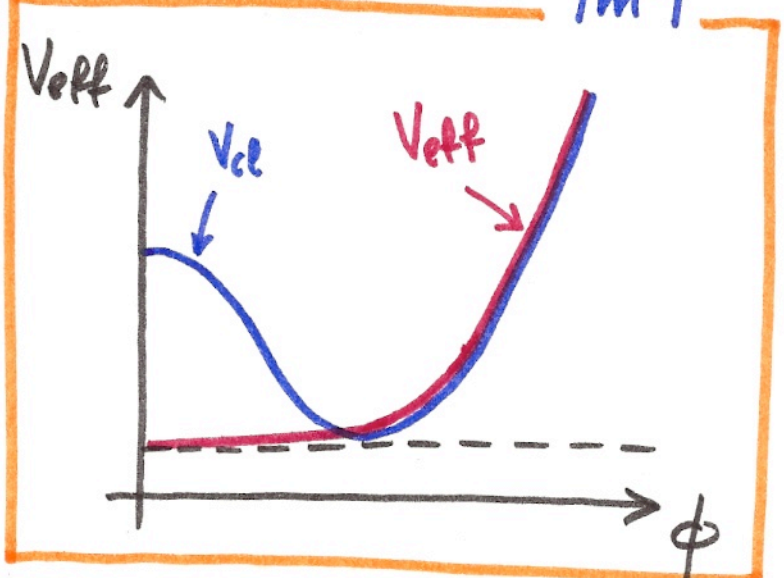
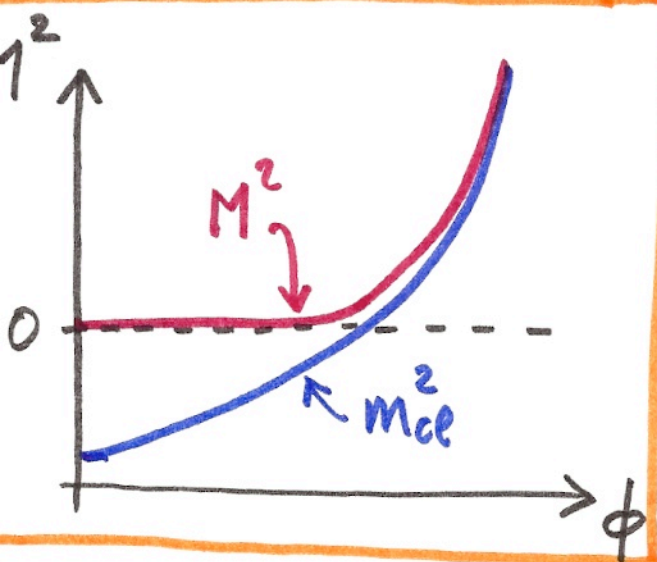
# Effective potential : symmetry restoration

[Serreau ('11)]

$$M^2(\phi^2) = \frac{m_{ce}^2(\phi^2)}{2} + \sqrt{\left(\frac{m_{ce}^2(\phi^2)}{2}\right)^2 + c\lambda H^4}$$

$$V_{\text{eff}}(\phi) = \int_0^{\phi^2} du M^2(u)$$

- ①  $m_{ce}^2(\phi^2) > 0 \rightsquigarrow M^2(\phi^2) \approx m_{ce}^2(\phi^2)$
- ②  $m_{ce}^2(\phi^2) = 0 \rightsquigarrow M^2(\phi^2) \approx \sqrt{c\lambda} H^2$   
[Starobinsky, Yokoyama ('94)]
- ③  $m_{ce}^2(\phi^2) < 0 \rightsquigarrow M^2(\phi^2) \approx \frac{c\lambda H^4}{|m_{ce}^2|}$



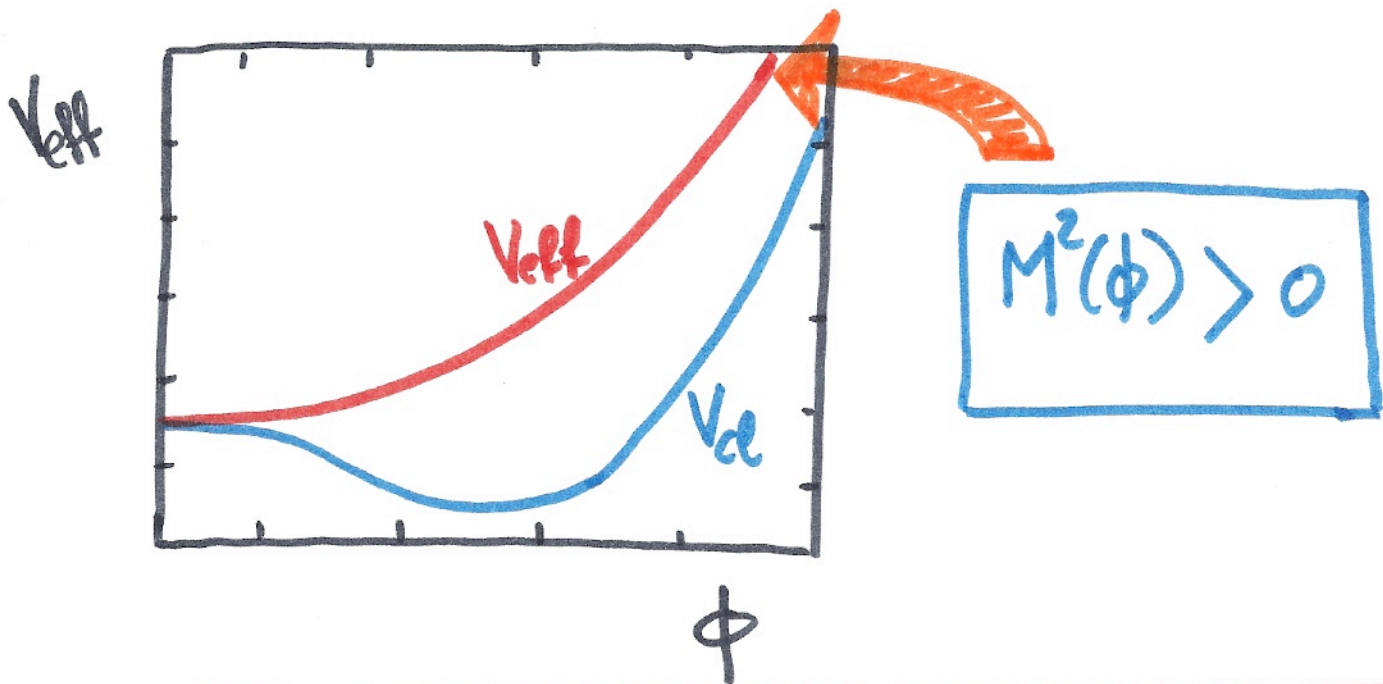


# Large N : Effective potential

[Serreau, PRL (11)]

$$V_{\text{eff}}(\phi) = \int_0^{\phi^2} dx \frac{M^2(x)}{2}$$

$$= \frac{3}{2\lambda} (M^4(\phi) - M^4(0)) + \frac{3H^4}{16\pi^2} \ln \frac{M^2(\phi)}{M^2(0)}$$



Radiative symmetry restoration

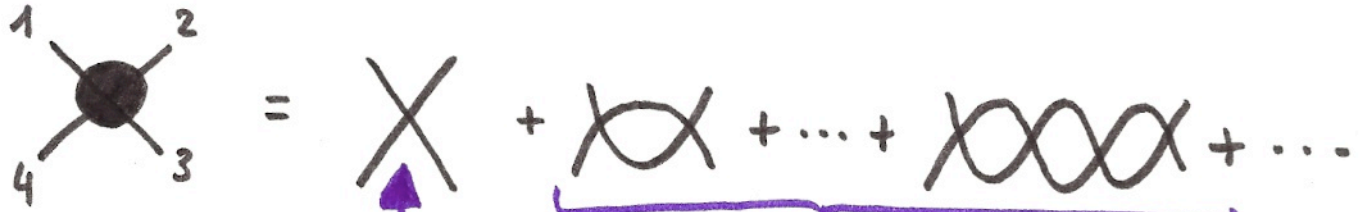
$\forall d, \forall H$

[See also: Ratra ('85); Mazzitelli et al. (89), Lazzari et al. (13)]



# Four-point vertex : infrared logs.

[Serreau, Parentani ('13)]



$$\Gamma_{1234}^{(4)} = -\lambda \delta_{12} \delta_{13} \delta_{14} + i \delta_{12} \boxed{\Pi_{13}} \delta_{34} + \text{perm.}$$

$$\mathbf{I}(x, x') = \Pi(x, x') + i \int d^D z \Pi(x, z) \mathbf{I}(z, x')$$

with  $\Pi(x, x') = -\frac{\lambda}{6} G^2(x, x') = x \text{ --- } \text{bubble} \text{ --- } x'$

## P-representation :

$$\hat{\Pi}(p, p') = -\frac{\lambda}{6} \int \frac{d^d q}{(2\pi)^d} \frac{\hat{G}(q, p, p')}{q} \frac{\hat{G}(r, p, p')}{r}$$

$r = |\vec{q} + \vec{e}|$ ,  $\vec{e}$  unit vector

$$\hat{\mathbf{I}}(p, p') = \hat{\Pi}(p, p') - i \int_C ds \hat{\Pi}(p, s) \hat{\mathbf{I}}(s, p')$$

- ➡ Each bubble brings IR logs  $\sim \ln(P/H)$
- ➡ The integral (SD) equation resums the infinite series of bubble diagrams



The IR dynamics is one-dimensional

IR modes  $q, p \lesssim H$  dominate the loop integral for IR external momenta  $p, p' \lesssim H$

$$v = \sqrt{\frac{d^2}{4} - \frac{M^2}{H^2}} \equiv \frac{d}{2} - \epsilon \quad ; \quad \epsilon \ll 1$$

$$\hat{\Pi}_F(p, p') = \frac{\pi_F}{(pp')^{\kappa+1/2}} \quad ; \quad \hat{\Pi}_p(p, p') = \frac{\pi_p}{\sqrt{pp'}} P_v^\epsilon(\ln p/p')$$

$$\kappa = v - \epsilon \quad ; \quad P_v^\epsilon(x) = \frac{\text{sh } vx}{v} e^{-\epsilon|x|}$$

The bubble summation equation rewrites

$$\hat{\Pi}_F(p, p') = \frac{\pi_F H^{2\kappa}}{\sqrt{pp'}} \bar{A}(\ln \frac{p}{H}) \bar{A}(\ln \frac{p'}{H})$$

$$\hat{\Pi}_p(p, p') = \frac{\pi_p}{\sqrt{pp'}} \bar{I}(\ln p/p')$$

$$\bar{I}(x) = P_v^\epsilon(x) + \pi_p \int_x^0 dy P_v^\epsilon(x-y) \bar{I}(y)$$

$$\bar{A}(x) = e^{-\kappa x} + \pi_p \int_x^0 dy \bar{I}(x-y) e^{-\kappa y}$$

One-dimensional integral equation

## Exact Solution

$$P_v^\varepsilon(x) = \frac{\text{sh}(vx)}{v} e^{-\varepsilon|x|}$$

$$\bar{I}(x) = P_v^\varepsilon(x) + \pi\rho \int_x^0 dy P_v^\varepsilon(x-y) \bar{I}(y)$$

$$\bar{I}(x) = P_{\bar{v}}^\varepsilon(x) \quad \text{with} \quad \bar{v} = \sqrt{v^2 + \pi\rho}$$

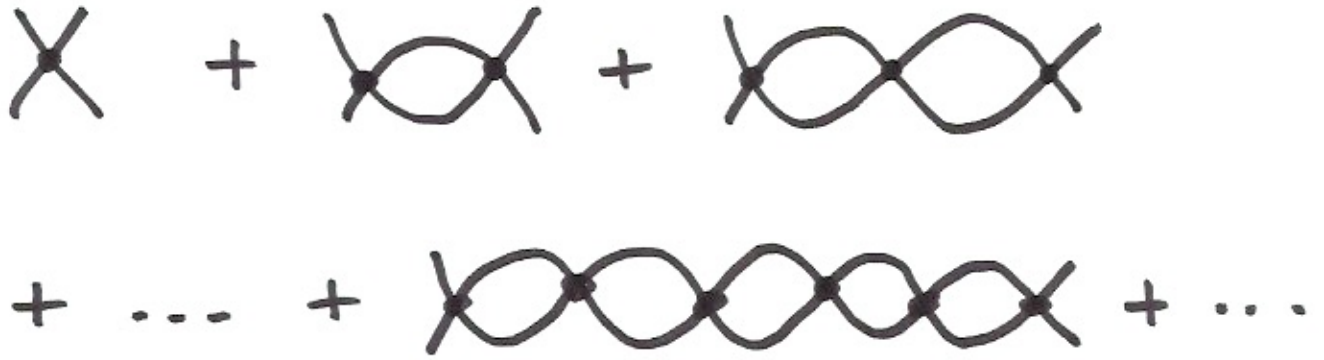
$$\hat{\Pi}_F(p, p') = \frac{\pi F}{(pp')^{\bar{\kappa} + 1/2}} ; \quad \bar{\kappa} = \bar{v} - \varepsilon$$

$$\hat{\Pi}_\rho(p, p') = \frac{\pi\rho}{\sqrt{pp'}} P_{\bar{v}}^\varepsilon(\ln p/p')$$

The dangerous logs resum to a modified power law

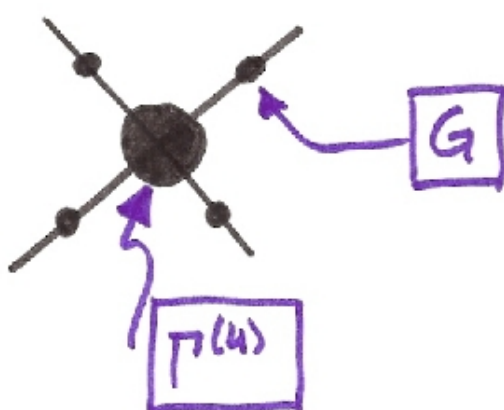
- Akin to anomalous dimension in critical phenomena
- Akin to mass correction in dS





# Four-point correlator : non Gaussianities

[Serreau (13)]

$$\langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle =$$


$$= \frac{\lambda}{3N} \frac{F_v^3}{2v} \frac{(-\eta)^{2-4v}}{(k_1 \dots k_4)^{2v}} \text{dab dcd} \quad g\left(\ln \frac{\eta}{\eta_0}, \vec{k}_i\right) + \text{perm.}$$

ex: massless case ( $m_{\text{eff}}^2 = 0$ ) in the deep IR

$$g(x, \vec{k}_i) = -\frac{1}{4\varepsilon} (k_1^{2v} + \dots + k_4^{2v}) + \frac{1}{8\varepsilon} \frac{(k_1^{2v} + k_2^{2v})(k_3^{2v} + k_4^{2v})}{|\vec{k}_1 + \vec{k}_2|^{2v}}$$

➡ Non perturbative enhancement of loop contributions due to IR effects

➡ Loop contrib's are of the same order as tree level ones. Riotto, Slotk (08)

# Dyson-Schwinger Eqs. (DSE)

## Nonlocal IR logarithms



A Feynman diagram showing a horizontal line representing a propagator. A loop is attached to this line, with the label  $\Sigma$  above it. The two external legs of the loop are labeled  $G_0$ .

$$\sim \lambda^2 H^2 \ln(P/H)$$

➔ need to resum the series



A series of diagrams representing the Dyson equation. It starts with a single circle on a line, followed by a plus sign, then two circles on a line, followed by a plus sign and an ellipsis.

DSE :

$$G^{-1} = G_0^{-1} - \Sigma$$

$$(\square + m^2)G(x,y) - \int_z \Sigma(x,z)G(z,y) = -i\delta(x-y)$$

Integro-diff. eqn. with nontrivial metric

[Akhmedov et al. (12), (14); Gauthier, J.S (13)  
Garbrecht, Rigopoulos (11)]



# DSE : p-representation

$$\underline{G}(x, x') = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \underline{\tilde{G}}(t, t', k)$$

$$\underline{G}(t, t', k) = \frac{\hat{G}(p, p')}{k [a(t) a(t')]^{\frac{D-2}{2}}}$$

$$p = \frac{k}{a(t)}, \quad p' = \frac{k}{a(t')}$$

Similarly :  $\underline{\tilde{\Sigma}}(t, t', k) = k^3 \frac{\hat{\Sigma}(p, p')}{[a(t) a(t')]^{\frac{D+2}{2}}}$

$$\left( \partial_p^2 + 1 - \frac{v^2 - 1/4}{p^2} \right) \hat{G}(p, p')$$

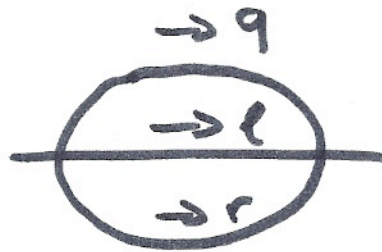
$$+ \int dp'' \hat{\Sigma}(p, p'') \hat{G}(p'', p') = i \delta(p - p')$$

$$v = \sqrt{\frac{d^2}{4} - \frac{M^2}{H^2}}$$

d+1 dimensional  
system

DSE : two-loop

[Gautier, J.S., PLB (1983)]



$$\hat{\Sigma}(p, p') \propto \lambda^2 \int_{q, l, r} \frac{\hat{G}_0(q, p, q, p') \hat{G}_0(l, p, l, p') \hat{G}_0(r, p, r, p')}{q l r}$$

Exact analytical solution  
of DSE for  $p, p' \ll 1$

$$\hat{G}(p, p') = C_+ G_{M_+}^{\text{free}}(p, p') + C_- G_{M_-}^{\text{free}}(p, p')$$

$$C_+ + C_- = 1$$

IR logs. resum to modified  
power laws + "splitting"



DSE :  $1/N$  expansion

$$\Sigma^{NLO} = \text{[circle with horizontal line]} + \text{[circle with two internal lines]} + \dots + \text{[circle with four internal lines]} + \dots$$

$$= \text{[circle with dashed top arc]} \sim \frac{1}{N}$$

➔ Exact expression for IR momenta

➔ Exact solution of the DSE at NLO in  $1/N$

[Gautier, Serreca ('15)]



# Two-point correlator

$$G(z) \propto \frac{C_+}{M_+^2 z^{M_+^2/d}} + \frac{C_-}{M_-^2 z^{M_-^2/d}}$$

⇒ Coincidence limit :  $z = 1$

$$\langle \varphi^2(x) \rangle = G(1) \propto \frac{C_+}{M_+^2} + \frac{C_-}{M_-^2} \equiv \frac{1}{M_{\text{dyn}}^2}$$

Agrees with the Stochastic approach

⇒ Deep infrared (late time) limit

$$G(z) \sim \frac{C_+}{M_+^2} \frac{1}{z^{M_+^2/d}}$$

Amplitude

Power-law decay

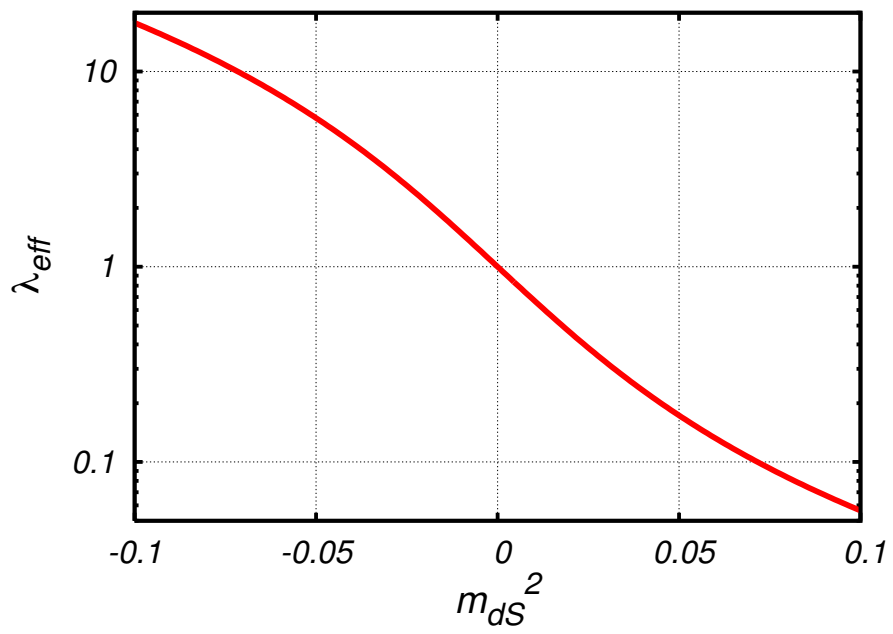
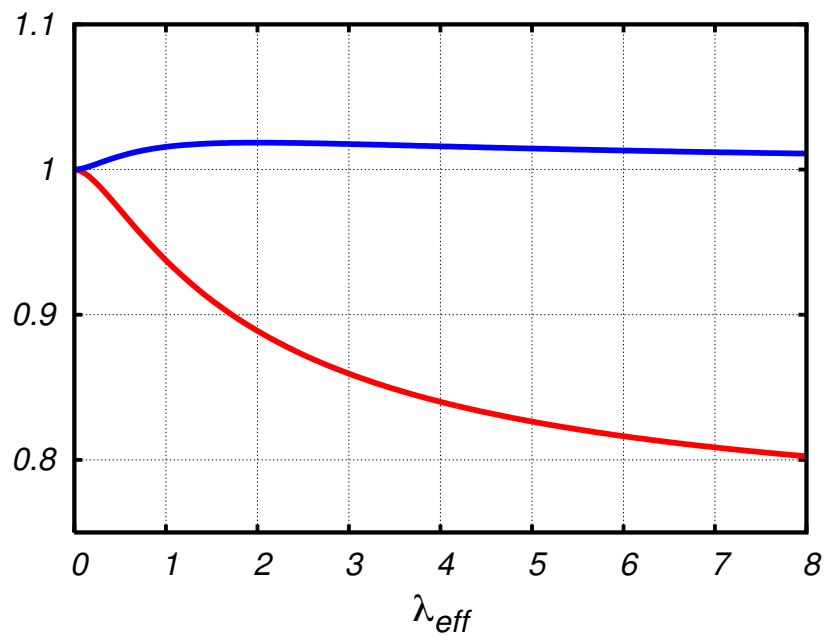
# Results at NLO in $1/N$

$$\frac{M_{\text{dyn}}^2}{M_{\text{LO}}^2} = 1 + \frac{2}{N} \frac{\lambda_{\text{eff}}}{(1 + \lambda_{\text{eff}})^2}$$

$$\frac{M_+^2}{M_{\text{LO}}^2} = 1 + \frac{1}{N} \frac{\lambda_{\text{eff}}(2 - \lambda_{\text{eff}})}{(1 + \lambda_{\text{eff}})^2}$$

$$\frac{M_+^2}{C_+ M_{\text{LO}}^2} = 1 + \frac{1}{2N} \frac{\lambda_{\text{eff}}(4 + 5\lambda_{\text{eff}})}{(1 + \lambda_{\text{eff}})^3}$$

$$\lambda_{\text{eff}} = \frac{\lambda}{6\Omega_{\text{DHI}} M_{\text{LO}}^4}$$



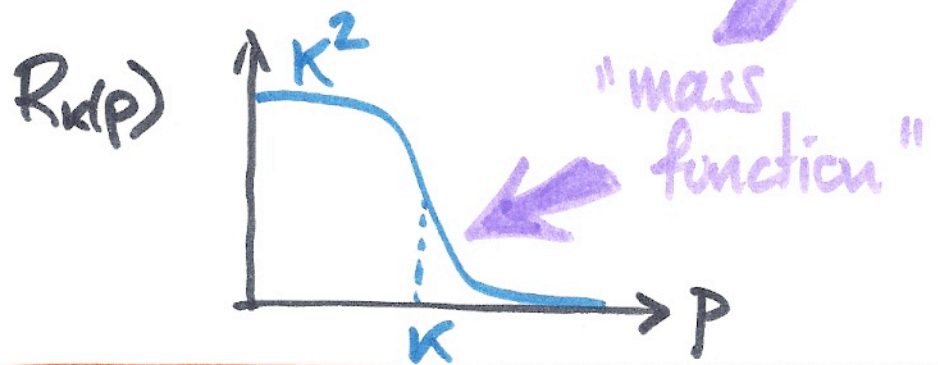


# Nonperturbative Renormalization Group (NPRG)

[Kaya ('13); Serreau ('14)]

## Infrared regulator:

$$S[\varphi] \longrightarrow S[\varphi] + \frac{1}{2} \int_{x,y} \varphi(x) R_\kappa(x,y) \varphi(y)$$



## Regulated effective action $\Gamma_\kappa[\phi]$

$$\partial_\kappa \Gamma_\kappa[\phi] = \frac{1}{2} \text{Tr} \left\{ \partial_\kappa R_\kappa \cdot (\Gamma_\kappa^{(2)} + R_\kappa)^{-1} \right\}$$

[Wetterich ('93)]



# NPRG : Local Potential Approx. (LPA)

$$\Gamma_k[\phi] = - \int_x \left\{ \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + V_k(\phi) \right\}$$

Full field dependence

+ choose  $R_k(t, t', p) = \delta(t - t') (\kappa^2 - p^2) \Theta(\kappa^2 - p^2)$

$$\kappa \partial_\kappa V_k = \frac{C_d \kappa^{d+2}}{\kappa^2 + V_k''} \mathcal{B}_d(V_k, \kappa)$$

$$C_d = \frac{\pi}{16d} \frac{\Omega_d}{(2\pi)^d}, \quad v_k = \sqrt{\frac{d^2}{4} - V_k''}$$

$$\mathcal{B}_d(v, \kappa) = e^{-\pi \text{Im}(v)} \left\{ (d^2 - 2v^2 + 2\kappa^2) |H_v(\kappa)|^2 + 2\kappa^2 |H'_v(\kappa)|^2 - 2d\kappa \text{Re}[H_v^*(\kappa) H'_v(\kappa)] \right\}$$

Kaya ('13) Guilleux, Serreau ('15)



From UV to IR : onset  
of gravitational effects

→  $\kappa \gtrsim H$

$$\kappa \partial_\kappa V_\kappa \approx \frac{8C_d}{\pi} \frac{\kappa^{d+2}}{\sqrt{\kappa^2 + V_\kappa''}}$$

Minkowski Regime

→  $\kappa \lesssim H$  and  $|V_\kappa''| \ll H^2$

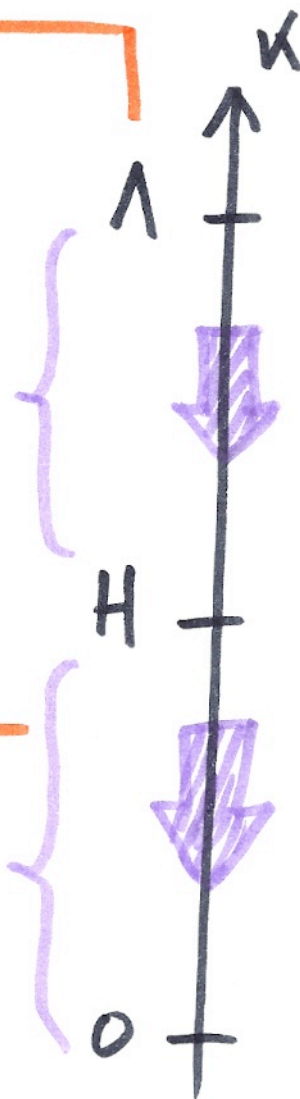
$$\kappa \partial_\kappa V_\kappa \approx \frac{1}{\sqrt{2d+1}} \frac{\kappa^2}{\kappa^2 + V_\kappa''}$$

[Compare to  $\frac{\kappa^{d+2}}{\kappa^2 + V_\kappa''}$  in flat Euclidean space]



Effective dimensional reduction

$$D_{\text{eff}} = 0$$



# Zero-dimensional field theory

$$e^{-\mathcal{L}_{\text{DH}} W_\kappa(J)} = \int d\varphi e^{-\mathcal{L}_{\text{DH}} (V_{\text{eff}}(\varphi) + J\varphi + \frac{\kappa^2}{2} \varphi^2)}$$

$$V_\kappa(\phi) = W_\kappa(J) - J\phi - \frac{\kappa^2}{2} \phi^2$$

with  $W'_\kappa(J) \equiv \phi$

$$\kappa \partial_\kappa V_\kappa(\phi) = \frac{1}{\mathcal{L}_{\text{DH}}} \frac{\kappa^2}{\kappa^2 + V_\kappa''(\phi)}$$

Adjust  $V_{\text{eff}}$  by matching  
at the scale  $\kappa \approx H$

$$V_{\text{eff}}(\varphi) \approx V_H(\varphi)$$



## Relation to the stochastic approach

[Starobinsky, Yokoyama ('94)]

Effective Langevin eqn. for IR modes

$$\dot{\varphi} + \frac{1}{d} V'_{\text{soft}}(\varphi) = \xi \quad ; \quad \langle \xi(t) \xi(t') \rangle = \frac{2}{d \Omega_{D+1}} \delta(t-t')$$

Fokker-Planck equation

$$\partial_t P(\varphi, t) = \frac{1}{d} \frac{\partial}{\partial \varphi} \left\{ V'_{\text{soft}} P + \frac{1}{\Omega_{D+1}} P' \right\}$$

$$P(\varphi, t \rightarrow \infty) \propto \exp \left\{ -\Omega_{D+1} V_{\text{soft}} \right\}$$

$$\langle \varphi^2 \rangle = \int d\varphi P(\varphi) \varphi^2 = \frac{1}{\Omega_{D+1}} \underline{\underline{W''_{k=0}(\mathcal{J}=0)}}$$

provided  $V_{\text{soft}} = V_{\text{eff}} \approx V_H$

[see also Rigopoulos, Prokopec ('18)]

# Relation to Euclidean de Sitter

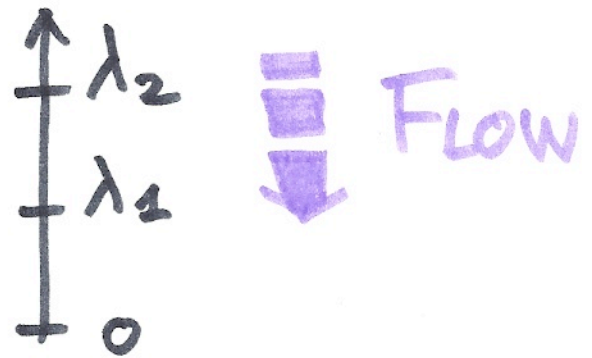
[Rajaraman ('80); Beneke, Pock ('13); Benedetti ('15)]

$$\text{NPRG on } S_D : S \rightarrow S + \frac{1}{2} \int_{xy} \varphi \underline{R}_\kappa \varphi$$

Compact space  $\Rightarrow$  discrete spectrum

$$\square Y_L^2 = -\lambda_L Y_L^2$$

$$\lambda_L = L(L+D-1)H^2$$



For  $\kappa^2 < \lambda_1$ , heavy modes decouple  
 $\Rightarrow$  zero mode only  $\bar{\varphi}$

$$e^{-\Omega_{D+1} \bar{W}_\kappa(\mathbb{I})} = \int d\bar{\varphi} e^{-\Omega_{D+1} (\bar{V}_{\text{eff}}(\bar{\varphi}) + \mathbb{I} \bar{\varphi} + \frac{\kappa^2}{2} \bar{\varphi}^2)}$$

$$\text{with } e^{-\Omega_{D+1} \bar{V}_{\text{eff}}(\bar{\varphi})} = \int \mathcal{D}\hat{\varphi} e^{-\bar{S}[\bar{\varphi}, \hat{\varphi}]}$$



From UV to IR : onset  
of gravitational effects

■ Massive theory in UV ( $\kappa \gtrsim H$ )  
(i.e. Minkowski symmetric phase)

⇒  $V_H'' \gtrsim H^2$  : Minkowski flow

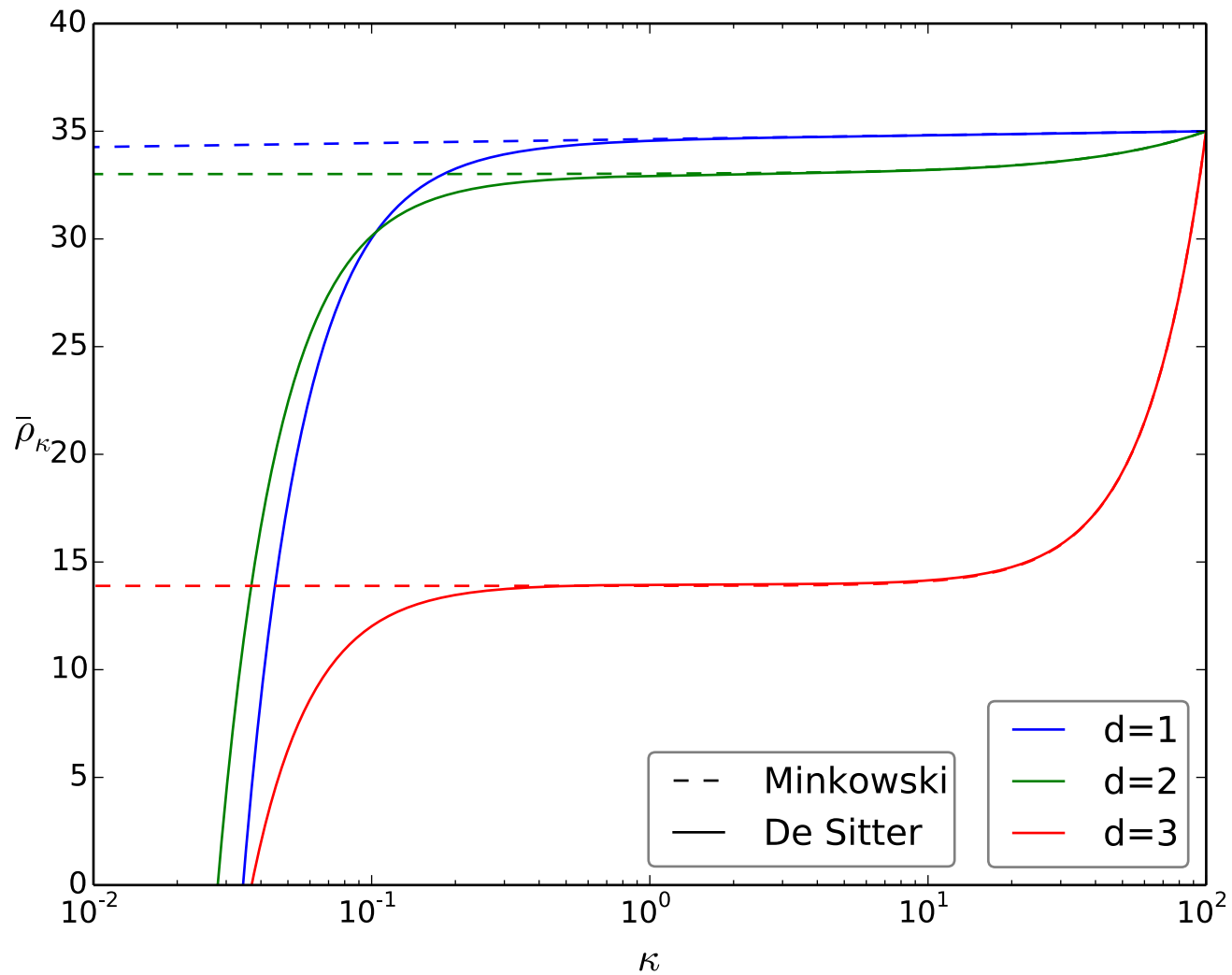
■ Massless modes / flat potential  
(e.g. Minkowski critical theory or  
broken phase)

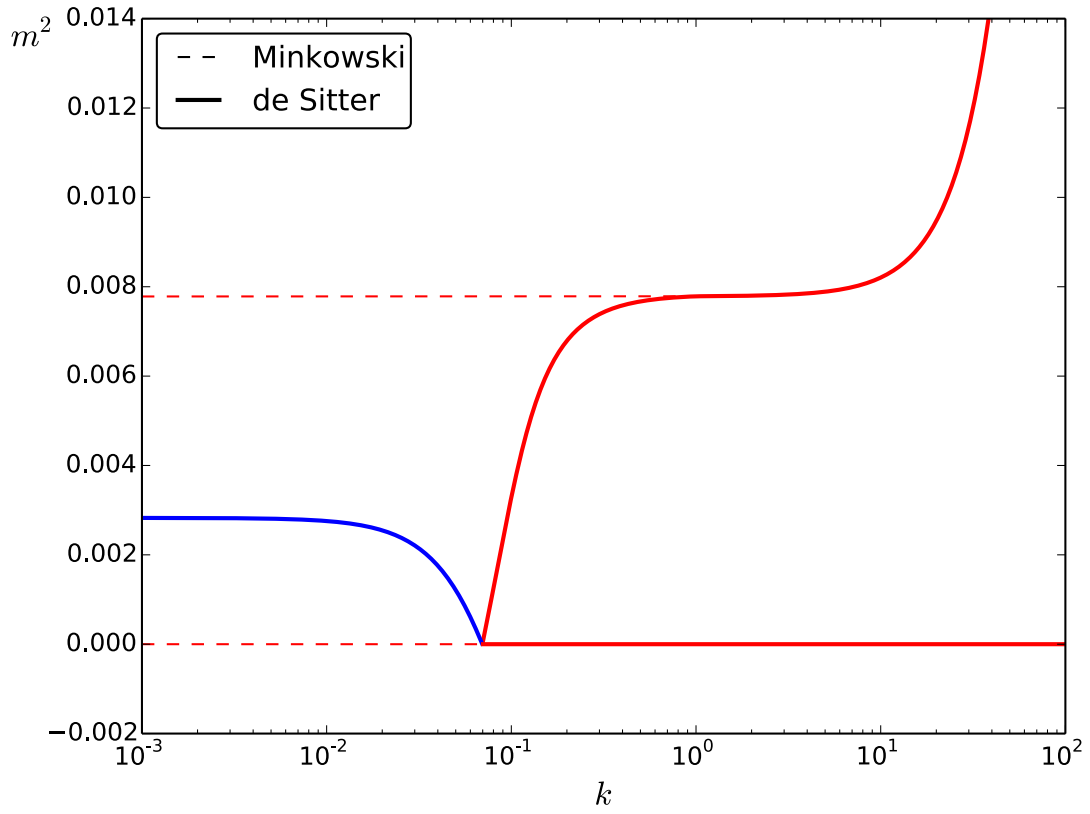
↪ Regime of dimensional reduction



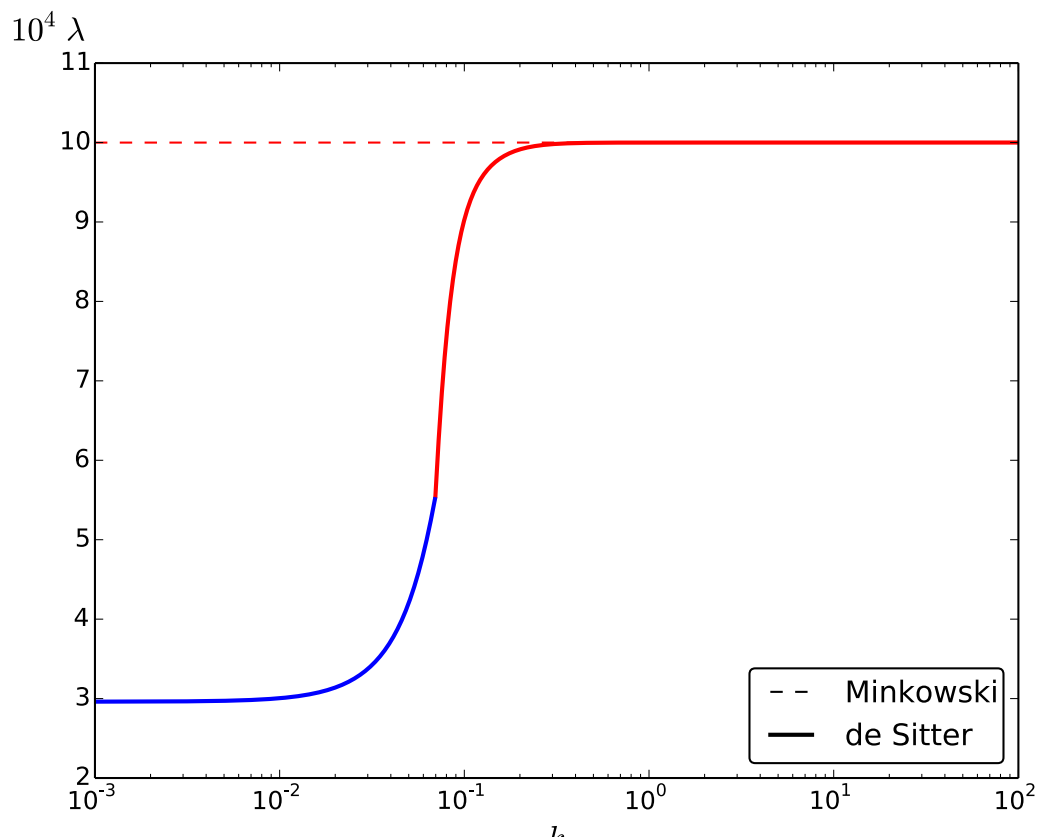
Symmetry restoration  
Mass (re)generation

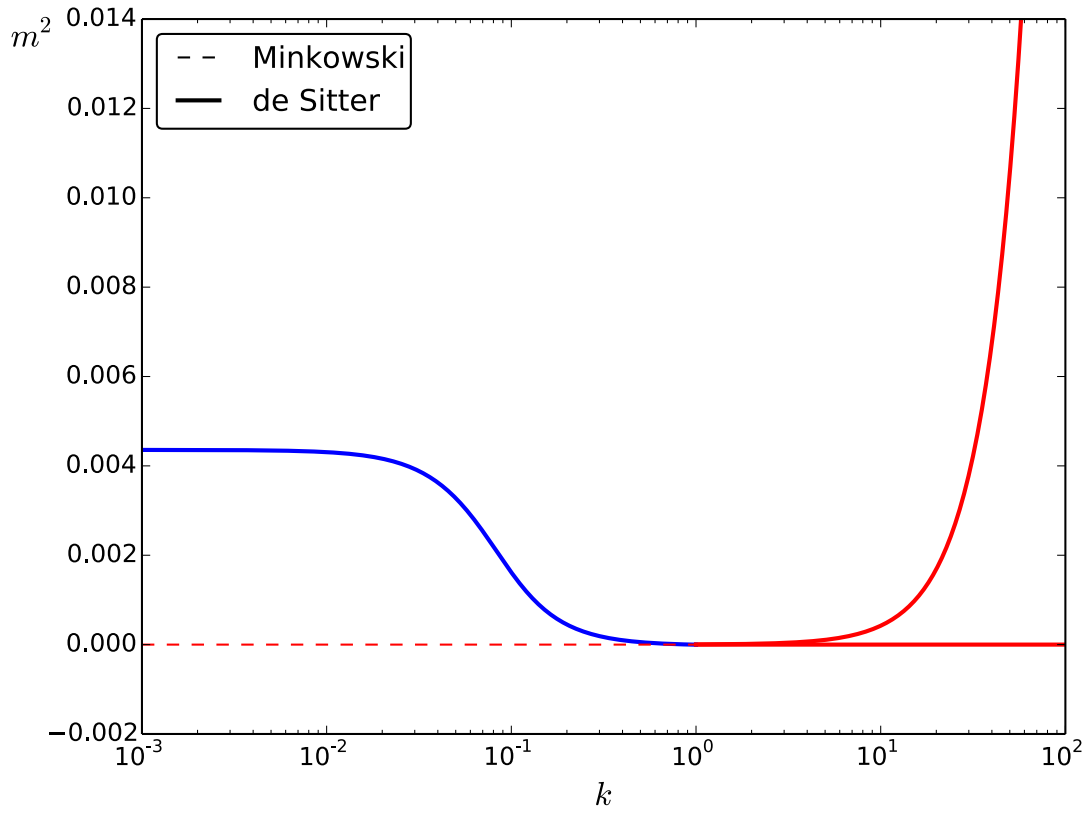




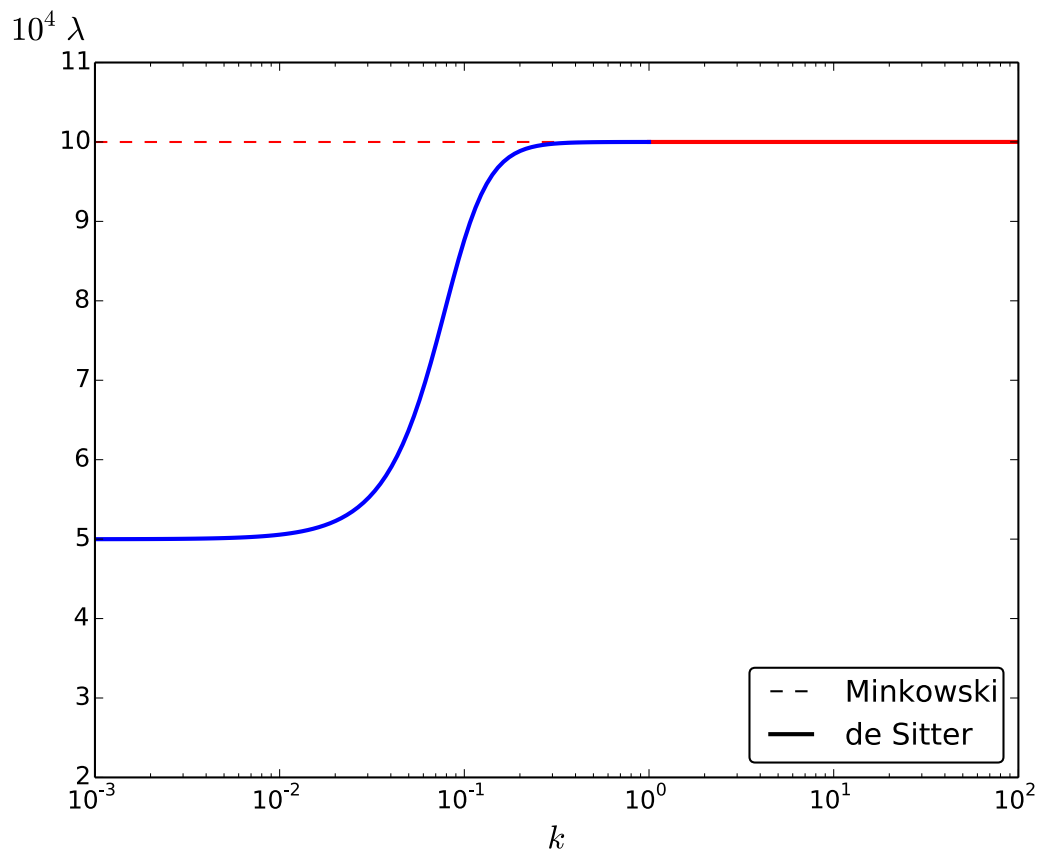


$N \rightarrow \infty$ , “broken symmetry” case

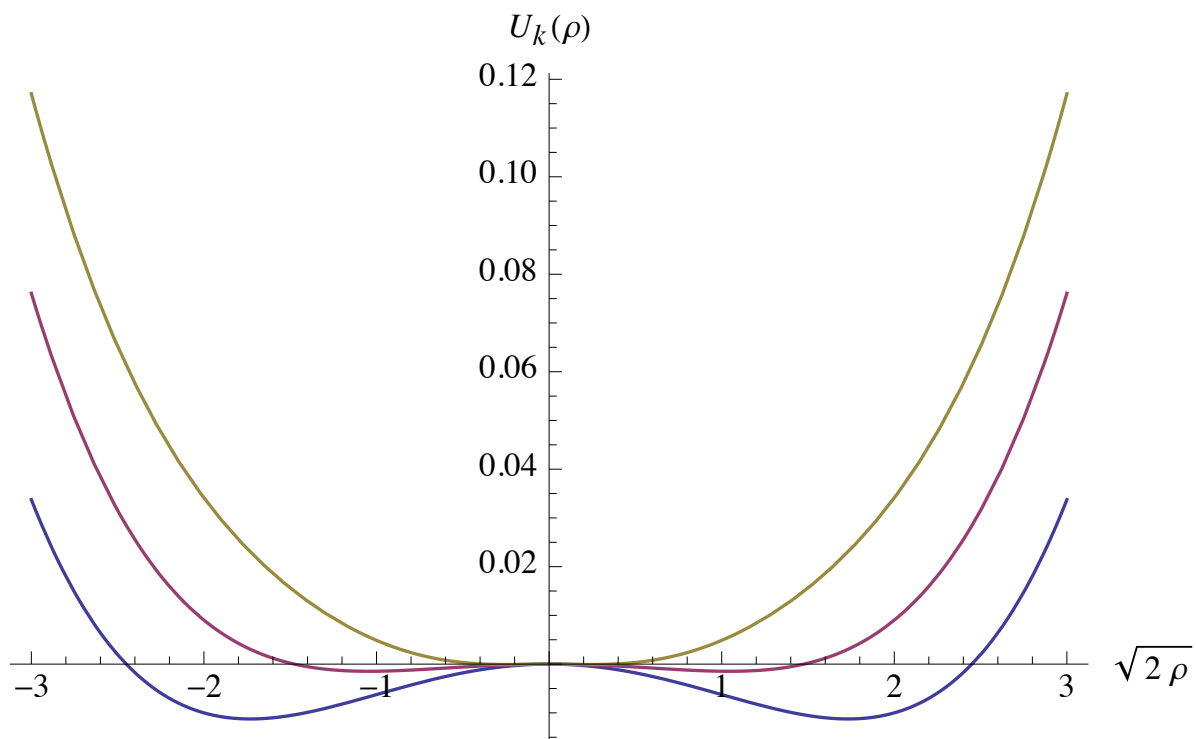




$N \rightarrow \infty$ , “critical” case







$$N = 1$$

## OUTLOOK

- Rich, nonperturbative IR dynamics for light scalar fields in de Sitter space
- Nonperturbative / resummation techniques  $\Rightarrow$  nontrivial results
- stochastic, large  $N$ , DSE, NPRG...
- mass generation, sym. restoration,  $\mathcal{O}(1)$  loop corrections (non Gaussianities), nontrivial structure of correlators, dimensional reduction, relation between stochastic and RG flow ...