Free scalar correlators in de Sitter spacetime via the stochastic approach beyond slow roll

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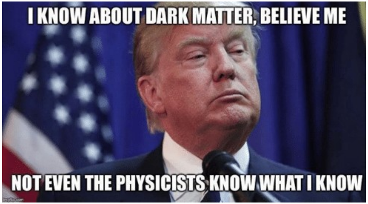
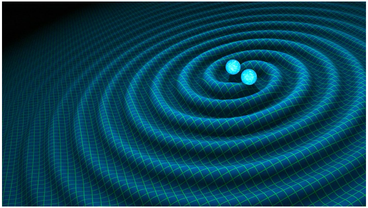
A.C., A. Rajantie; arXiv:2011.00907

Why are we interested in de Sitter spacetime?

- Inflationary dynamics
  - Formal interest
  - Gravitational wave background anisotropies
  - Dark matter generation
  - EW vacuum decay
- Current spacetime of the Universe



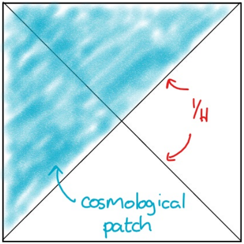


Cosmological de Sitter spacetime

Metric:

$$ds^2 = dt^2 - a(t)^2(dr^2 + r^2 d\Omega_2^2) \qquad ; \qquad a(t) = e^{Ht}$$

- Horizon at $R_H = 1/H$.
- Subhorizon: wavelengths $< 1/H$
  Superhorizon: wavelengths $> 1/H$



Spectator scalar field in de Sitter

- Action:

$$S[\phi] = \int d^4x a(t)^3 \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a(t)^2}(\nabla\phi)^2 - V(\phi) + 12\xi H^2\phi^2 \right]$$

- Equation of motion:

$$\begin{pmatrix} \dot{\phi} \\ \dot{\pi} \end{pmatrix} = \begin{pmatrix} \pi \\ -3H\pi + \frac{1}{a(t)^2}\nabla^2\phi - V'(\phi) - 12\xi H^2\phi \end{pmatrix}$$

State of play

- QFT in a curved spacetime can be used to calculate de Sitter correlators in free field theory; however, introducing self-interactions results in a breakdown of perturbation theory due to IR divergences.

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- Alternative approach: the stochastic method

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- QFT in a curved spacetime can be used to calculate de Sitter correlators in free field theory; however, introducing self-interactions results in a breakdown of perturbation theory due to IR divergences.

- Alternative approach: the stochastic method

Quantum modes can be summarised by a statistical noise contribution to the classical equations of motion, for large spacetime separations.

- In the massless limit $m^2/H^2 << 1$, this is achieved by introducing a hard cut-off between subhorizon (quantum) and superhorizon (classical) modes in order to calculate the noise *[Starobinsky, Yokoyama; 1994]* This *does not work* for massive fields.

Notable work

Development of the stochastic approach

- Early work - *Starobinsky, Yokoyama; 1994*
- Path integral method - *Rigopolous; 2016 & Moss, Rigopoulos; 2017*
- Multifield generalisation - *Pinol, Renaux-Petel, Tada; 2019 & 2020*

Applications

- Stochastic inflation - *Vennin, Starobinsky; 2015 & Grain, Vennin; 2017*
- Dark energy - *Glavan, Prokopec, Starobinsky; 2018*
- Vacuum decay - *Markkanen, Rajantie; 2020*

Aim

To extend the stochastic approach beyond the massless limit

How?

- We will calculate the stochastic correlators for free fields i.e. $V(\phi) + 12\xi H^2\phi^2 = \frac{1}{2}m^2\phi^2$, for a general noise term.
- These will be matched to the exact solutions from the QFT in order to calculate the noise.
- Introducing self-interactions is left for future work.

The massless $(m^2/H^2 << 1)$ limit

[Starobinsky, Yokoyama; 1994]

In the massless limit, the equation of motions simplifies to

$$0 = 3H\dot{\phi} + m^2\phi.$$

The field is split as $\phi = \overline{\phi} + \delta\phi$, where $\overline{\phi}$ contains the classical modes and

$$\delta\phi(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\theta(k - \epsilon a(t)H)\hat{\phi}_k(t, \mathbf{x}).$$

ϵ is introduced as a cut-off parameter which is sufficiently small such that $\mathcal{O}(\epsilon^2)$ is negligible.

The massless ($m^2/H^2 << 1$) limit

The equation of motion now becomes

$$0 = 3H\dot{\overline{\phi}} + m^2\overline{\phi} - \hat{\xi}$$

where

$$\hat{\xi}(t, \mathbf{x}) = \epsilon a(t) H^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(k - \epsilon a(t) H) \hat{\phi}_k(t, \mathbf{x}).$$

The equation of motion can be approximated by a Langevin equation

$$0 = 3H\dot{\phi} + m^2\phi - \xi$$

with a stochastic white noise contribution

$$\left\langle \xi(t, \mathbf{x}) \xi(t', \mathbf{x}) \right\rangle = \lim_{\epsilon \to 0} \left\langle \hat{\xi}(t, \mathbf{x}) \hat{\xi}(t', \mathbf{x}) \right\rangle = \frac{9H^5}{4\pi^2} \delta(t - t').$$

The massless ($m^2/H^2 << 1$) limit

The time evolution of the probability distribution function (PDF) $P(\phi; t)$ is given by the Fokker-Planck equation associated with the Langevin equation

$$\frac{\partial P(\phi;t)}{\partial t} = \frac{m^2}{3H} P(\phi;t) + \frac{m^2}{3H}\phi\frac{\partial P(\phi;t)}{\partial \phi} + \frac{H^3}{8\pi^2}\frac{\partial^2 P(\phi;t)}{\partial \phi^2}.$$

From this, the overdamped correlators can be calculated via a spectral expansion *[Markkanen et. al.; 2019]*.

Extended SY approach

To extend beyond the massless limit, we also make the split $\pi = \overline{\pi} + \delta\pi$ where

$$\delta\pi(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \theta(k - \epsilon a(t) H) \hat{\pi}_k(t, \mathbf{x}).$$

Combining this with the ϕ split given, the equation of motion reads

$$\begin{pmatrix} \dot{\overline{\phi}} \\ \dot{\overline{\pi}} \end{pmatrix} = \begin{pmatrix} \overline{\pi} \\ -3H\overline{\pi} - m^2\phi \end{pmatrix} + \begin{pmatrix} \hat{\xi}_\phi(t, \mathbf{x}) \\ \hat{\xi}_\pi(t, \mathbf{x}) \end{pmatrix}$$

where

$$\hat{\xi}_\pi(t, \mathbf{x}) = \epsilon a(t) H^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \delta(k - \epsilon a(t) H) \hat{\pi}_k(t, \mathbf{x}).$$

Extended SY approach

Again, the equation of motion can be approximated by a Langevin equation

$$\begin{pmatrix} \dot{\phi} \\ \dot{\pi} \end{pmatrix} = \begin{pmatrix} \pi \\ -3H\pi - m^2\phi \end{pmatrix} + \begin{pmatrix} \xi_\phi(t, \mathbf{x}) \\ \xi_\pi(t, \mathbf{x}) \end{pmatrix}$$

with a stochastic white noise contribution

$$\left\langle \xi_i(t, \mathbf{x})\xi_j(t', \mathbf{x}) \right\rangle = \left\langle \hat{\xi}_i(t, \mathbf{x})\hat{\xi}_j(t', \mathbf{x}) \right\rangle = \sigma^2_{SY,ij}\delta(t - t').$$

where $i, j \in \{\phi, \pi\}$. $\sigma^2_{SY,ij}$ are calculated using the mode functions in the Bunch-Davies vacuum *[Bunch, Davies; 1978]*.

The Fokker-Planck equation

Now, we consider a general white noise with the form

$$\left\langle \xi_i(t,\mathbf{x})\xi_j(t',\mathbf{x})\right\rangle = \sigma_{ij}^2\delta(t-t').$$

The time evolution of the probability distribution function (PDF) $P(\phi,\pi;t)$ is given by the Fokker-Planck equation associated with the Langevin equation

$$\begin{aligned}\frac{\partial P(\phi,\pi;t)}{\partial t} =& 3HP(\phi,\pi;t) - \pi\frac{\partial P(\phi,\pi;t)}{\partial \phi} + \left(3H\pi + m^2\phi\right)\frac{\partial P(\phi,\pi;t)}{\partial \pi} \\ &+ \frac{1}{2}\sigma_{\phi\phi}^2\frac{\partial^2 P(\phi,\pi;t)}{\partial \phi^2} + \sigma_{\phi\pi}^2\frac{\partial^2 P(\phi,\pi;t)}{\partial\phi\partial\pi} + \frac{1}{2}\sigma_{\pi\pi}^2\frac{\partial^2 P(\phi,\pi;t)}{\partial\pi^2}.\end{aligned}$$

Equilibrium solution

In equilibrium, $\dfrac{\partial P(\phi, \pi; t)}{\partial t} = 0$, which can be solved to give

$$P_{eq}(\phi, \pi) \propto e^{-Q^2}$$

where

$$Q^2 = \frac{3H\big(((9H^2+m^2)\sigma_{\phi\phi}^2 + 6H\sigma_{\phi\pi}^2 + \sigma_{\pi\pi}^2)\pi^2 + 6Hm^2\sigma_{\phi\phi}^2\phi\pi + (m^2\sigma_{\phi\phi}^2 + \sigma_{\pi\pi}^2)m^2\phi^2\big)}{(m^2\sigma_{\phi\phi}^2 + 3H\sigma_{\phi\pi}^2 + \sigma_{\pi\pi}^2)^2 + 9H^2(\sigma_{\phi\phi}^2\sigma_{\pi\pi}^2 - \sigma_{\phi\pi}^4)}$$

with normalisation $\int d\phi \int d\pi P_{eq}(\phi, \pi) = 1$.

Solving the Fokker-Planck equation: Outline

- Write (ϕ, π) in terms of a new set of dynamical variables (q, p), where the Fokker-Planck equation for (q, p) is simpler than that of (ϕ, π).

- Solve the (q, p) Fokker-Planck equation and thus calculate the (q, p)-correlators.

- Write the (ϕ, π) correlators in terms of their (q, p) counterparts and hence evaluate them.

New dynamical variables (q, p)

We define (q, p) in terms of (ϕ, π) as

$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{\sqrt{1-\frac{\alpha}{\beta}}} \begin{pmatrix} 1 & \alpha H \\ \frac{1}{\beta H} & 1 \end{pmatrix} \begin{pmatrix} \pi \\ \phi \end{pmatrix}$$

where $\alpha = 3/2 - \nu$ and $\beta = 3/2 + \nu$, with $\nu = \sqrt{\dfrac{9}{4} - \dfrac{m^2}{H^2}}$.

Stochastic field correlator

We want to calculate

$$\begin{aligned}\langle\phi(0,\mathbf{x})\phi(t,\mathbf{x})\rangle =&\frac{1}{1-\frac{\alpha}{\beta}}\left(\frac{1}{\beta^2H^2}\left\langle p(0)p(t)\right\rangle-\frac{1}{\beta H}\left\langle q(0)p(t)\right\rangle\right.\\ &\left.-\frac{1}{\beta H}\left\langle p(0)q(t)\right\rangle+\left\langle q(0)q(t)\right\rangle\right).\end{aligned}$$

We can similarly find the $\pi-\phi$ and $\pi-\pi$ correlators but we won't here.

(q, p) Langevin equation

In these new variables, the Langevin equation is

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} -\alpha H q \\ -\beta H p \end{pmatrix} + \begin{pmatrix} \xi_q \\ \xi_p \end{pmatrix} .$$

These are 1-dimensional!

(q,p) Fokker-Planck equation

The associated Fokker-Planck equation is

$$\begin{aligned}\frac{\partial P(q,p;t)}{\partial t} =&\alpha H P(q,p;t) + \alpha H q\frac{\partial P(q,p;t)}{\partial q} + \frac{1}{2}\sigma_{qq}^2\frac{\partial^2 P(q,p;t)}{\partial q^2}\\ &+ \beta H P(q,p;t) + \beta H p\frac{\partial P(q,p;t)}{\partial p} + \frac{1}{2}\sigma_{pp}^2\frac{\partial^2 P(q,p;t)}{\partial p^2}\\ &+ \sigma_{qp}^2\frac{\partial^2 P(q,p;t)}{\partial q\partial p}\end{aligned}$$

where $\langle \xi_i(t,\mathbf{x})\xi_j(t,\mathbf{x})\rangle = \sigma_{ij}^2\delta(t-t')$; now, $i,j \in q,p$.

The equilibrium solution is

$$P_{eq}(q,p) \propto e^{-\frac{9H(\sigma_{qq}^2\beta p^2 - \frac{4}{3}\sigma_{qp}^2\alpha\beta qp + \sigma_{pp}^2\alpha q^2)}{9\sigma_{qq}^2\sigma_{pp}^2 - 4\alpha\beta\sigma_{qp}^4}}.$$

Time-evolution operator

To calculate the (q,p) correlators, we need the time-evolution operator $U(q_0,q,p_0,p;t)$, which is defined as the Green's function of the (q,p) Fokker-Planck equation and therefore obeys

$$\begin{aligned}\frac{\partial U(q_0,q,p_0,p;t)}{\partial t} =& \alpha H U(q_0,q,p_0,p;t) + \alpha H q \frac{\partial U(q_0,q,p_0,p;t)}{\partial q} \\ &+ \frac{1}{2}\sigma_{qq}^2 \frac{\partial^2 U(q_0,q,p_0,p;t)}{\partial q^2} + \beta H U(q_0,q,p_0,p;t) \\ &+ \beta H p \frac{\partial U(q_0,q,p_0,p;t)}{\partial p} + \frac{1}{2}\sigma_{pp}^2 \frac{\partial^2 U(q_0,q,p_0,p;t)}{\partial p^2} \\ &+ \sigma_{qp}^2 \frac{\partial^2 U(q_0,q,p_0,p;t)}{\partial q \partial p}.\end{aligned}$$

Time-evolution operator

Since we only need to calculate the $q-q$, $q-p$ and $p-p$ correlators, we only need to evolve q or p forward in time - not both - for any given correlator.

Hence, we need the separate q and p time-evolution operators, $U_q(q_0, q; t)$ and $U_p(p_0, p; t)$ respectively, which obey the 1-d Fokker-Planck equations

$$\frac{\partial U_q(q_0, q; t)}{\partial t} = \alpha H U_q(q_0, q; t) + \alpha H q \frac{\partial U_q(q_0, q; t)}{\partial q} + \frac{1}{2}\sigma_{qq}^2 \frac{\partial^2 U_q(q_0, q; t)}{\partial q^2},$$

$$\frac{\partial U_p(p_0, p; t)}{\partial t} = \beta H U_p(p_0, p; t) + \beta H p \frac{\partial U_p(p_0, p; t)}{\partial p} + \frac{1}{2}\sigma_{pp}^2 \frac{\partial^2 U_p(p_0, p; t)}{\partial p^2}.$$

These can be solved via a spectral expansion.

(q,p) correlators

The (q,p) correlators are thus given by

$$\begin{aligned}
\langle q(0)q(t)\rangle &= \int dp_0 \int dq \int dq_0 P_{eq}(q_0,p_0)U_q(q_0,q;t)q_0 q = \frac{\sigma_{qq}^2}{2\alpha H}e^{-\alpha H t},\\
\langle p(0)q(t)\rangle &= \int dp_0 \int dq \int dq_0 P_{eq}(q_0,p_0)U_q(q_0,q;t)p_0 q = \frac{\sigma_{qp}^2}{3H}e^{-\alpha H t},\\
\langle q(0)p(t)\rangle &= \int dq_0 \int dp \int dp_0 P_{eq}(q_0,p_0)U_p(p_0,p;t)q_0 p = \frac{\sigma_{qp}^2}{3H}e^{-\beta H t},\\
\langle p(0)p(t)\rangle &= \int dp \int dp_0 \int dq_0 P_{eq}(q_0,p_0)U_p(p_0,p;t)p_0 p = \frac{\sigma_{pp}^2}{2\beta H}e^{-\beta H t}.
\end{aligned}$$

Stochastic field correlator

The equal-space stochastic correlator is given by

$$\langle\phi(0,\mathbf{x})\phi(t,\mathbf{x})\rangle = \left(\frac{m^2\sigma_{qq}^2}{4H^3\nu\alpha^2} - \frac{\sigma_{qp}^2}{6H^2\nu}\right)e^{-\alpha Ht} + \left(\frac{\sigma_{pp}^2}{4H^3\nu\beta^2} - \frac{\sigma_{qp}^2}{6H^2\nu}\right)e^{-\beta Ht}.$$

Due to the de Sitter symmetry, the equal-time correlator is given by

$$\langle\phi(0,\mathbf{0})\phi(0,\mathbf{x})\rangle = \left(\frac{m^2\sigma_{qq}^2}{4H^3\nu\alpha^2} - \frac{\sigma_{qp}^2}{6H^2\nu}\right)|H\mathbf{x}|^{-2\alpha} + \left(\frac{\sigma_{pp}^2}{4H^3\nu\beta^2} - \frac{\sigma_{qp}^2}{6H^2\nu}\right)|H\mathbf{x}|^{-2\beta}.$$

Quantum field correlator

In the Bunch-Davies vacuum,

$$\left\langle \hat{\phi}(0,\mathbf{0})\hat{\phi}(0,\mathbf{x})\right\rangle = \frac{H^2}{16\pi^2}\Gamma\left(\frac{3}{2}+\nu\right)\Gamma\left(\frac{3}{2}-\nu\right){}_2F_1\left(\frac{3}{2}+\nu,\frac{3}{2}-\nu,2;1-\frac{\left|H\mathbf{x}\right|^2}{4}\right).$$

[Bunch, Davies; 1978]

Stochastic vs quantum field correlator

To leading order in the large spacetime separations,

- Quantum

$$\langle\hat{\phi}(0,\mathbf{0})\hat{\phi}(0,\mathbf{x})\rangle = \frac{H^2}{16\pi^2}\left[\frac{\Gamma(\frac{3}{2}-\nu)\Gamma(2\nu)4^{\frac{3}{2}-\nu}}{\Gamma(\frac{1}{2}+\nu)}|H\mathbf{x}|^{-2\alpha}+\frac{\Gamma(-2\nu)\Gamma(\frac{3}{2}+\nu)4^{\frac{3}{2}+\nu}}{\Gamma(\frac{1}{2}-\nu)}|H\mathbf{x}|^{-2\beta}\right]$$

- Stochastic

$$\langle\phi(0,\mathbf{0})\phi(0,\mathbf{x})\rangle = \left(\frac{m^2\sigma_{qq}^2}{4H^3\nu\alpha^2}-\frac{\sigma_{qp}^2}{6H^2\nu}\right)|H\mathbf{x}|^{-2\alpha}+\left(\frac{\sigma_{pp}^2}{4H^3\nu\beta^2}-\frac{\sigma_{qp}^2}{6H^2\nu}\right)|H\mathbf{x}|^{-2\beta}$$

Extended SY correlator

If we use the extended SY approach to calculate the noise, the stochastic field correlator is given by

$$\begin{aligned}\langle\phi(0,\mathbf{0})\phi(0,\mathbf{x})\rangle_{SY} =& \frac{H^3\epsilon^3}{384\pi\alpha}\Big[\epsilon\Big(\mathcal{H}^{(1)}_{\nu-1}(\epsilon)-\mathcal{H}^{(1)}_{\nu+1}(\epsilon)\Big)\Big(\epsilon\mathcal{H}^{(2)}_{\nu-1}(\epsilon)-3\mathcal{H}^{(2)}_{\nu}(\epsilon)-\epsilon\mathcal{H}^{(2)}_{\nu+1}(\epsilon)\Big)\\ &+\mathcal{H}^{(1)}_{\nu}(\epsilon)\Big(-3\epsilon\Big(\mathcal{H}^{(2)}_{\nu-1}(\epsilon)-\mathcal{H}^{(2)}_{\nu+1}(\epsilon)\Big)+4\nu(3-\nu)\mathcal{H}^{(2)}_{\nu}(\epsilon)\Big)\Big]|H\mathbf{x}|^{-2\alpha}\\ &-\frac{H^3\epsilon^3}{384\pi\beta}\Big[\epsilon\Big(\mathcal{H}^{(1)}_{\nu-1}(\epsilon)-\mathcal{H}^{(1)}_{\nu+1}(\epsilon)\Big)\Big(\epsilon\mathcal{H}^{(2)}_{\nu-1}(\epsilon)-3\mathcal{H}^{(2)}_{\nu}(\epsilon)-\epsilon\mathcal{H}^{(2)}_{\nu+1}(\epsilon)\Big)\\ &-\mathcal{H}^{(1)}_{\nu}(\epsilon)\Big(3\epsilon\Big(\mathcal{H}^{(2)}_{\nu-1}(\epsilon)-\mathcal{H}^{(2)}_{\nu+1}(\epsilon)\Big)+4\nu(3+\nu)\mathcal{H}^{(2)}_{\nu}(\epsilon)\Big)\Big]|H\mathbf{x}|^{-2\beta}.\end{aligned}$$

This does not reproduce the quantum correlator at all masses for any ϵ.

Matched noise

However, we can match the noises to the QFT result, giving

$$\sigma^2_{M,qq} = \frac{H^3\nu}{2\pi^2\beta}\frac{\Gamma(2\nu)\Gamma(\frac{5}{2}-\nu)4^{1-\nu}}{\Gamma(\frac{1}{2}+\nu)},$$

$$\sigma^2_{M,pp} = \frac{H^5\beta\nu}{2\pi^2}\frac{\Gamma(-2\nu)\Gamma(\frac{5}{2}+\nu)4^{1+\nu}}{\Gamma(\frac{1}{2}-\nu)},$$

$$\sigma^2_{M,qp} = \sigma^2_{M,pq} = 0.$$

This choice reproduces *all* free QFT correlators.

Matched vs SY noise

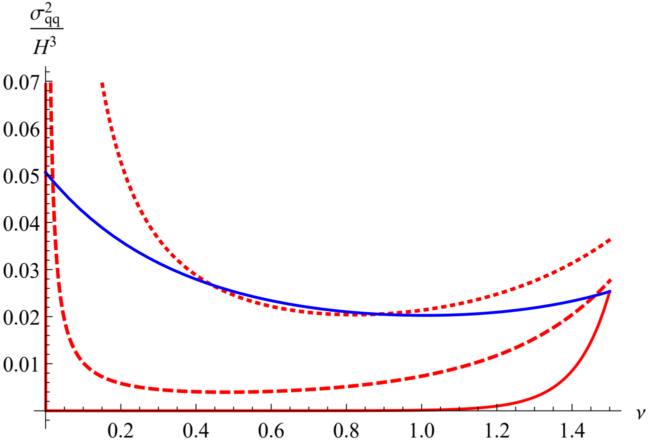

Figure: The matched (blue) $\sigma^2_{M,qq}$ and extended SY (red) noises $\sigma^2_{SY,qq}$ with $\epsilon = 0.01$ (solid), $\epsilon = 0.5$ (dashed) and $\epsilon = 0.99$ (dotted)

Summary

- The stochastic approach *can* be used to reproduce all quantum correlators to leading order in large spacetime separations beyond the massless limit.

- It requires us to match the noise with the exact result.

- How do we extend to include self-interactions?

Physical interpretation

In the massless limit, $\sigma_{pp}^2 = \sigma_{qp}^2 = 0$ and $\sigma_{qq}^2 = \frac{9H^5}{4\pi^2}$. If we look closer,

$$\sigma^2 = 2 \times \underbrace{3H}_{\text{friction}} \times \underbrace{\left(\frac{4\pi}{3H^3}\right)}_{\text{volume}}^{-1} \times \underbrace{\frac{H}{2\pi}}_{\text{temperature}}$$

Quantum modes are summarised as a thermal contribution, with de Sitter temperature *[Rigopolous; 2016]*.

Physical interpretation

- At leading order in mass, the stochastic correlator is true for any spacetime separations
  $\implies$ the thermal interpretation is a manifestation of the thermal nature of the Bunch-Davies vacuum
- Beyond the massless limit, the stochastic correlator only gives results at leading order in large spacetime separations
  $\implies$ information is lost and therefore the noise is not pure thermal