Disforming the Kerr metric

GReCO seminar, IAP

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based on arXiv:2006.06461, TA, E. Babichev, C. Charmousis, M. Hassaine

Introduction

- The Kerr solution describes rotating black holes in general relativity
- It is interesting to construct deformations of the Kerr spacetime, in order to compare to experiments and potentially find signatures of modified theories of gravity
- Ad hoc deformations of the Kerr spacetime have been introduced in the past (Psaltis+, 2011; Johannsen, 2013; Papadopoulos+, 2018...)
- Using the disformal map, we present a deformed version of the Kerr spacetime which is a solution to a higher order scalar-tensor theory

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1. Properties of the Kerr metric

Kerr solution

- Vacuum solution of GR describing a rotating black hole (Kerr, 1963). The metric g verifies $R_{\mu\nu} = 0$.
- In Boyer-Lindquist coordinates, the metric tensor is:

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2} dt d\varphi + \frac{\sin^2\theta}{\rho^2}\left[(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta\right] d\varphi^2$$
$$+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

where M is the mass, a is the angular momentum per unit mass, and

$$\rho^2 = r^2 + a^2 \cos^2\theta \ ,$$
$$\Delta = r^2 + a^2 - 2Mr \ .$$

- $R_{\mu\nu\alpha\sigma} R^{\mu\nu\alpha\sigma}$ is singular at $\rho = \sqrt{r^2 + a^2 \cos^2\theta} = 0$, so there is a ring singularity at

$$r = 0 \qquad \text{and} \qquad \theta = \frac{\pi}{2}$$

Symmetries and circularity

- The metric is stationary and axi-symmetric, which corresponds to 2 Killing directions

$$\xi_{(t)} = \partial_t \qquad \text{and} \qquad \xi_{(\varphi)} = \partial_\varphi$$

- The spacetime is circular, i.e. symmetric under the reflection $(t, \varphi) \to (-t, -\varphi)$, because the Killing fields verify the condition

$$\xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(t)} = \xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(\varphi)} = 0 \; .$$

- The Kerr spacetime also admits a nontrivial Killing 2-tensor K verifying the equation

$$\nabla_{(\mu} K_{\nu\sigma)} = 0 \; .$$

- This defines a third nontrivial constant of motion along geodesics (Carter's constant). The geodesic equations thus reduce to a first order system.

Stationary observers

- Consider constant (r, θ) observers, with a 4-velocity

$$u = \partial_t + \omega \partial_\varphi$$

- The condition $u^2 \leq 0$ implies $\omega \in [\omega_-, \omega_+]$, where

$$\omega_\pm = \frac{|g_{t\varphi}|}{g_{\varphi\varphi}} \left(1 \pm \sqrt{1 - \frac{g_{tt} g_{\varphi\varphi}}{g_{t\varphi}^2}} \right)$$

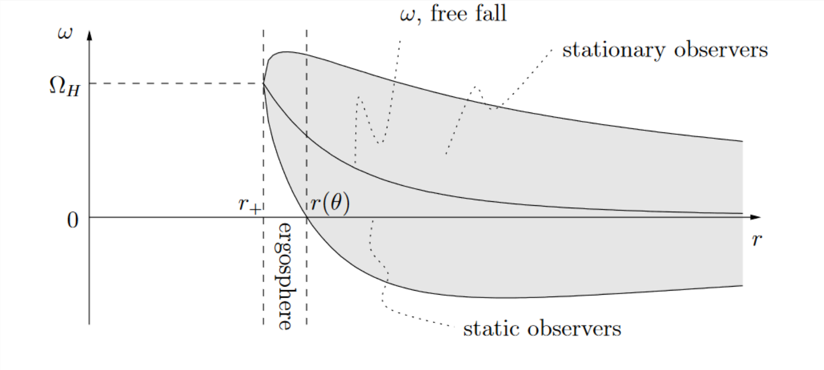
- Inside the *ergosphere*, where $g_{tt} > 0$, one necessarily has $\omega_- > 0$
- This surface is defined by $g_{tt} = 0$, which implies

$$r_E = M + \sqrt{M^2 - a^2 \cos^2 \theta}$$

- These observers stop to exist at the outer event horizon when $g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2 = 0$, at the radius

$$r_+ = M + \sqrt{M^2 - a^2}$$

Stationary observers



Graf, GR lecture notes

Killing horizon

- Rigidity theoreom (Hawking): The event horizon $\mathcal{H}$ of a real analytic, stationary, regular, vacuum spacetime is a Killing horizon: $\exists$ a Killing field k normal to $\mathcal{H}$ which verifies $k^2 = 0$ on $\mathcal{H}$.
- For the outer horizon of the Kerr spacetime, this Killing vector is

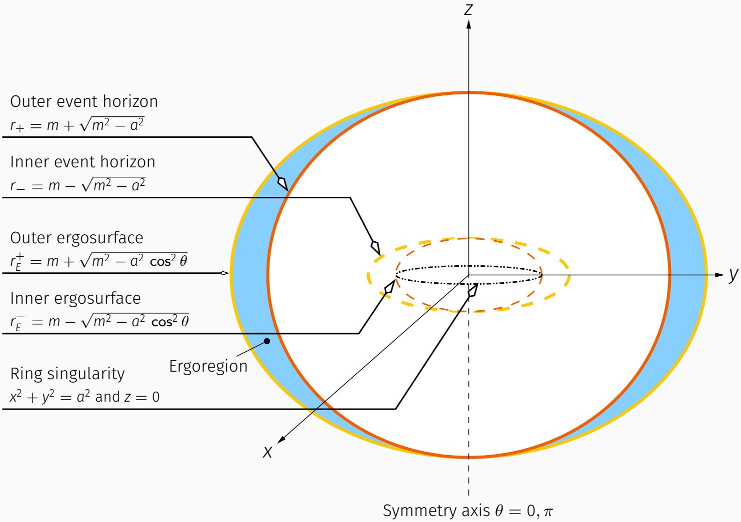
$$k = \partial_t + \frac{a}{2Mr_+}\partial_\varphi$$

- One can define the surface gravity κ_+ of $\mathcal{H}$ as

$$k^\mu \nabla_\mu k^\nu = \kappa_+ k^\nu$$

- The surface gravity is constant on $\mathcal{H}$ and is related to the Hawking temperature $T_H = \kappa_+/2\pi$

Kerr summary



Visser, 2007

2. Stealth-Kerr solution in DHOST theories

Degenerate Higher Order Scalar-Tensor (DHOST) theories

$$S = M_P^2 \int \mathrm{d}^4x \sqrt{-g} \left(f(\phi, X) R + K(\phi, X) - G_3(\phi, X) \Box\phi + \sum_{i=1}^{5} A_i(\phi, X) \mathcal{L}_i \right) + S_\mathrm{m} \left[g_{\mu\nu}, \psi_\mathrm{m} \right]$$

$$\mathcal{L}_1 = \phi_{\mu\nu}\phi^{\mu\nu}, \quad \mathcal{L}_2 = (\Box\phi)^2, \quad \mathcal{L}_3 = \phi_{\mu\nu}\phi^\mu\phi^\nu\Box\phi,$$
$$\mathcal{L}_4 = \phi_\mu\phi^\nu\phi^{\mu\alpha}\phi_{\nu\alpha}, \quad \mathcal{L}_5 = (\phi_{\mu\nu}\phi^\mu\phi^\nu)^2$$
$$X = \phi^\mu\phi_\mu$$

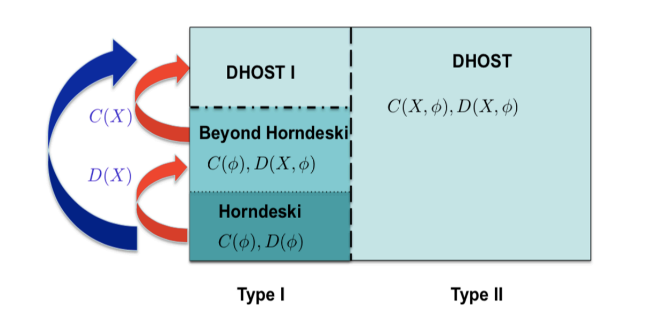
- Different classes of DHOST theories can be obtained (Langlois, Noui; Crisostomi+, 2016), but only one (subclass Ia) is viable for phenomenology:

$$A_2 = -A_1$$
$$A_4 = \frac{-16XA_1^3 + 4(3f + 16Xf_X)A_1^2 - X^2 f A_3^2 - (16X^2 f_X - 12Xf)A_3A_1 - 16f_X(3f + 4Xf_X)A_1 + 8f(Xf_X - f)A_3 + 48ff_X^2}{8(f - XA_1)^2}$$
$$A_5 = \frac{(4f_X - 2A_1 + XA_3)(-2A_1^2 - 3XA_1A_3 + 4f_XA_1 + 4fA_3)}{8(f - XA_1)^2}$$

Stability of the DHOST class under the disformal map

- These theories can be obtained from Horndeski theories by a disformal transformation of the metric (Ben Achour+; Crisostomi+, 2016...):

$$\tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \partial_\mu \phi \partial_\nu \phi$$



Langlois, 2018

Stealth-Kerr solution

- A stealth-Kerr solution was constructed (Charmousis+, 2019), where the scalar field is the Hamilton-Jacobi potential of the Kerr spacetime

$$g = g_{\mathrm{Kerr}}$$

$$\phi = -Et + L_z\varphi \pm \int \frac{\sqrt{\mathcal{R}(r)}}{\Delta}\mathrm{d}r \pm \int \Theta(\theta)\mathrm{d}\theta$$

- One looks for a solution to the Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -q^2$$

- A separable solution exists because the Kerr solution admits a Killing tensor K, linked to the Carter constant $\mathcal{Q}$

$$\mathcal{Q} = K^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (aE - L_z)^2$$

- The resulting scalar defines a geodesic because one has

$$\nabla^\mu\phi\nabla_\nu\nabla_\mu\phi = \nabla^\mu\phi\nabla_\mu\nabla_\nu\phi = 0$$

Stealth-Kerr solution

- In order for $\partial_\mu \phi$ to be regular at the poles, one must set $L_z = 0$, which implies

$$\eta \equiv -\frac{E}{q}$$
$$\mathcal{R}(r) = q^2 \left(r^2 + a^2\right)\left(\eta^2(r^2 + a^2) - \Delta\right)$$
$$\Theta(\theta) = a^2 q^2 \sin^2\theta(1 - \eta^2)$$

- In the following, we set $\eta = 1$, so that the scalar field depends on r only

$$E = -q$$
$$\mathcal{R}(r) = 2Mrq^2\left(r^2 + a^2\right)$$
$$\Theta(\theta) = 0$$

3. Disformed Kerr metric

Disformed Kerr metric

- Starting from the Kerr solution, we perform the transformation:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{D}{q^2}\,\partial_\mu\phi\,\partial_\nu\phi\,,$$
$$\phi = q\left[t + \int \frac{\sqrt{2Mr(a^2+r^2)}}{\Delta}\mathrm{d}r\right]\,.$$

- The line element is now

$$\mathrm{d}\tilde{s}^2 = -\left(1-\frac{2\tilde{M}r}{\rho^2}\right)\mathrm{d}t^2 - 2D\frac{\sqrt{2\tilde{M}r(a^2+r^2)}}{\Delta}\mathrm{d}t\mathrm{d}r + \frac{\rho^2\Delta - 2\tilde{M}(1+D)rD(a^2+r^2)}{\Delta^2}\mathrm{d}r^2$$
$$-\frac{4\sqrt{1+D}\tilde{M}ar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\varphi^2 + \rho^2\mathrm{d}\theta^2$$

with $\tilde{M} = M/(1+D)$ and the rescaling $t \to \sqrt{1+D}t$

- The scalar again defines a geodesic direction, since

$$\tilde{X} = \tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{X}{1+D}$$

Regular solution

- The disformed metric has the following curvature scalars

$$\tilde{R} = -\frac{Da^2 Mr[1+3\cos(2\theta)]}{(1+D)\rho^6}, \quad \tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} = \frac{M^2 Q_2(r,\theta)}{\rho^{12}(r^2+a^2)(1+D)^2},$$

- The solution is not Ricci-flat, but the only singularity is at $\rho = 0$, like Kerr. To verify this, one changes coordinates to

$$t \to v - r - \int \frac{2Mr}{\Delta} \mathrm{d}r\,, \qquad \varphi \to -\chi - a\int \frac{\mathrm{d}r}{\Delta}$$

- The metric components are regular in these coordinates, and the scalar field reads

$$\phi = q\left(v - r + \int \frac{\mathrm{d}r}{1+\sqrt{\frac{r^2+a^2}{2Mr}}}\right)$$

Non-circularity in the general case

- If $a = 0$, there exists a diffeomorphism $\mathrm{d}t \to \mathrm{d}T + f(r)\mathrm{d}r$ that brings the metric to the form (Babichev+, 2017; Achour+, 2019)

$$\mathrm{d}\tilde{s}^2 = -\left(1 - \frac{2\tilde{M}}{r}\right)\mathrm{d}T^2 + \left(1 - \frac{2\tilde{M}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \,.$$

- In the general case, we still have the two Killing vectors

$$\xi_{(t)} = \partial_t \qquad \text{and} \qquad \xi_{(\varphi)} = \partial_\varphi$$

- However, we now have

$$\xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(t)} = -D\frac{4a^2\tilde{M}r\sqrt{2\tilde{M}r(a^2+r^2)}\cos\theta\sin^3\theta}{\rho^4}\mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi$$

- This means we cannot write the metric in a form that is invariant under $(t, \varphi) \to (-t, -\varphi)$

Asymptotically similar to Kerr

- Asymptotically, the Kerr metric can be written

$$\mathrm{d}s^2_{\mathrm{Kerr}} = -\left[1 - \frac{2\tilde{M}}{r} + \mathcal{O}\left(\frac{1}{r^3}\right)\right]\mathrm{d}T^2 - \left[\frac{4\tilde{a}\tilde{M}}{r^3} + \mathcal{O}\left(\frac{1}{r^5}\right)\right][x\mathrm{d}y - y\mathrm{d}x]\,\mathrm{d}T + \left[1 + \mathcal{O}\left(\frac{1}{r}\right)\right]\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]$$

- After a coordinate transformation, one can write the disformal metric as

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}s^2_{\mathrm{Kerr}} + \frac{D}{1+D}\left[\mathcal{O}\left(\frac{\tilde{a}^2\tilde{M}}{r^3}\right)\mathrm{d}T^2 + \mathcal{O}\left(\frac{\tilde{a}^2\tilde{M}^{3/2}}{r^{7/2}}\right)\alpha_i\mathrm{d}T\mathrm{d}x^i + \mathcal{O}\left(\frac{\tilde{a}^2}{r^2}\right)\beta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right].$$

with $\tilde{a} = a\sqrt{1+D}$ and $\alpha_i, \beta_{ij} \sim \mathcal{O}(1)$.

- The corrections subleading corrections in $\mathrm{d}T\mathrm{d}x^i$ terms are larger than what is expected for the Kerr spacetime

Stationary observers

- Consider constant (r, θ) observers, with a 4-velocity

$$u = \partial_t + \omega \partial_\varphi$$

- The condition $u^2 \leq 0$ implies $\omega \in [\omega_-, \omega_+]$, where

$$\omega_\pm = \frac{1}{\tilde{g}_{\varphi\varphi}} \left(-\tilde{g}_{t\varphi} \pm \sqrt{\tilde{g}_{t\varphi}^2 - \tilde{g}_{tt}\tilde{g}_{\varphi\varphi}} \right)$$

- Inside the *static limit* defined by $\tilde{g}_{tt} = 0$, one necessarily has $\omega_- > 0$
- These observers no longer exist when $\tilde{g}_{t\varphi}^2 - \tilde{g}_{tt}\tilde{g}_{\varphi\varphi} = 0$, which happens when

$$P(r, \theta) \equiv r^2 + a^2 - 2\tilde{M}r + \frac{2\tilde{M}Da^2 r \sin^2\theta}{\rho^2(r, \theta)} = 0$$

- The outermost surface $r = R_0(\theta)$ which satisfies $P(R_0(\theta), \theta) = 0$ is called the *stationary limit*

Nature of the stationary limit

- When $D = 0$, the stationary limit coincides with the event horizon
- In the general case, the normal vector N to this surface is

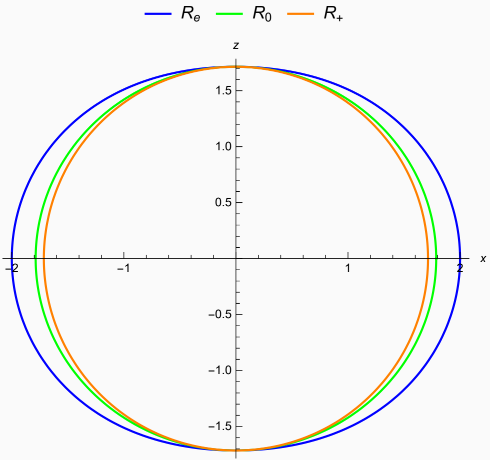
$$N_\mu = \big(0, 1, -R_0'(\theta), 0\big)$$

- One can check that

$$N^2|_{r=R_0} = \tilde{g}^{rr} + \tilde{g}^{\theta\theta} R_0'^2 > 0$$

- Hence the surface is timelike and cannot be the event horizon in th general case
- All Killing vectors of the form $\partial_t + \omega\partial_\varphi$ are spacelike inside this surface

- The ergosphere and stationary limit surface touch at the poles
- For $D = -0.2$ and $a = 0.7$, we have the following picture, with $R_+ \equiv \tilde{M}^2 + \sqrt{\tilde{M}^2 - a^2}$



Event horizon ?

- For Kerr, the horizons are found by solving $g^{rr} = 0 \implies \Delta = 0$ which admits constant r solutions. In our case, we have $\tilde{g}^{rr} = 0 \implies P = 0$, which doesn't admit constant r solutions when $D \neq 0$
- We look for more general null hypersurface of the form $r = R(\theta)$. The normal has components

$$n_\mu = (0, 1, -R'(\theta), 0)$$

- The condition $n^2 = 0$ yields

$$R'(\theta)^2 + P(R, \theta) = R'(\theta)^2 + R^2 + a^2 - 2\tilde{M}R + \frac{2\tilde{M}Da^2 R \sin^2\theta}{\rho^2(R, \theta)} = 0$$

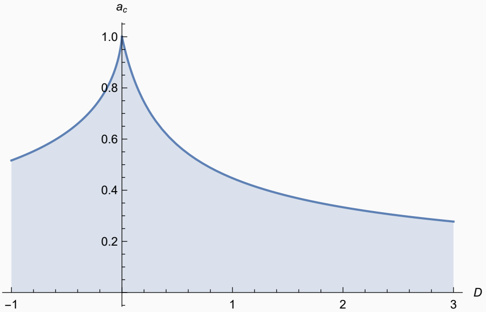
- To have a smooth solution, we must have

$$R'(0) = R'(\frac{\pi}{2}) = 0$$

Bounds on the rotation parameter

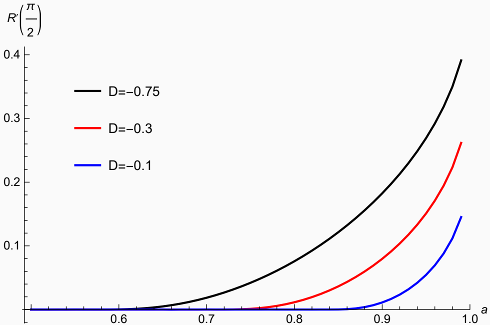
- After imposing $R'(\frac{\pi}{2}) = 0$, an expansion around $\theta = \frac{\pi}{2}$ yields a necessary condition to have $R''(\frac{\pi}{2}) \in \mathbb{R}$ (and similar arguments at $\theta = 0$).
- In units where $\tilde{M} = 1$, one must have $a < a_c$, where

$$\begin{aligned} Q_4(a_c^2) &= 0, & D &< 0 \\ a_c &= \frac{1}{\sqrt{1+4D}}, & D &> 0 \end{aligned}$$



Numerical bounds on a

- Numerically, one can start the integration at $\theta = 0$, and check if $R'(\frac{\pi}{2}) = 0$ (if $D > 0$ one instead integrates from $\theta = \pi/2$ to $\theta = \pi$)



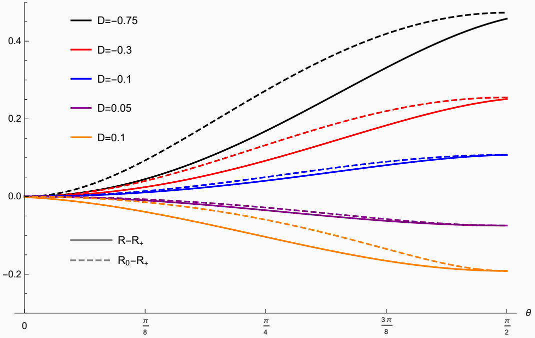
- As one increases numerical precision, this becomes consistent from the bounds coming from the expansion around. $\theta = \pi/2$.

Numerical integration

- For $\theta = 0$, the black hole looks like Kerr, and we have

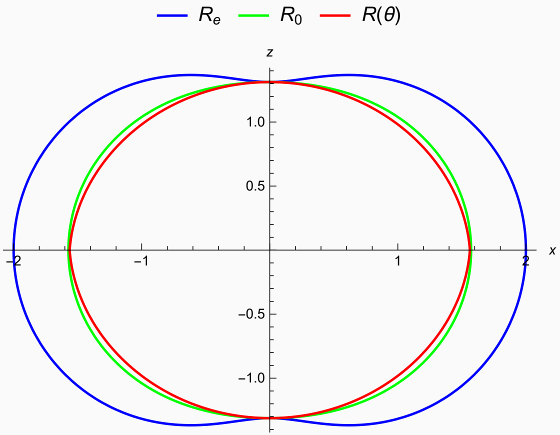
$$R(0) = R_+ \equiv \tilde{M}^2 + \sqrt{\tilde{M}^2 - a^2}$$

- This initial condition guarantees $R'(0) = 0$, and a numerical integration with $a = 0.9$ yields



- What happens in the region between R and R_0 ?

- $D = -0.2$ and $a = 0.95$ (not smooth)



Event horizon ?

- In the Kerr spacetime, the event horizon is located at $r = R_+$. By considering the hypersurfaces $r = R_+ + \zeta$, one can show that these surfaces are timelike outside the event horizon, and become spacelike between the horizons
- Similarly, we introduce the family of surfaces

$$R_\zeta(\theta) = R(\theta) + \zeta$$

- Under the assumption $R(\theta) \geq \tilde{M}$, one can show that $\exists\ \zeta_0$ such that the surfaces $r = R_\zeta(\theta)$ are

$$\begin{aligned} \text{timelike for} \qquad & \zeta > 0 \\ \text{spacelike for} \qquad & \zeta_0 < \zeta < 0 \end{aligned}$$

- These correspond to coordinates adapted to the horizon, in which the horizon is located at $\zeta = 0$

Killing tensor ?

- Another important feature of the Kerr metric is the existence of a Killing tensor, which allows to separate the Hamilton-Jacobi equation.
- We have considered small deformations $D \ll 1$ and checked that the disformed spacetime does not even admit an approximate Killing tensor $\tilde{K} = K + D\delta K$ satisfying the Killing equation at first order in D, meaning that

$$\tilde{\nabla}_{(\mu}\tilde{K}_{\nu\sigma)} = \mathcal{O}\left(D^2\right)$$

- There are papers implying that a separable spacetime should be circular (Benenti+, 1979...)
- Even if there is no Killing tensor, it is possible to study geodesics numerically, or consider equatorial geodesics for which only 3 constants are needed

Conclusion

- Alternatives to the Kerr spacetime are interesting to detect possible effects of modified theories of gravity
- We have constructed a solution of a particular DHOST theory by performing a disformal transformation of the Kerr solution using a geodesic scalar
- While asymptotically very similar to Kerr, the solution presents many interesting properties: non-circularity, horizon not located at constant r and not a Killing horizon, the stationary limit is distinct from the event horizon...
- These aspects are worthy of study, along with the geodesics of this new spacetime
- Other papers have studied some aspects of these solutions: the particular DHOST theories that these objects are a solution of (Achour+, 2020); shadows of this black hole (Long+, 2020)

Thank you for your attention.

Regular coordinates

- In regular coordinates, the disformed Kerr metric reads

$$
\begin{aligned}
d\tilde{s}^2 = &-\left(1+D-\frac{2Mr}{\rho^2}\right)dv^2 + 2\left(1+D-\frac{D}{1+\sqrt{\frac{r^2+a^2}{2Mr}}}\right)dv dr - D\left(1-\frac{1}{1+\sqrt{\frac{a^2+r^2}{2Mr}}}\right)^2 dr^2 \\
&+ \frac{4aMr\sin^2\theta}{\rho^2}dv d\chi + 2a\sin^2\theta dr d\chi + \rho^2 d\theta^2 \\
&+ \frac{\sin^2\theta\left(2a^4\cos^2\theta + 4a^2Mr\sin^2\theta + a^2r^2\left[3+2\cos(2\theta)\right] + 2r^4\right)}{2\rho^2}d\chi^2 \,.
\end{aligned}
$$