Quantum singularity avoidance and Bianchi I clocks

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Motivations: (quantum) cosmology

Homogeneous & isotropic metric (FLRW): $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right]$

spatial curvature

Hubble rate $H \equiv \frac{\dot{a}}{a}$

Matter component: perfect fluid $T_{\mu\nu} = p g_{\mu\nu} + (\rho + p)\, u_\mu u_\nu$

equation of state $p = w\rho \rightarrow \begin{cases} w = 0 & \textit{dust} \\ w = \frac{1}{3} & \textit{radiation} \end{cases}$

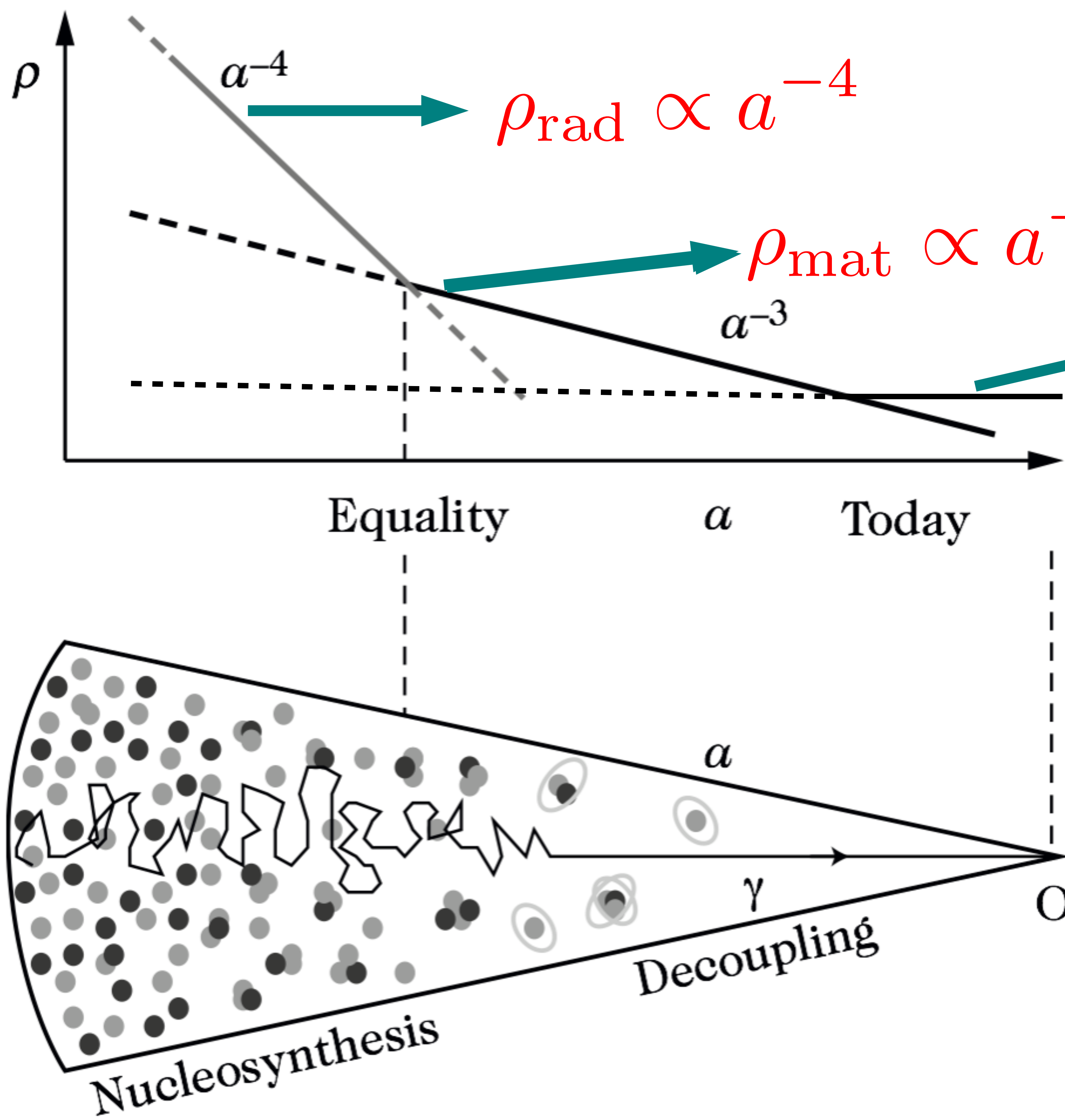
+ cosmological constant = Einstein equations

$$\begin{cases} H^2 + \frac{\mathcal{K}}{a^2} = \frac{1}{3}\left(8\pi G_{\mathrm{N}}\rho + \Lambda\right) \\ \frac{\ddot{a}}{a} = \frac{1}{3}\left[\Lambda - 4\pi G_{\mathrm{N}}\left(\rho + 3p\right)\right] \end{cases}$$

Particular solution: dust and radiation

integrate conservation equation

$$\rho\left[a\left(t\right)\right] = \rho_{\text{ini}} \exp\left\{-3\int\left[1+w(a)\right]\mathrm{d}\,\ln a\right\} \underset{w\to\text{cst}}{=} \rho_{\text{ini}}\left(\frac{a}{a_{\text{ini}}}\right)^{-3(1+w)}$$



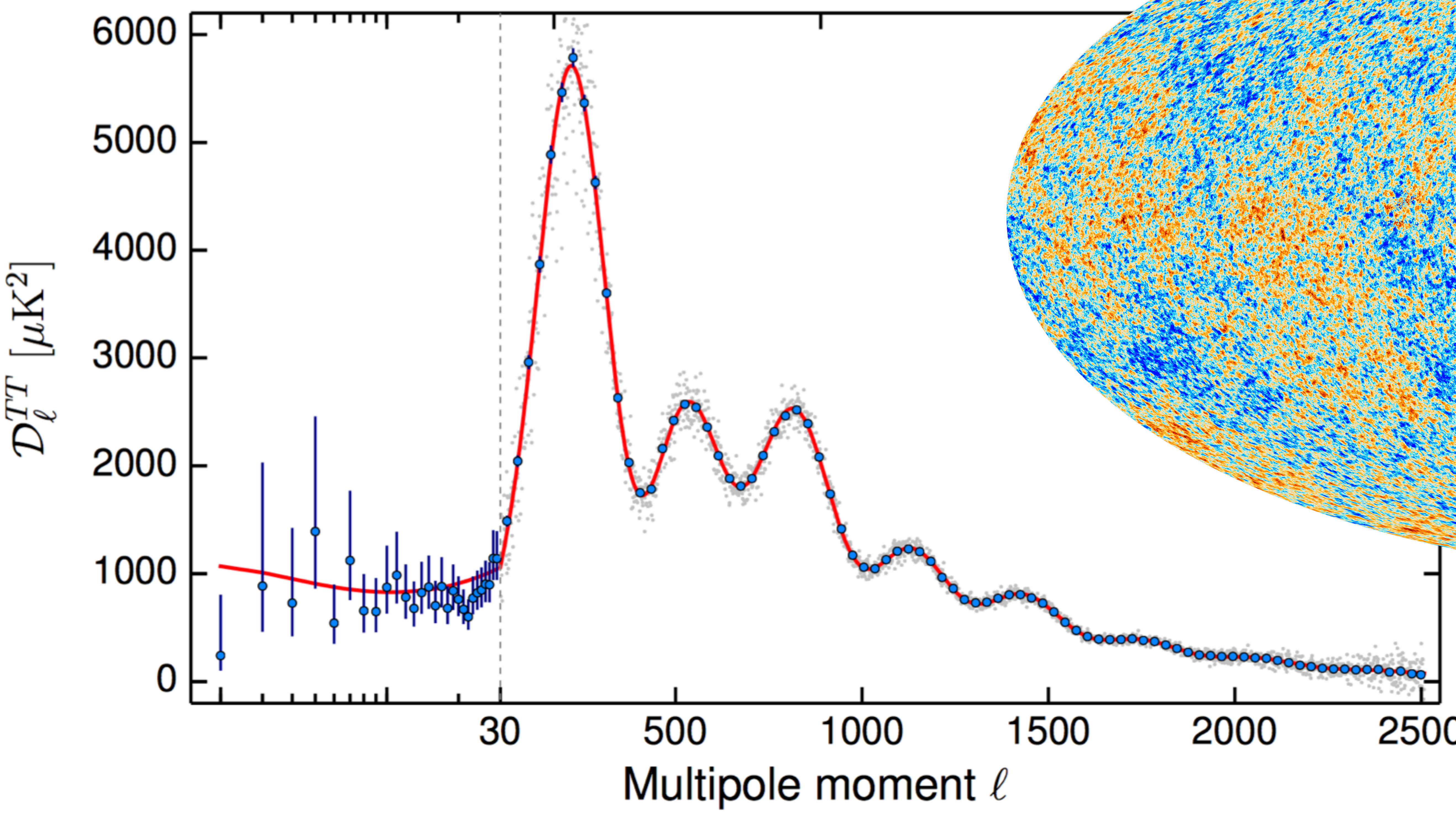
$\rho_{\text{rad}} \propto a^{-4}$

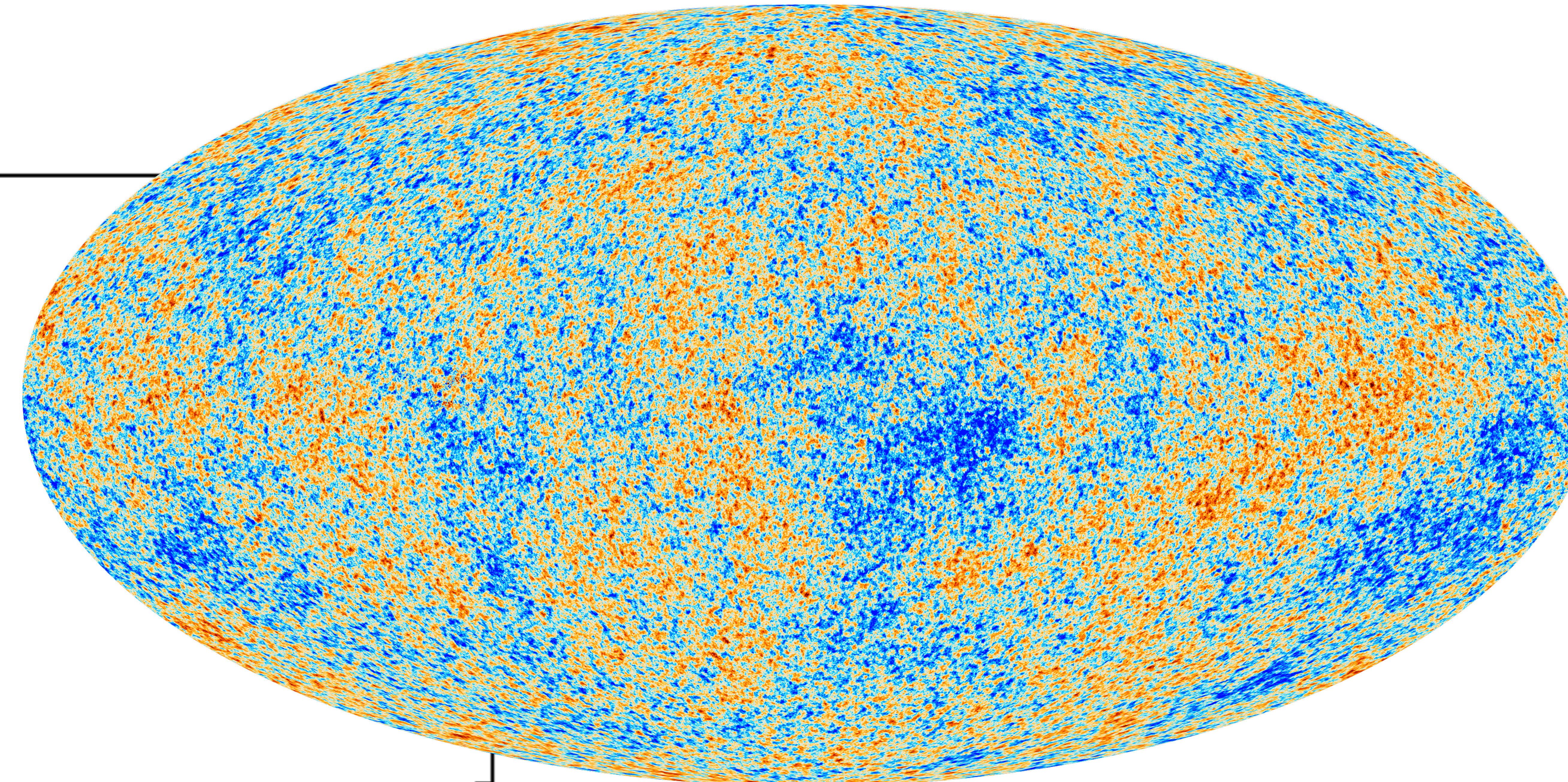
$\rho_{\text{mat}} \propto a^{-3}$

$\rho_{\Lambda} \propto a^{0}$

Phenomenologically valid description for 14 Gyrs!!!!

Planck 2015





$\Omega_{\mathcal{K}} = 0.000 \pm 0.005$

$n_{\rm s} = 0.9639 \pm 0.0047$ almost $\underbrace{\text{scale invariant}}_{\text{excluded}}$

$f_{\rm NL}^{\rm loc} = 0.8 \pm 5$

$f_{\rm NL}^{\rm eq} = -4 \pm 43$

$f_{\rm NL}^{\rm ort} = -26 \pm 21$

} gaussian signal

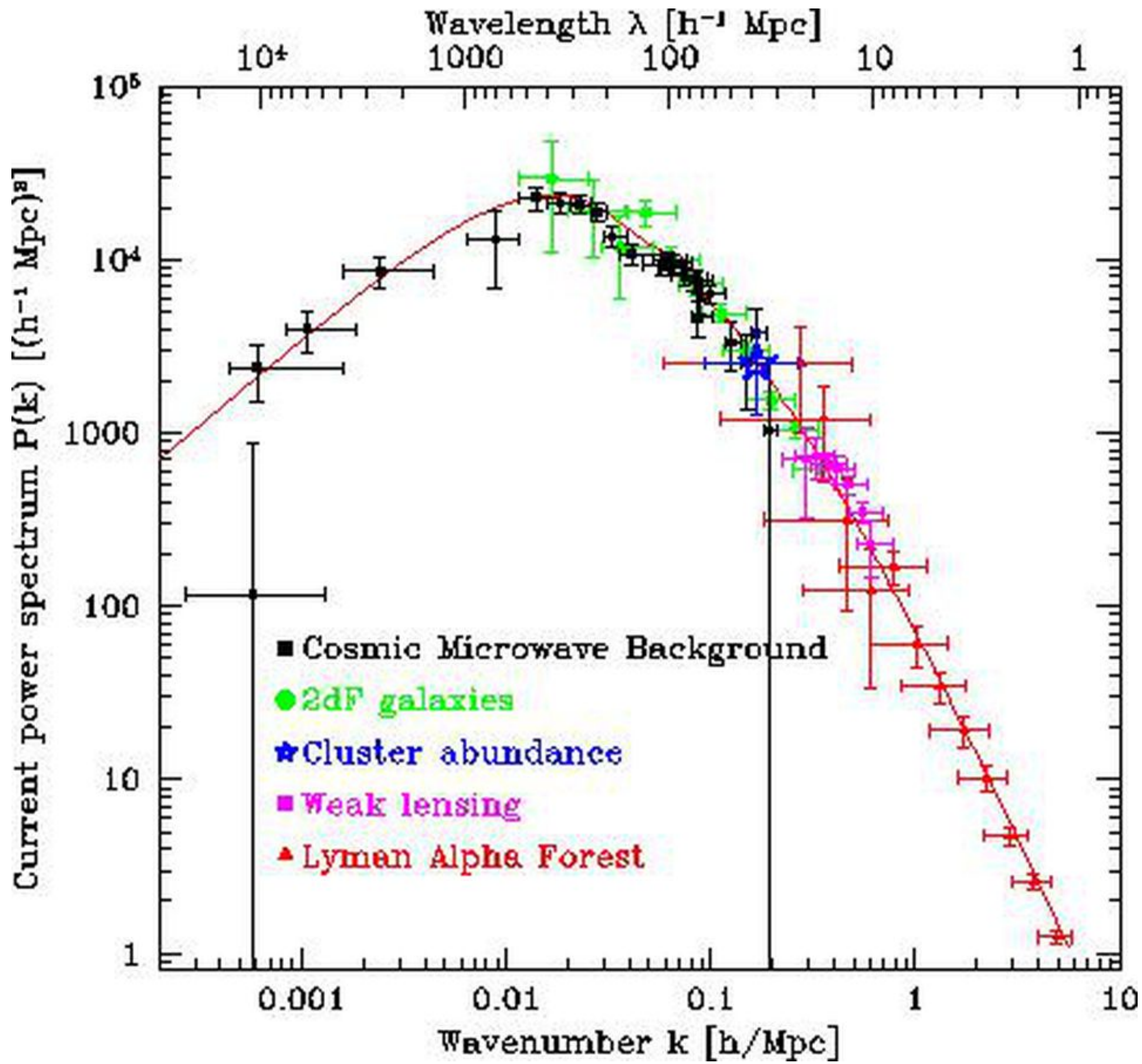
isocurvature $\lesssim 1\%$

$r < 0.08$

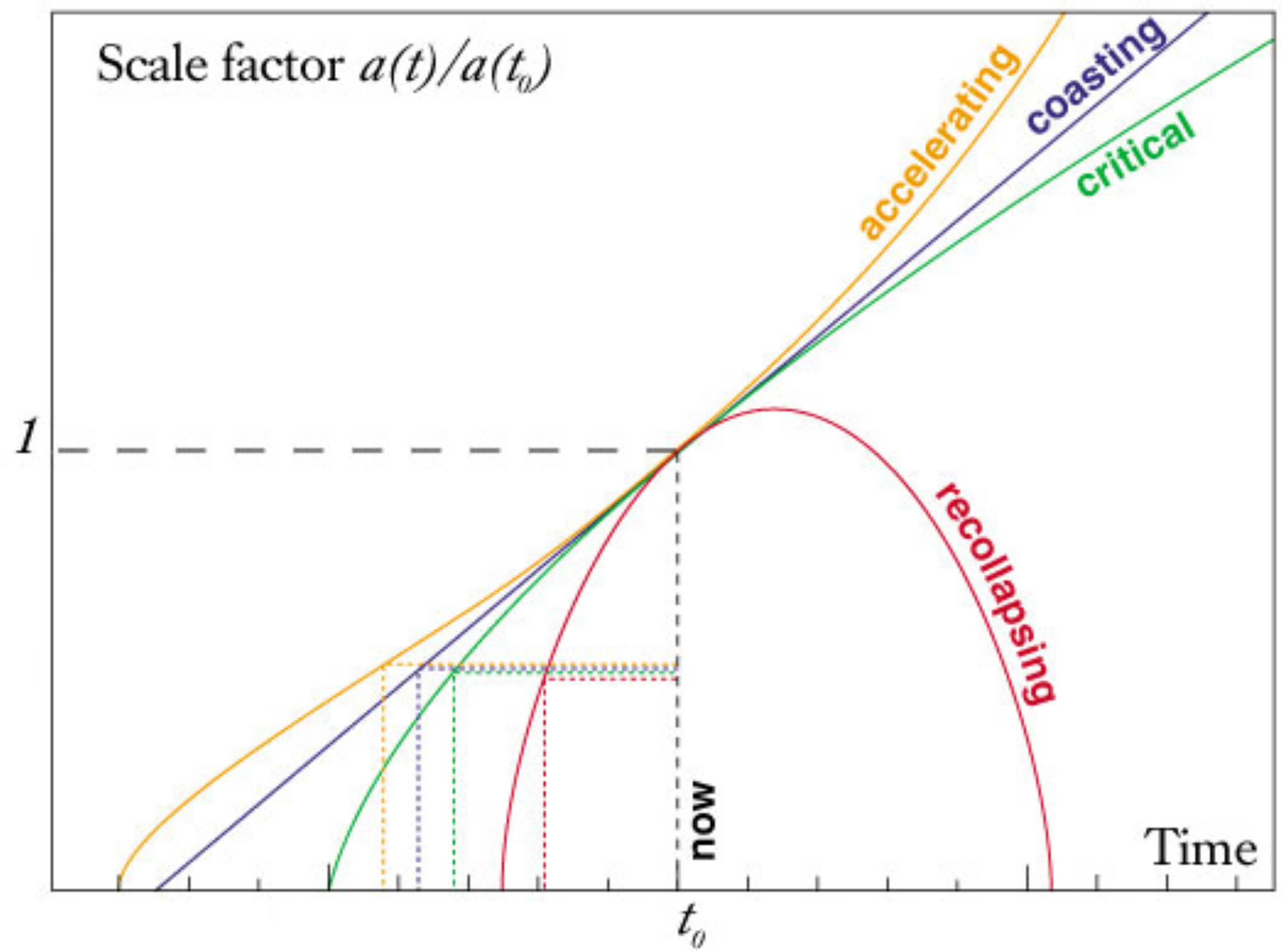
quantum vacuum fluctuations of a single scalar d.o.f

compatible with ***INFLATION***

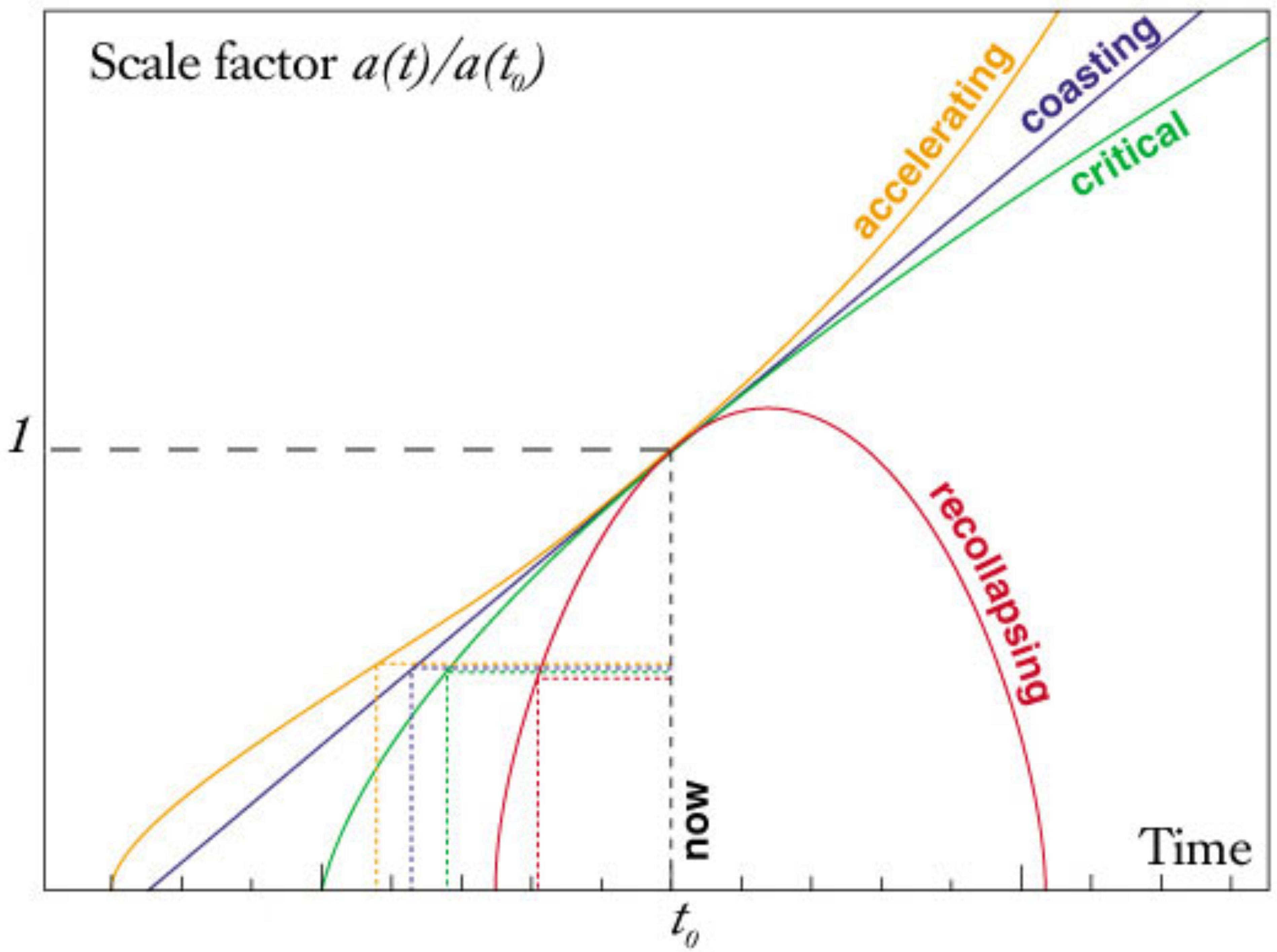
Numerical simulation for large scale structure formation...



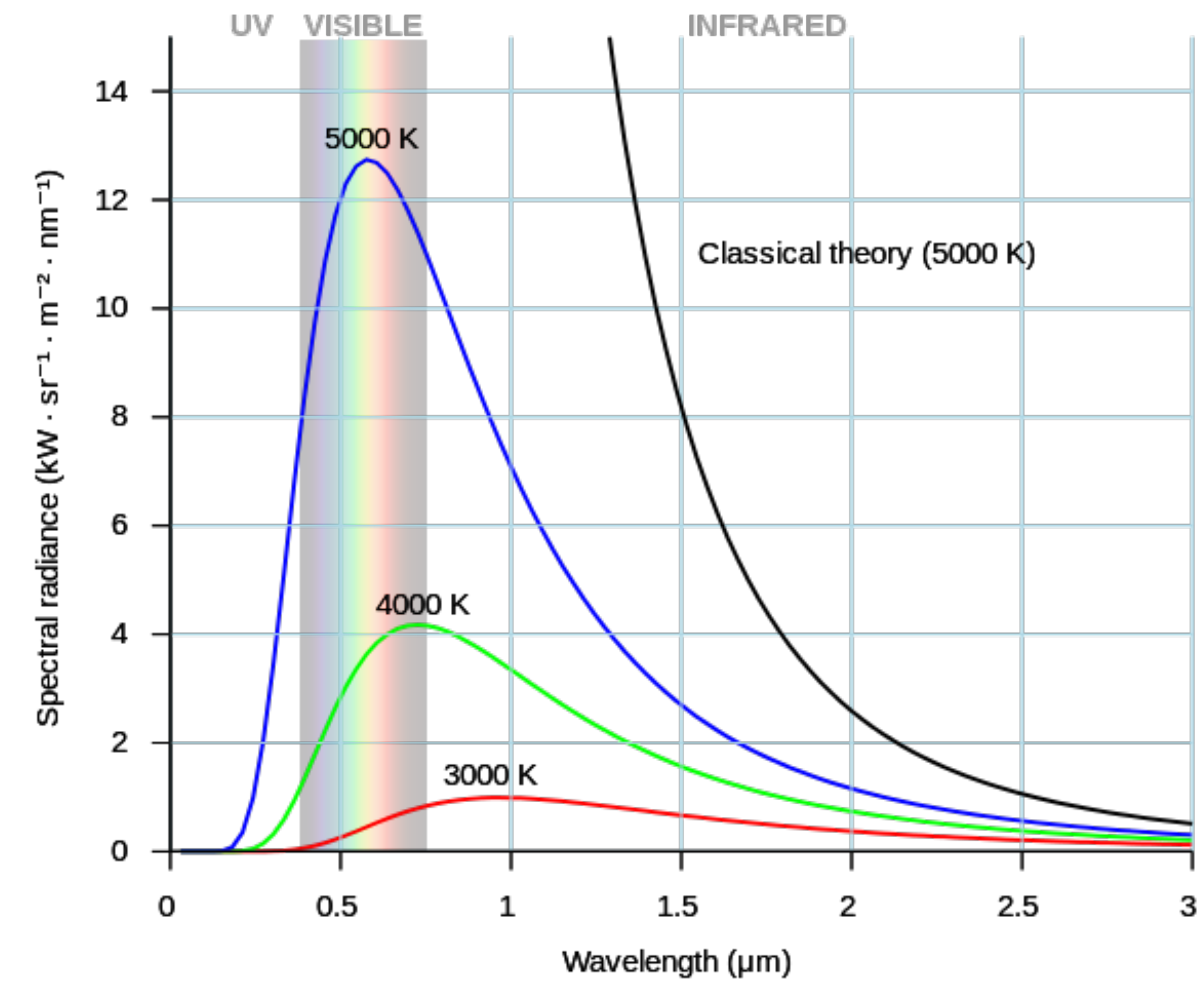
A central problem (though not often formulated thus...): the singularity

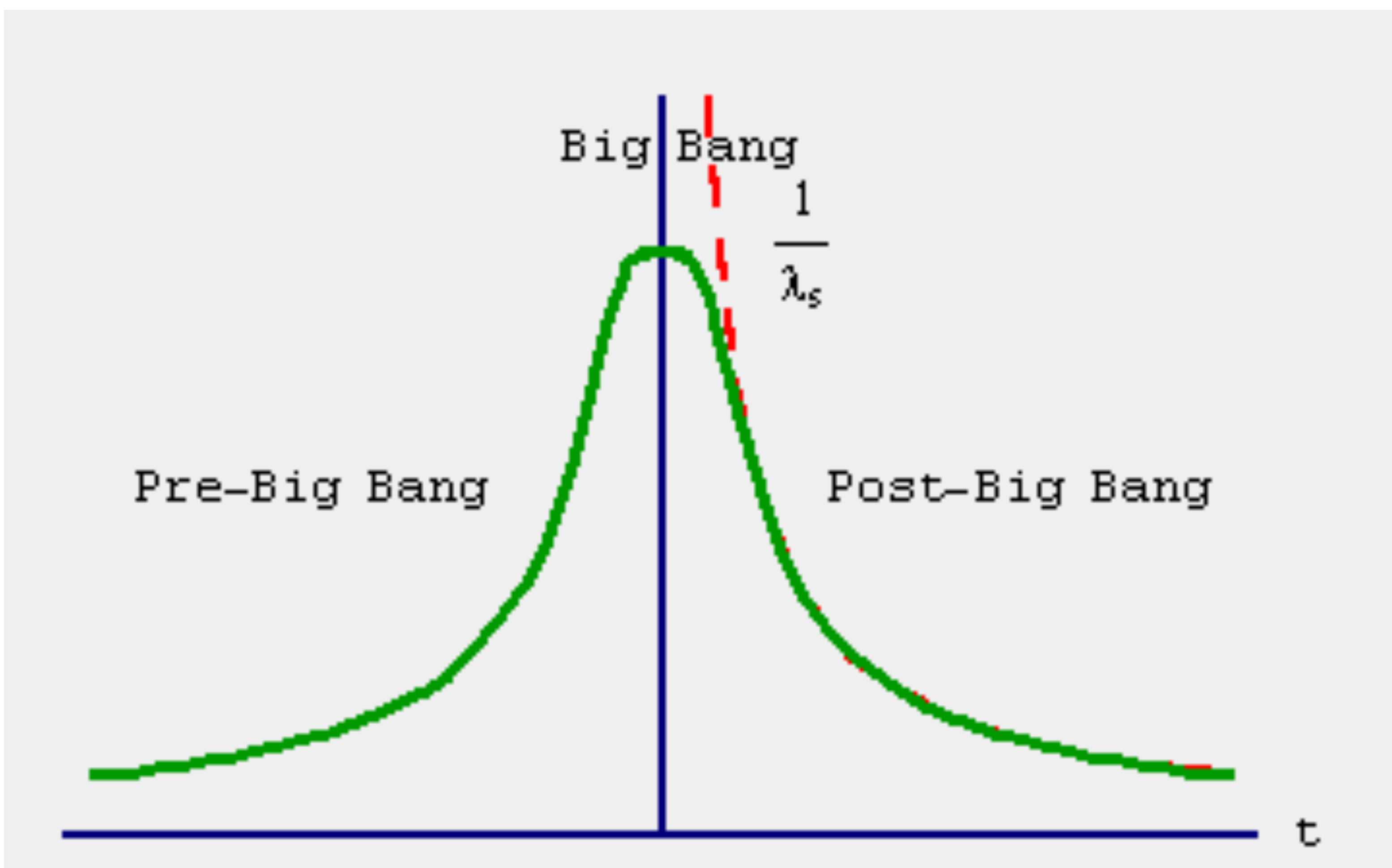


Singularity problem...



a quantum effect?





Quantum cosmology

Hamiltonian GR (3+1)

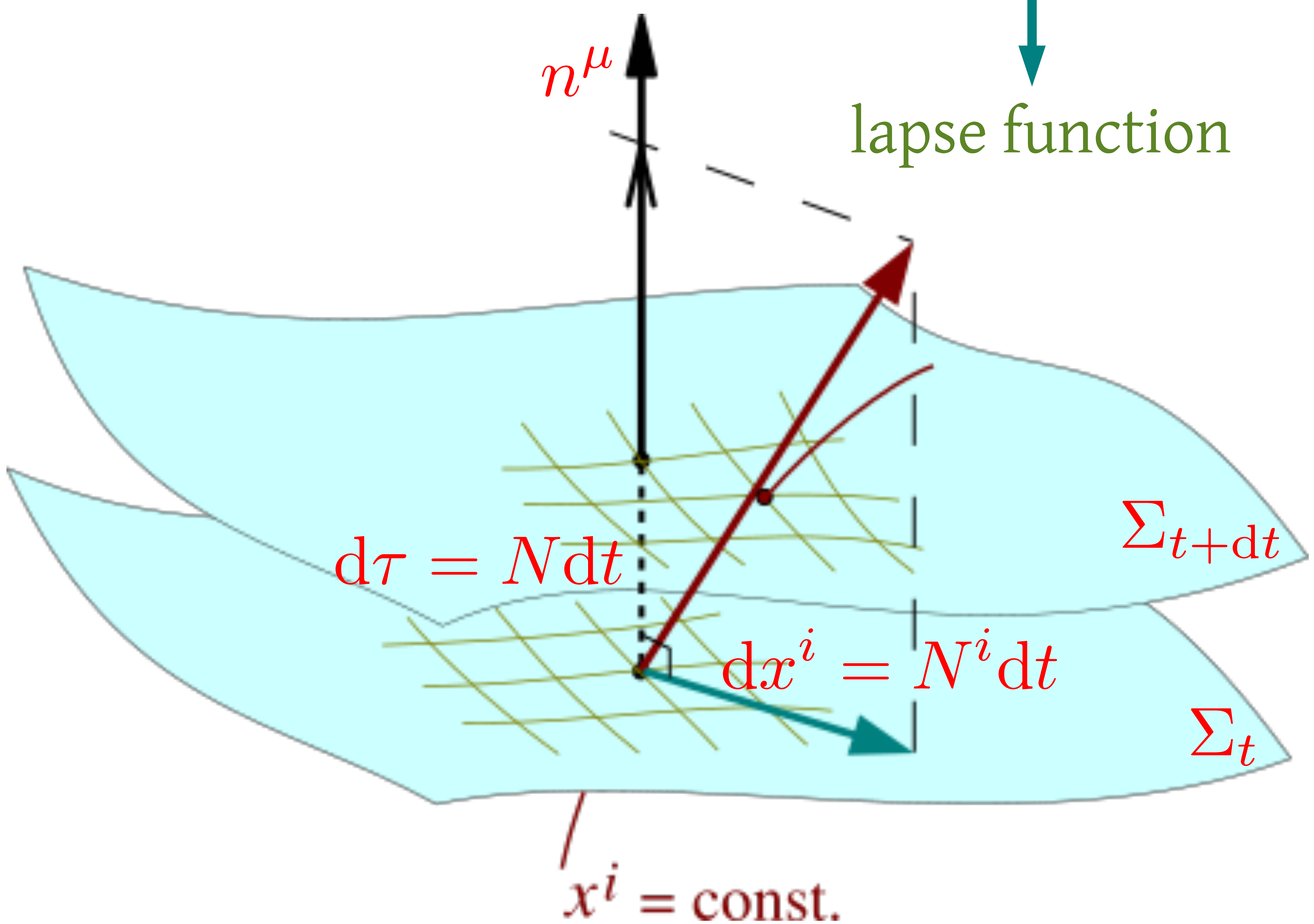
$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)$$

lapse function

intrinsic metric =
first fundamental form

intrinsic curvature tensor
$^3R^i{}_{jkl}(h)$

shift vector



extrinsic curvature =
second fundamental form:

$$K_{ij} = -\nabla_j^{(h)} n_i = \frac{1}{2N}\left(\nabla_j^{(h)} N_i + \nabla_i^{(h)} N_j - \frac{\partial h_{ij}}{\partial t}\right)$$

Action (Einstein-Hilbert, compact space):

$$\mathcal{S} = \frac{1}{16\pi G_{\mathrm{N}}}\left[\int_{\mathcal{M}}\sqrt{-g}\;\left(R-2\Lambda\right)\mathrm{d}^4x + 2\int_{\partial\mathcal{M}}\sqrt{h}K^i{}_i\mathrm{d}^3x\right] + \mathcal{S}_{\mathrm{matter}}\left[\Phi\left(x\right)\right]$$

$$\longrightarrow \mathcal{S} = \int L\mathrm{d}t = \frac{1}{16\pi G_{\mathrm{N}}}\int \mathrm{d}t\left[\int \mathrm{d}^3x\, N\sqrt{h}\left(K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda\right) + L_{\mathrm{matter}}\right]$$

Canonical momenta

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{h}}{16\pi G_{\mathrm{N}}}\left(K^{ij} - h^{ij}K\right)$$

$$\pi^{\Phi} \equiv \frac{\delta L}{\delta \dot{\Phi}} = -\frac{\sqrt{h}}{N}\left(\dot{\Phi} - N\frac{\partial\Phi}{\partial x^i}\right)$$

$$\left.\begin{aligned} \pi^0 &\equiv \frac{\delta L}{\delta \dot{N}} \approx 0 \\ \pi^i &\equiv \frac{\delta L}{\delta \dot{N}^i} \approx 0 \end{aligned}\right\}\ \text{primary constraints}$$

Hamiltonian $$H \equiv \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + \pi^{ij} \dot{h}_{ij} + \pi^\Phi \dot{\Phi} \right) - L$$

$$= \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + N\mathcal{H} + N_i \mathcal{H}^i \right)$$

$$\mathcal{H} = \frac{1}{\sqrt{h}} \left(h_{ik} h_{jl} - \frac{1}{2} h_{ij} h_{kl} \right) \pi^{ij} \pi^{kl} - \sqrt{h}\, {}^3\!R$$

$$\mathcal{H}^i = -2\sqrt{h} \nabla_j \left(\frac{\pi^{ij}}{\sqrt{h}} \right)$$

$$\left.\begin{array}{l} \text{variation wrt lapse: } \mathcal{H} = 0 \rightarrow \text{Hamiltonian constraint} \\ \text{variation wrt shift: } \mathcal{H}^i = 0 \rightarrow \text{momentum constraint} \end{array}\right\} \Longrightarrow \text{classical description complete}$$

Superspace & canonical quantization

relevant configuration space $\mathrm{Riem}\left(\Sigma\right) \equiv \left\{h_{ij}\left(x^{\mu}\right), \Phi\left(x^{\mu}\right) \middle| x \in \Sigma\right\}$

parameters

GR $\Longrightarrow$ invariance/diffeomorphisms $\Longrightarrow$ $\mathrm{Conf} = \dfrac{\mathrm{Riem}\left(\Sigma\right)}{\mathrm{Diff}\left(\Sigma\right)}$: superspace

Wave functional $\Psi\left[h_{ij}\left(x\right), \Phi\left(x\right)\right] = \langle h_{ij}, \Phi | \Psi \rangle$

+ Dirac canonical quantization procedure

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}} \qquad \pi^{\Phi} \to -i\frac{\delta}{\delta \Phi} \qquad \pi^{0} \to -i\frac{\delta}{\delta N} \qquad \pi^{i} \to -i\frac{\delta}{\delta N_i}$$

primary constraints
$$\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\[2ex] \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\rm N}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = \left[-16\pi G_{\rm N}\mathcal{G}_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \frac{\sqrt{h}}{16\pi G_{\rm N}}\left(-{}^3R + 2\Lambda + 16\pi G_{\rm N}\hat{T}^{00}\right)\right]\Psi = 0$$

Wheeler - De Witt equation

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$$

De Witt metric

primary constraints $\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\mathrm{N}}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = 0$$

time-independent Schrödinger equation

primary constraints $\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \qquad \Longrightarrow \qquad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\mathrm{N}}\hat{T}^{0i}\Psi$

same Ψ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = 0$$

TIMELESS Schrödinger equation

mini-superspace

restrict attention from an infinite dimensional configuration space to a 2 dimensional space = mini-superspace

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \mapsto \quad a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right]$$

WDW equation becomes Schrödinger like for $\Psi\left[a(t),\phi(t)\right]$

Conceptual & technical issues

infinite # d.o.f. to a few: mathematical consistency?

freeze momenta... Heisenberg uncertainties?

$[\text{quantization}, \text{minisuperspace}] \neq 0$

mini-superspace

restrict attention from an infinite dimensional configuration space to a 2 dimensional space = mini-superspace

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \mapsto \quad a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right]$$

WDW equation becomes Schrödinger like for $\Psi\left[a(t),\phi(t)\right]$

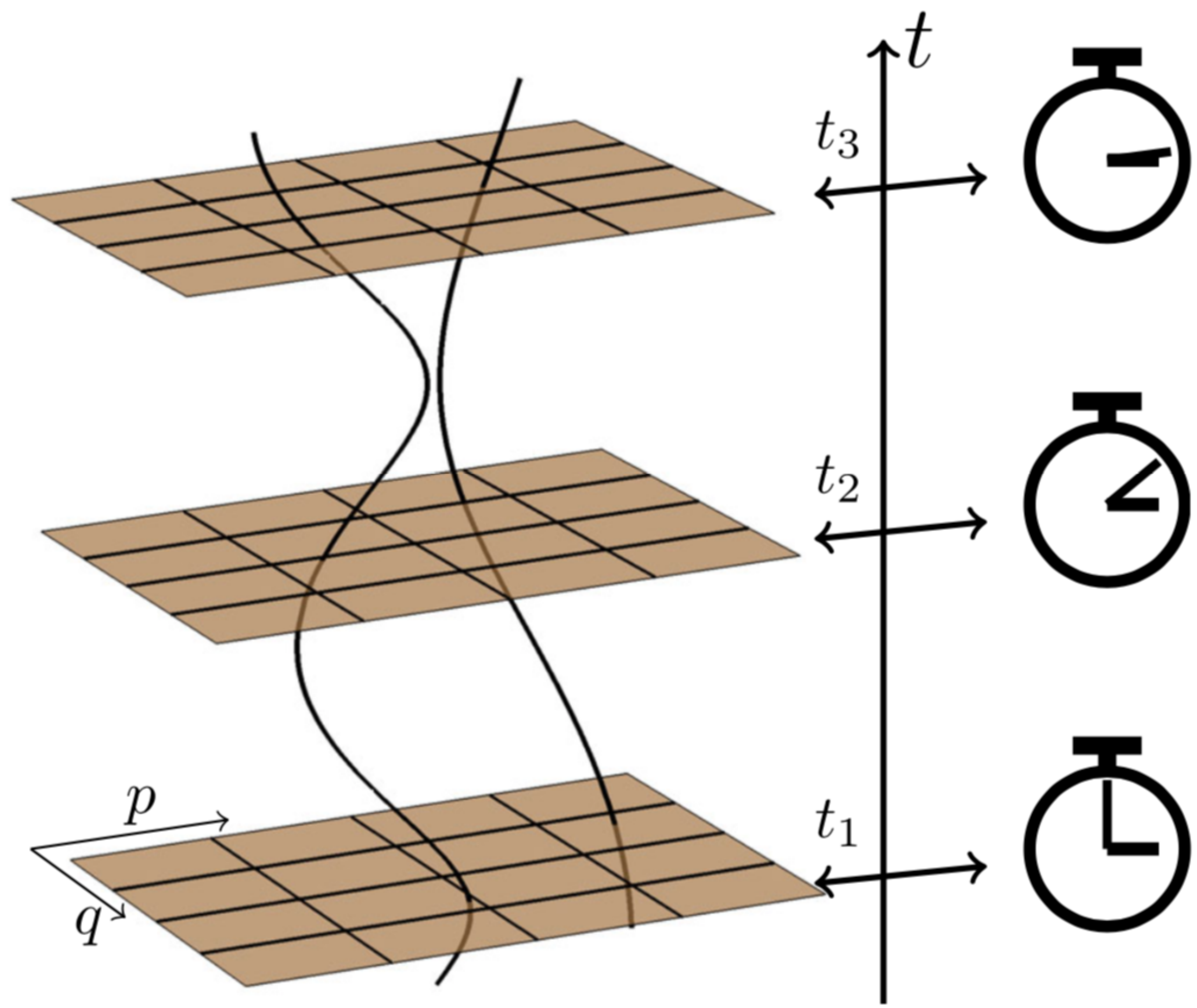
Conceptual & technical issues

ACTUALLY MAKE CALCULATIONS!

The clock issue in quantum cosmology

→ GR = constrained system: lack of external time

→ arbitrary degree of freedom: internal clock



Classical system q_i & p_i

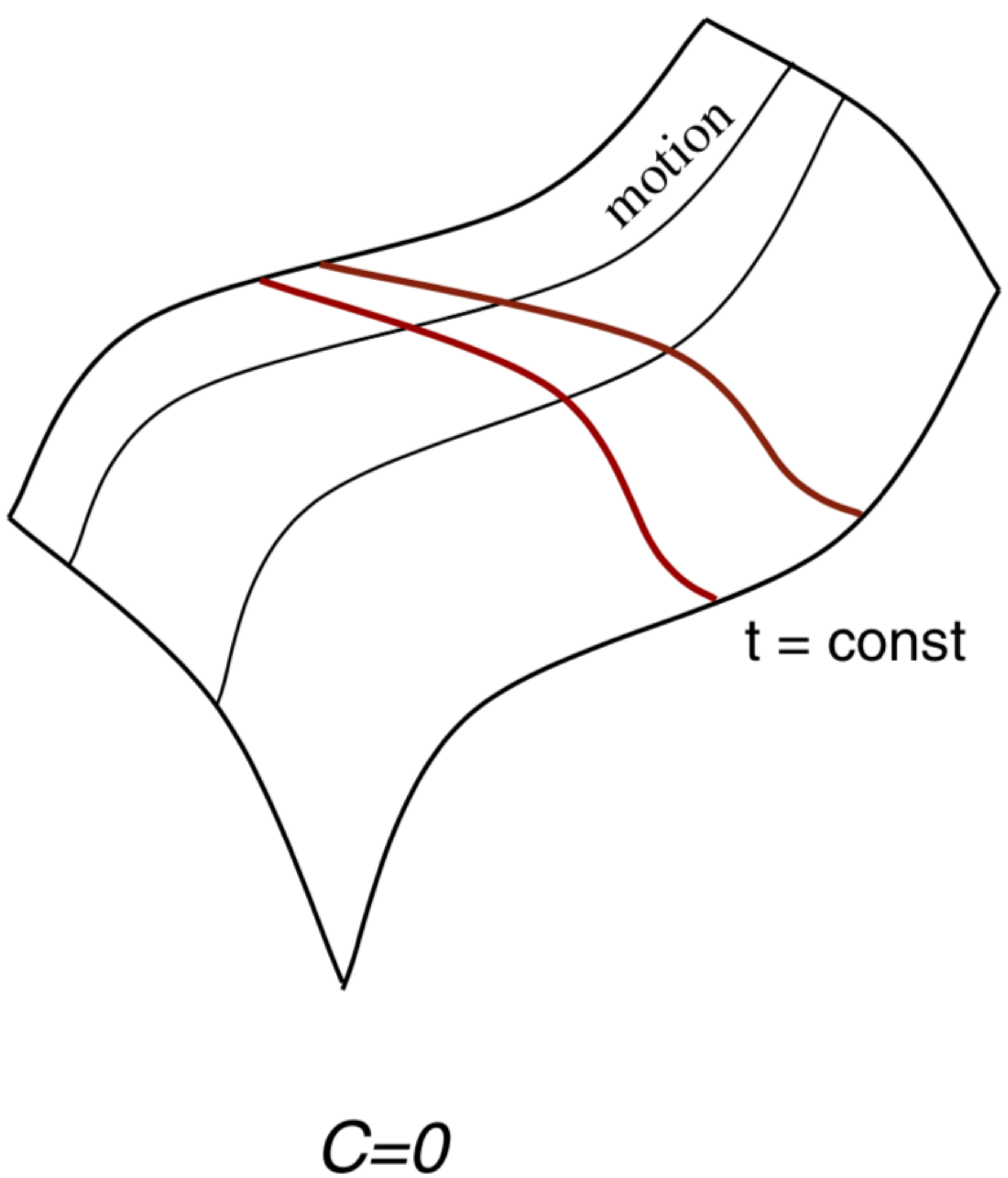
Constraint $C(q_i,p_i)=0$ & $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{O}(q_i,p_i)=\{\mathcal{O},C\}_{\mathrm{P.B}}$

evolution parameter (time)

observable

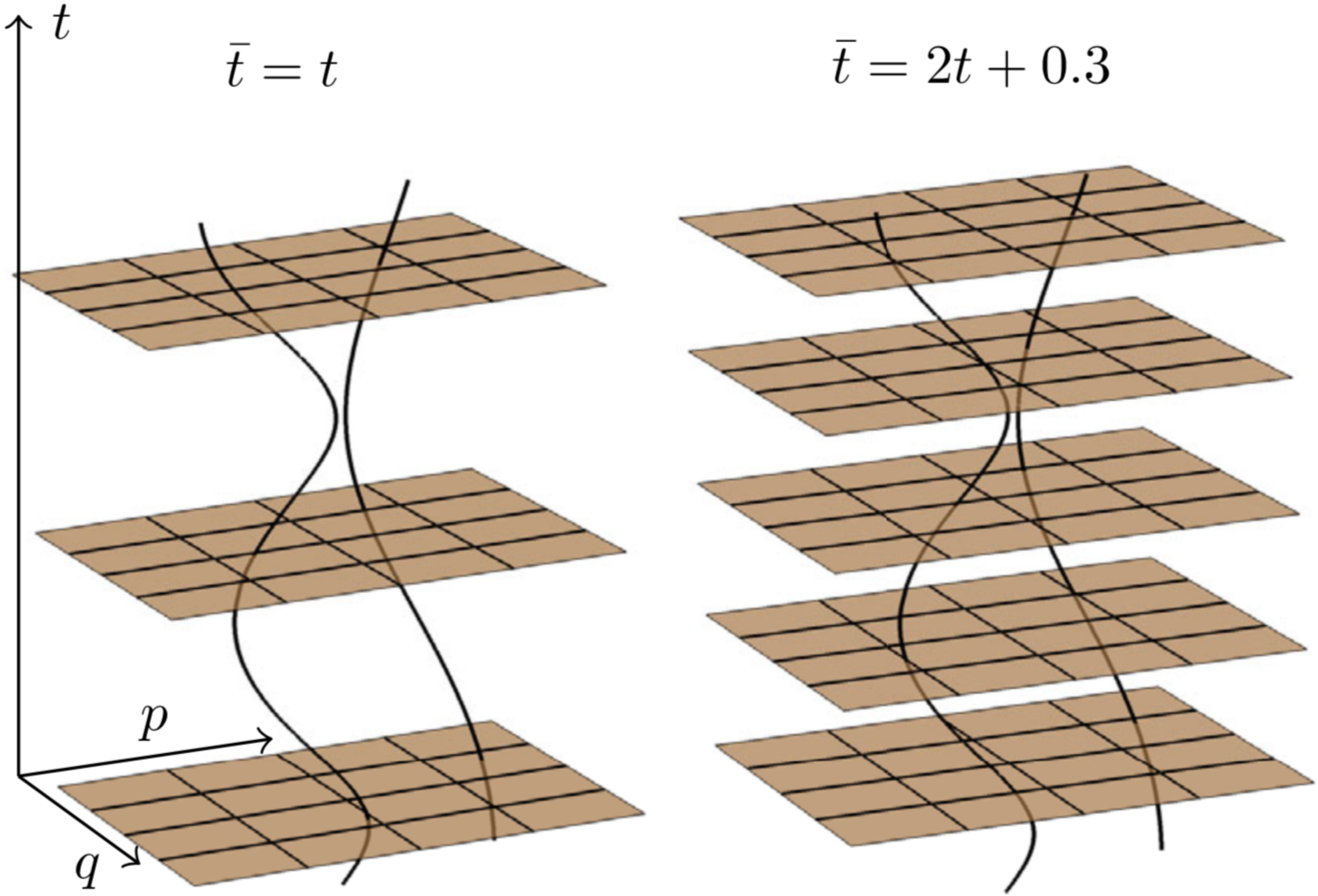
Time parametrization invariance $\tau \to \tau'$ $\longrightarrow$ $N(q_i,p_i,\tau)$

arbitrary non vanishing lapse function

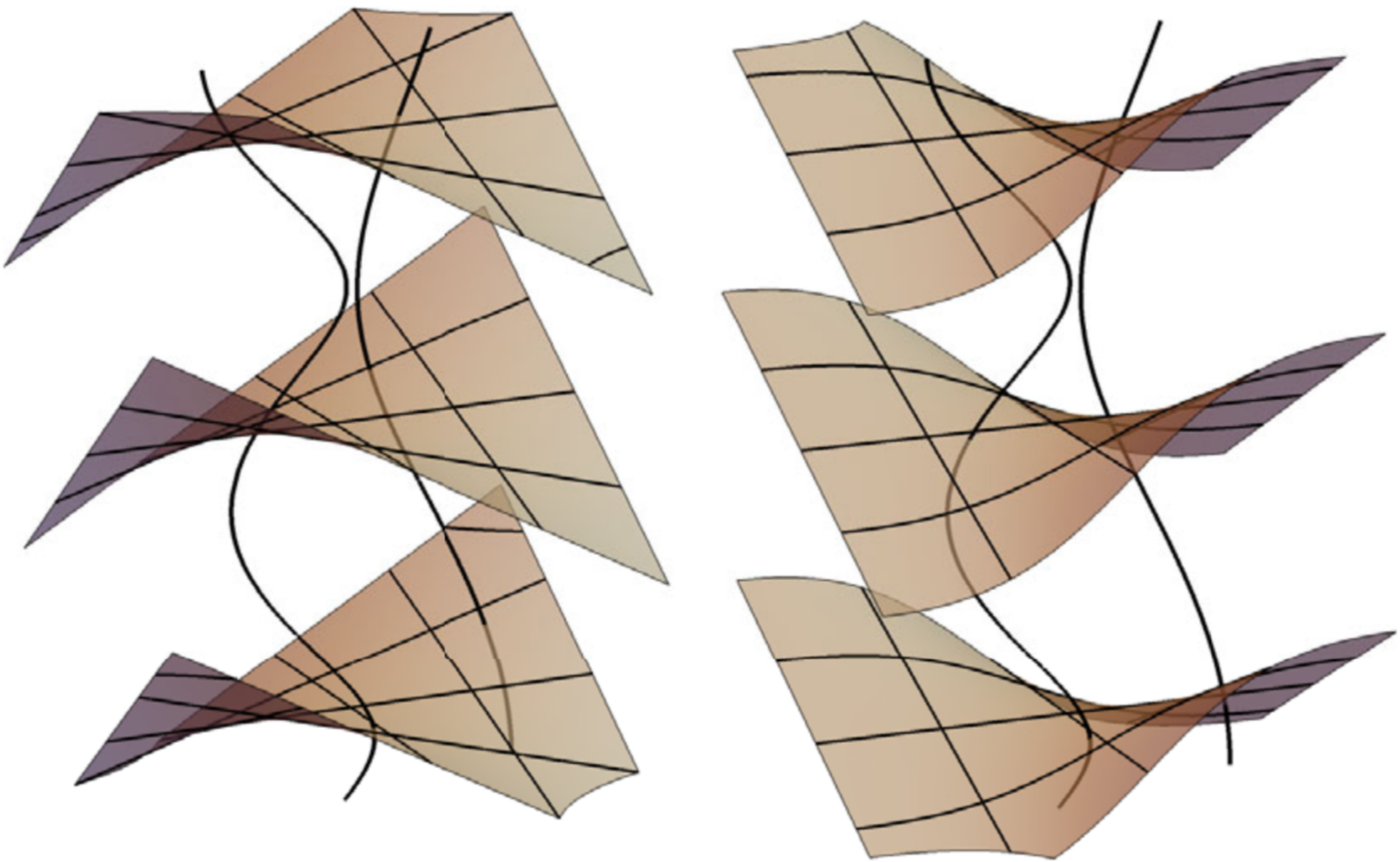


$$\mathrm{d}\tau = N\mathrm{d}\tau' \implies \frac{\mathrm{d}}{\mathrm{d}\tau'}\mathcal{O}(q_i,p_i)=\{\mathcal{O},NC\}_{\mathrm{P.B}}$$

hamiltonian $H=NC$


t
$\bar{t} = t$
$\bar{t} = 2t + 0.3$
p
q

$\bar{t} = t + qp$ $\qquad\qquad$ $\bar{t} = t - \frac{3qp}{3p^2+1}$



Constrained system $C\left(\left\{q^k\right\},\left\{p_k\right\}\right)=H_{\text{tot}}\left(\left\{q^k\right\},\left\{p_k\right\}\right)=0$

Canonical transformation $\left(\left\{q^k\right\},\left\{p_k\right\}\right)\mapsto\left(\left\{Q^a\right\},\left\{P_a\right\}\right)$

$\longrightarrow$ $\exists Q^\alpha$ such that $\left\{Q^\alpha, H_{\text{tot}}\right\}_{\text{P.B.}}=1=\dfrac{\mathrm{d}Q^\alpha}{\mathrm{d}t}$?

defines clock

Quantum system $\hat{H}_{\text{tot}}\Psi\equiv\hat{C}\Psi(Q^a)=0$

$\hat{C}$ linear in $\hat{P}_\alpha=-i\dfrac{\partial}{\partial Q^\alpha}$

will become time

$\longrightarrow$ $\hat{C}=\hat{P}_\alpha+\hat{H}(P_1,\cdots,P_{\alpha-1},P_{\alpha+1},\cdots,P_n,\{Q^a\})$

$\hat{C}\Psi(Q^a)=0\Longrightarrow$ time-dependent Schrödinger equation

A simple example

$$\mathcal{S} = \frac{1}{2}\int \left(\frac{\dot{x}^2}{z} - zx^2 - \frac{\dot{y}^2}{z} + zy^2\right) \mathrm{d}t$$

First, redefine time: $\mathrm{d}\tau = z\mathrm{d}t$ $\longrightarrow$ $\mathcal{S} = \frac{1}{2}\int \left[\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - x^2 - \left(\frac{\mathrm{d}y}{\tau}\right)^2 + y^2\right] \mathrm{d}\tau$

Classical EOMs $\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = -x \\ \dfrac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = -y \end{cases}$ $\longrightarrow$ 2 independent harmonic oscillators

$$H_{\text{tot}} = H + H_y$$

Canonical transformation $T = \arctan\left(\frac{p_y}{y}\right)$ & $P_T = -\frac{1}{2}\left(p_y^2 + y^2\right) = H_y$

$$\{T, P_T\}_{\text{P.B.}} = 1$$

$$\downarrow$$

$$\{T, H_{\text{tot}}\}_{\text{P.B.}} = 1 \longrightarrow \frac{\mathrm{d}T}{\mathrm{d}\tau} = 1 \longrightarrow \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}T} = p_x \\ \dfrac{\mathrm{d}p_x}{\mathrm{d}T} = -x \end{cases}$$

$H_{\text{tot}} = P_T + H$ on shell $H_{\text{tot}} \approx 0$

Quantization: x only! $\hat{H}_{\text{tot}}\psi(x,T) = 0 \implies i\frac{\partial \psi}{\partial T} = \hat{H}\psi(x,T)$

& $\int |\psi(x,T)|^2 \mathrm{d}x = 1$

y remains classical (clock)

Bianchi I case $\quad ds^2 = -N^2 d\tau^2 + \sum_{i=1}^{3} a_i^2 \left(dx^i\right)^2$

Scale factors $\quad \begin{cases} a_1 & = & e^{\beta_0+\beta_+ + \sqrt{3}\beta_-} \\ a_2 & = & e^{\beta_0+\beta_+ - \sqrt{3}\beta_-} \\ a_3 & = & e^{\beta_0 - 2\beta_+} \end{cases}$

Volume $V \equiv a_1 a_2 a_3 = e^{3\beta_0}$

$$d\beta_0 = \frac{1}{3} e^{-3\beta_0} dV$$

Action $\quad \mathcal{S} = \int d\tau \Big(\underbrace{p_0 \dot\beta_0 + p_+ \dot\beta_+ + p_- \dot\beta_-}_{\frac{d\theta}{d\tau}} - \underbrace{NC}_{H} \Big)$

canonical
one-form

constraint $\quad C = \frac{e^{-3\beta_0}}{24} \left(-p_0^2 + p_+^2 + p_-^2\right)$

ensure canonical one-form remains canonical $\quad p_V \equiv \frac{\mathrm{e}^{-3\beta_0}}{3} p_0$

$$\mathrm{d}\theta = p_V \mathrm{d}V + p_+ \mathrm{d}\beta_+ + p_- \mathrm{d}\beta_-$$

constraint $\quad C = \frac{3V}{8}\left(-p_V^2 + \frac{p_+^2 + p_-^2}{9V^2}\right)$

cyclic variable $\dot{p}_\pm = 0$ $\quad$ set $p_+ = k\cos\alpha$ and $p_- = k\sin\alpha$

$$\longrightarrow \mathrm{d}\theta = p_V \mathrm{d}V + p_k \mathrm{d}k + p_\alpha \mathrm{d}\alpha + \underbrace{\mathrm{d}\left(k\cos\alpha\beta_+ + k\sin\alpha\beta_-\right)}_{\rightarrow \text{ exact... ignore!}}$$

$$\begin{aligned} p_k &\equiv -\left(\cos\alpha\beta_+ + \sin\alpha\beta_-\right), \\ p_\alpha &\equiv \left(k\sin\alpha\beta_+ - k\cos\alpha\beta_-\right) \end{aligned}$$

neither α nor P_α in $H = NC$

the system reduces to

$$\begin{cases} \mathrm{d}\theta &= p_V \mathrm{d}V + p_k \mathrm{d}k \\ C &= \dfrac{3V}{8}\left(-p_V^2 + \dfrac{k^2}{9V^2}\right) \end{cases}$$

Hamilton equations

$$\begin{aligned} \dot{k} &= 0 \\ \dot{p}_k &= -N\frac{k}{12V} \\ \dot{V} &= -N\frac{3Vp_V}{4} \\ \dot{p}_V &= -N\left[\frac{3}{8}\left(-p_V^2 + \frac{k^2}{9V^2}\right) - \frac{k^2}{12V^2}\right] \end{aligned}$$

+ constraint

$$\frac{3V}{8}\left(-p_V^2 + \frac{k^2}{9V^2}\right) = 0$$

$\longrightarrow$ closed for V and p_V

Choosing a time

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(9\frac{p_k}{k}\right) = -\frac{3}{4}\frac{N}{V}$$ monotonically increasing function

valid time choice $\tau = \frac{9p_k}{k} \implies N = -\frac{4}{3}V$

Solving directly in the action

$$\mathcal{S} = \int \mathrm{d}\theta = \int \mathrm{d}\tau \left(p_V \dot{V} - \underbrace{\frac{V^2 p_V^2}{2}}_{H}\right)$$

classical unconstrained one dimensional system

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(V p_V) = 0 \implies V p_V = V_0 p_{V0}$$

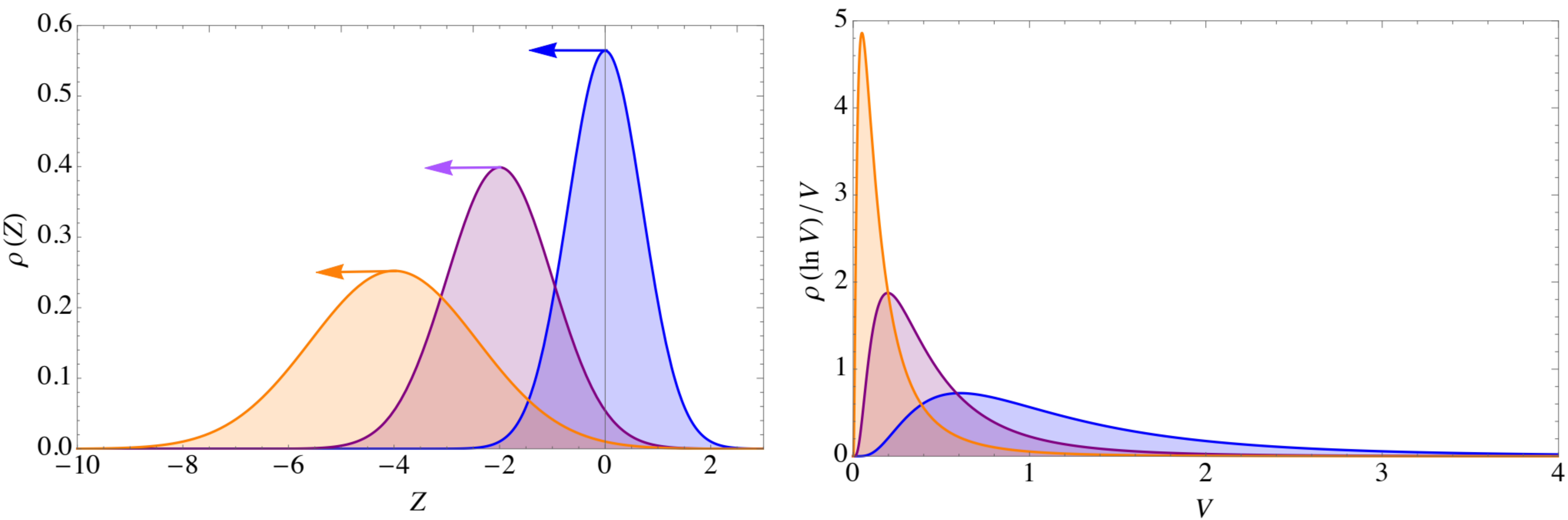
$V = V_0 \mathrm{e}^{(Vp_V)\tau}$ and $p_V = p_{V0}\mathrm{e}^{-(Vp_V)\cdot\tau}$

symmetric ordering choice

$$H = V^2 p_V^2 \quad \mapsto \quad \hat{H} = \sqrt{V}\frac{1}{i}\partial_V\sqrt{V}\cdot\sqrt{V}\frac{1}{i}\partial_V\sqrt{V}$$

coordinate transformation $V \mapsto Z = \ln V$

$$\longrightarrow \quad U\hat{H}U^{-1} = -\partial_Z^2, \quad \text{and} \quad Z \in \mathbb{R}$$



slow-gauge time
$$\mathrm{d}\theta \;=\; (Vp_V)\mathrm{d}V - \left(\frac{V^2p_V^2}{2}\right)\mathrm{d}\left(\frac{9p_k}{k} + \frac{V - \ln V}{Vp_V}\right)$$
$$+\mathrm{d}\left(\frac{9p_k}{2k}V^2p_V^2 + \frac{1}{2}V\ln Vp_V - \frac{1}{2}V^2p_V\right)$$

$\longrightarrow$ Action
$$\mathcal{S} = \int \mathrm{d}\theta = \int \mathrm{d}\eta\left(Vp_V\acute{V} - \frac{V^2p_V^2}{2}\right)$$

new time variable $\eta \equiv \dfrac{9p_k}{k} + \dfrac{V - \ln V}{Vp_V}$

$$\acute{V} \equiv \mathrm{d}V/\mathrm{d}\eta$$

canonical if
$$\pi_V = p_V V$$
$$\downarrow$$
$$H = \frac{1}{2}V^2p_V^2 = \frac{1}{2}\pi_V^2$$

freely moving particle...
$$(V, \pi_V) \in \mathbb{R}_+ \times \mathbb{R}$$
on the half line

Quantization: a gaussian wave packet

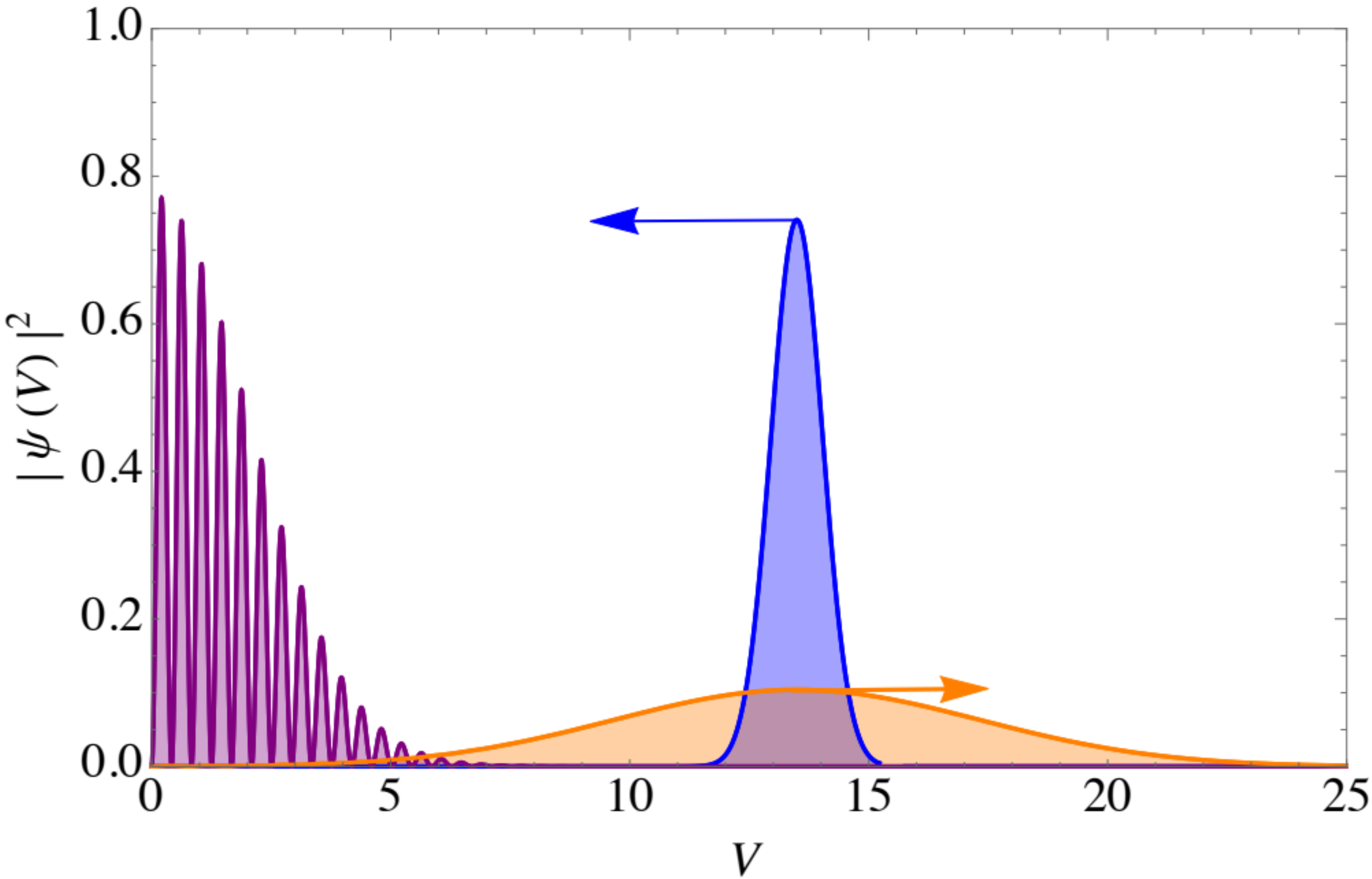
$$u(V,\eta) = \frac{\mathrm{e}^{-k^2/4}}{\sqrt{1+4i\eta}} \exp\left[-\frac{(V - ik/2)^2}{1+4i\eta}\right]$$

implement boundary conditions to ensure self-adjointness

$$\psi(V,\eta) = \frac{u(V+V_0,\eta) - u(-V+V_0,\eta)}{\left[\sqrt{\pi/2}\left(1-\mathrm{e}^{-V_0^2-k^2/2}\right)\right]^{1/2}}$$

$\longrightarrow$ ***solves the Schrödinger equation***

$$i\frac{\partial}{\partial\eta}\psi = -\,\triangle_D\,\psi$$



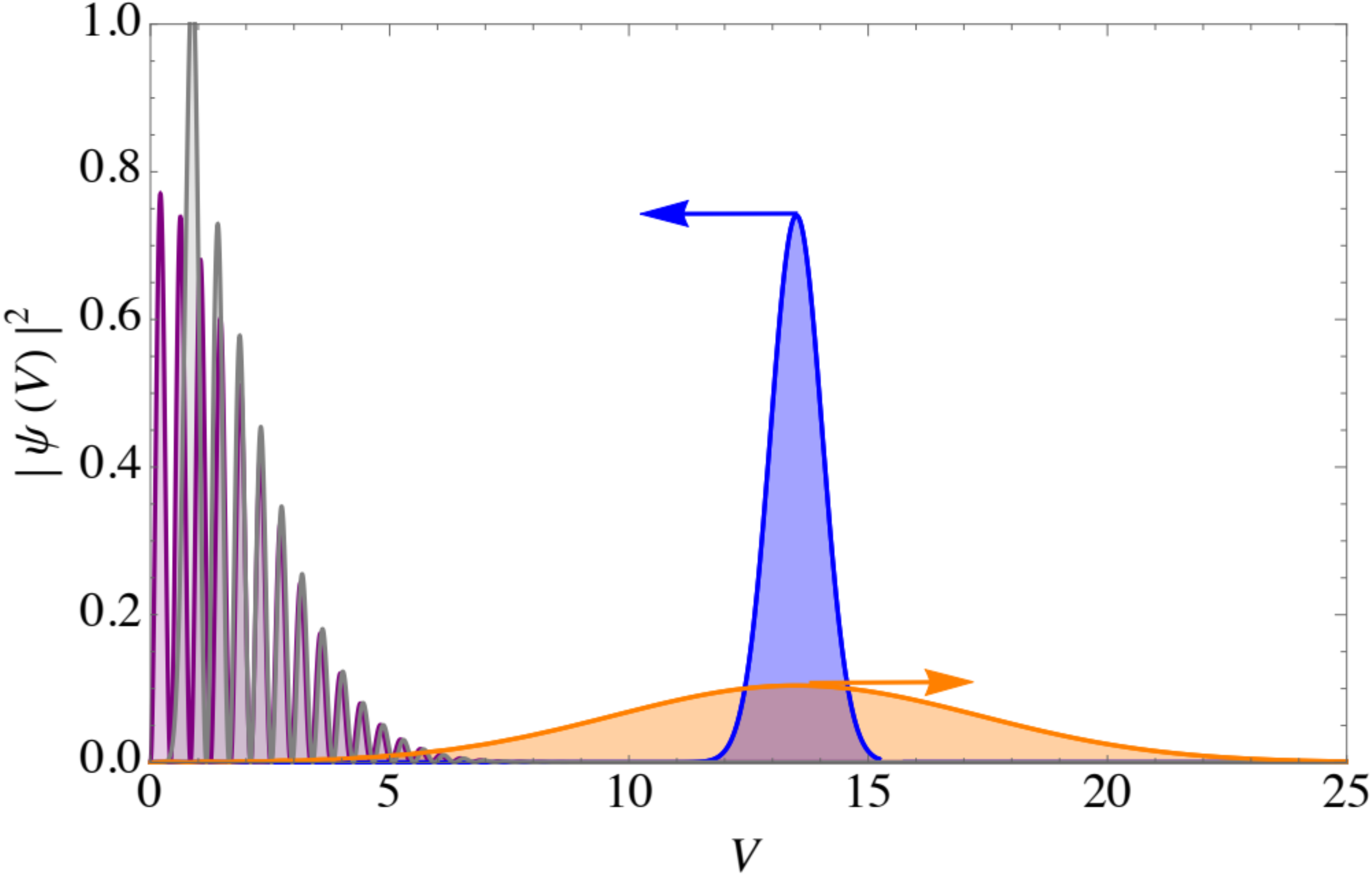
1.0
0.8
0.6
0.4
0.2
0.0
$|\psi(V)|^2$
0
5
10
15
20
25
V

Operator ordering ambiguity $\pi_V^2 \mapsto \hat{V}^s \hat{\pi}_V \hat{V}^{-2s} \hat{\pi}_V \hat{V}^s$

$\longrightarrow$ $\pi_V^2 \mapsto \hat{\pi}_V^2 + s\hat{V}^{-2}$

$\downarrow$

self-adjoint hamiltonian on the half-line $s > 3/4$



1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
V
$|\psi(V)|^2$

Closed algebra of operators

$$\begin{cases} [\hat{V}^2, \hat{H}] & = & 4i\hat{D}, \\ [\hat{D}, \hat{H}] & = & 2i\hat{H}, \\ [\hat{V}^2, \hat{D}] & = & 2i\hat{V}^2, \end{cases}$$

$$\hat{D} \equiv \frac{1}{2}\left(\hat{V}\hat{\pi}_V + \hat{\pi}_V\hat{V}\right)$$

→ Heisenberg equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{V}^2 = -i[\hat{V}^2, \hat{H}] = 4\hat{D}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{D} = -i[\hat{D}, \hat{H}] = 2\hat{H}$$

Heisenberg equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{V}^2 = -i[\hat{V}^2, \hat{H}] = 4\hat{D}$$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\hat{D} = -i[\hat{D}, \hat{H}] = 2\hat{H}$$

solution as time-dependent operators

$$\hat{D}(\eta) = 2\hat{H}\eta + \hat{D}(0) \qquad \hat{V}^2 = 4\hat{\eta}^2 + 4\hat{D}(0)\eta + \hat{V}^2(0)$$

expectation values follows similar equations…

semi-classical variables

$$\check{V}(t) = \sqrt{\langle \hat{V}^2(t) \rangle}$$

$$\check{\pi}_V(t) = \frac{\langle \hat{D}(t) \rangle}{\check{V}(t)}$$

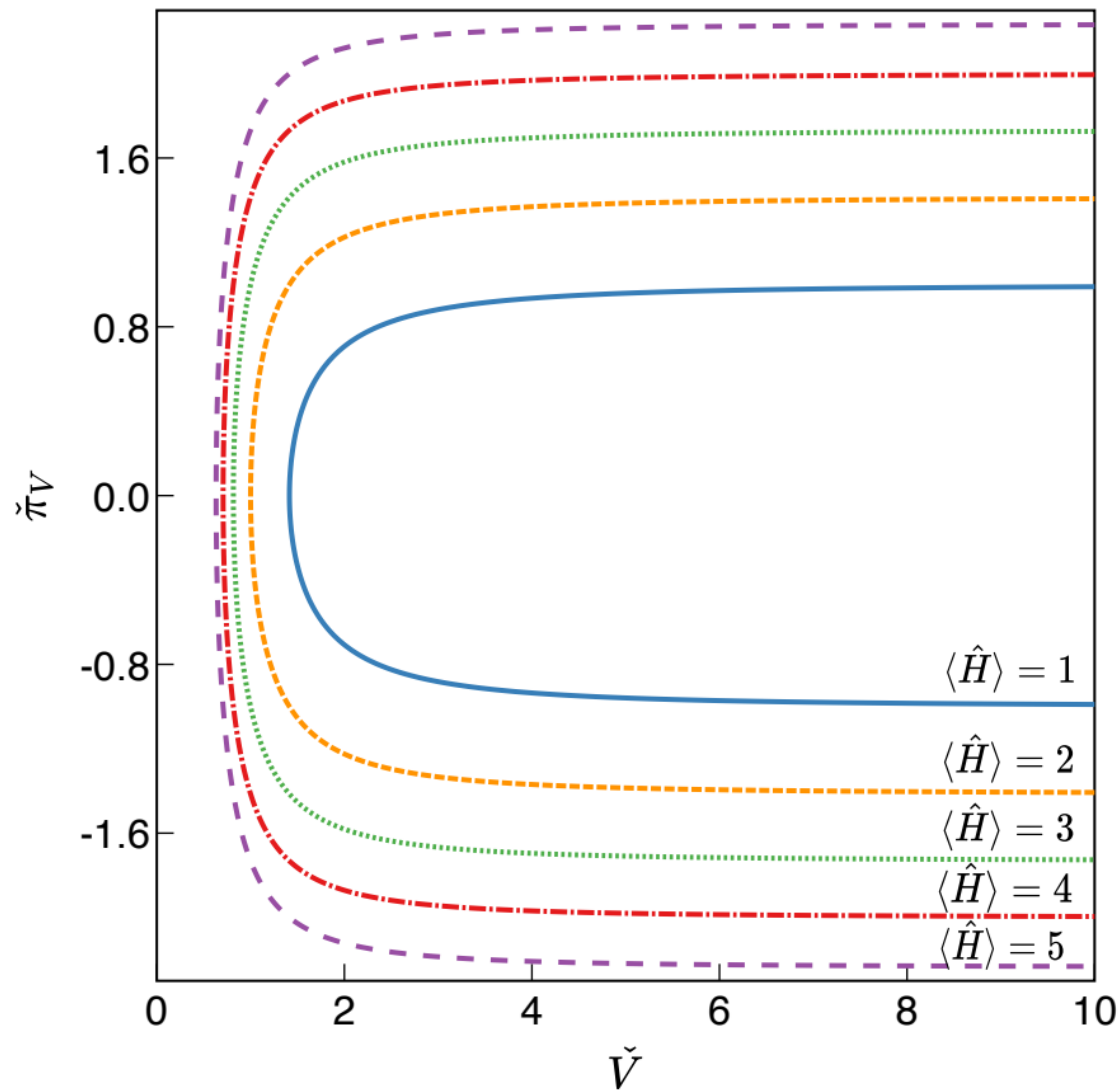
phase space solution

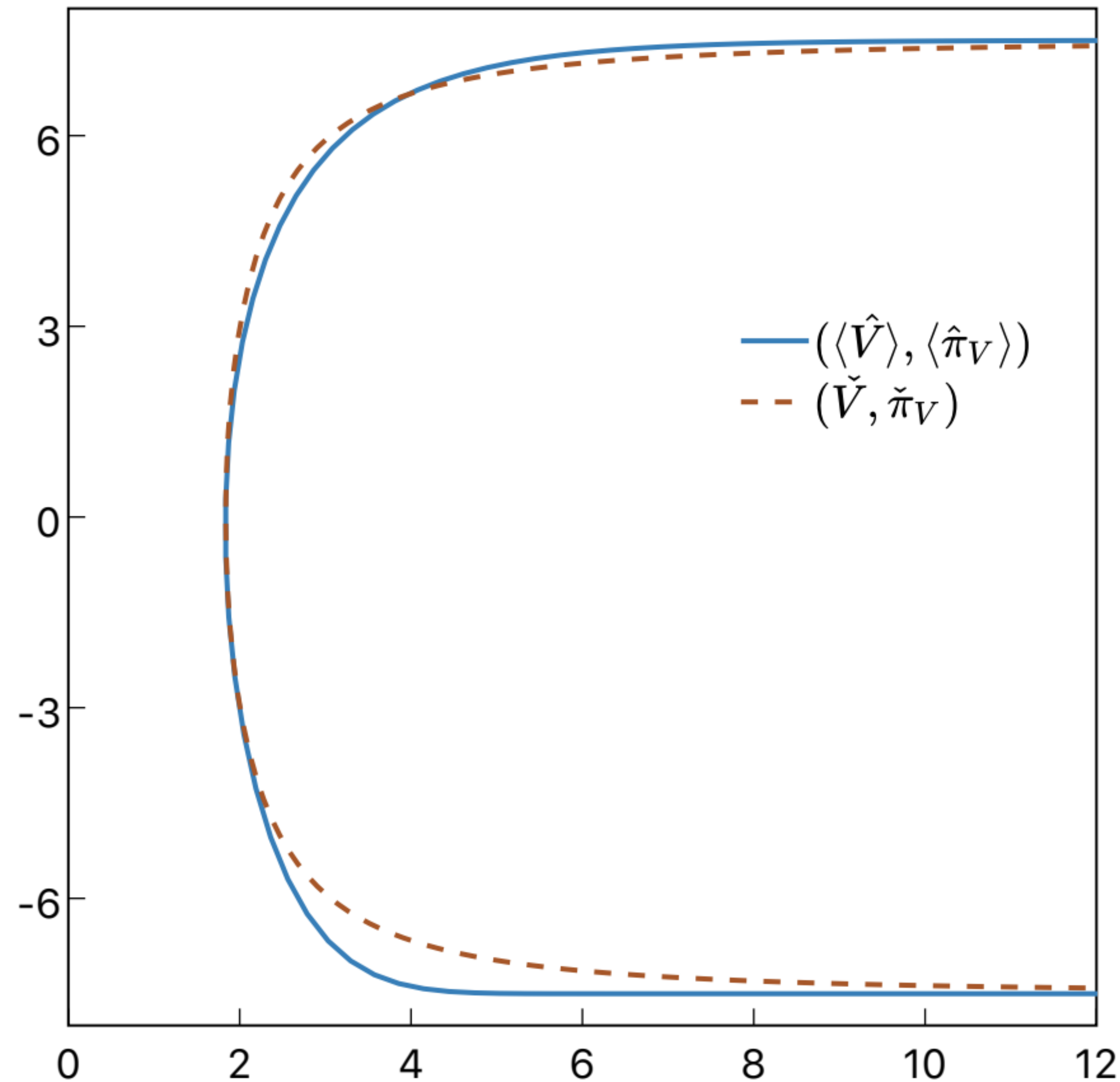
$$\check{V}(t) = \sqrt{4\langle\hat{H}\rangle t^2 + V_0^2},$$

$$\check{\pi}_V(t) = \frac{2\langle\hat{H}\rangle t}{\sqrt{4\langle\hat{H}\rangle t^2 + V_0^2}}.$$

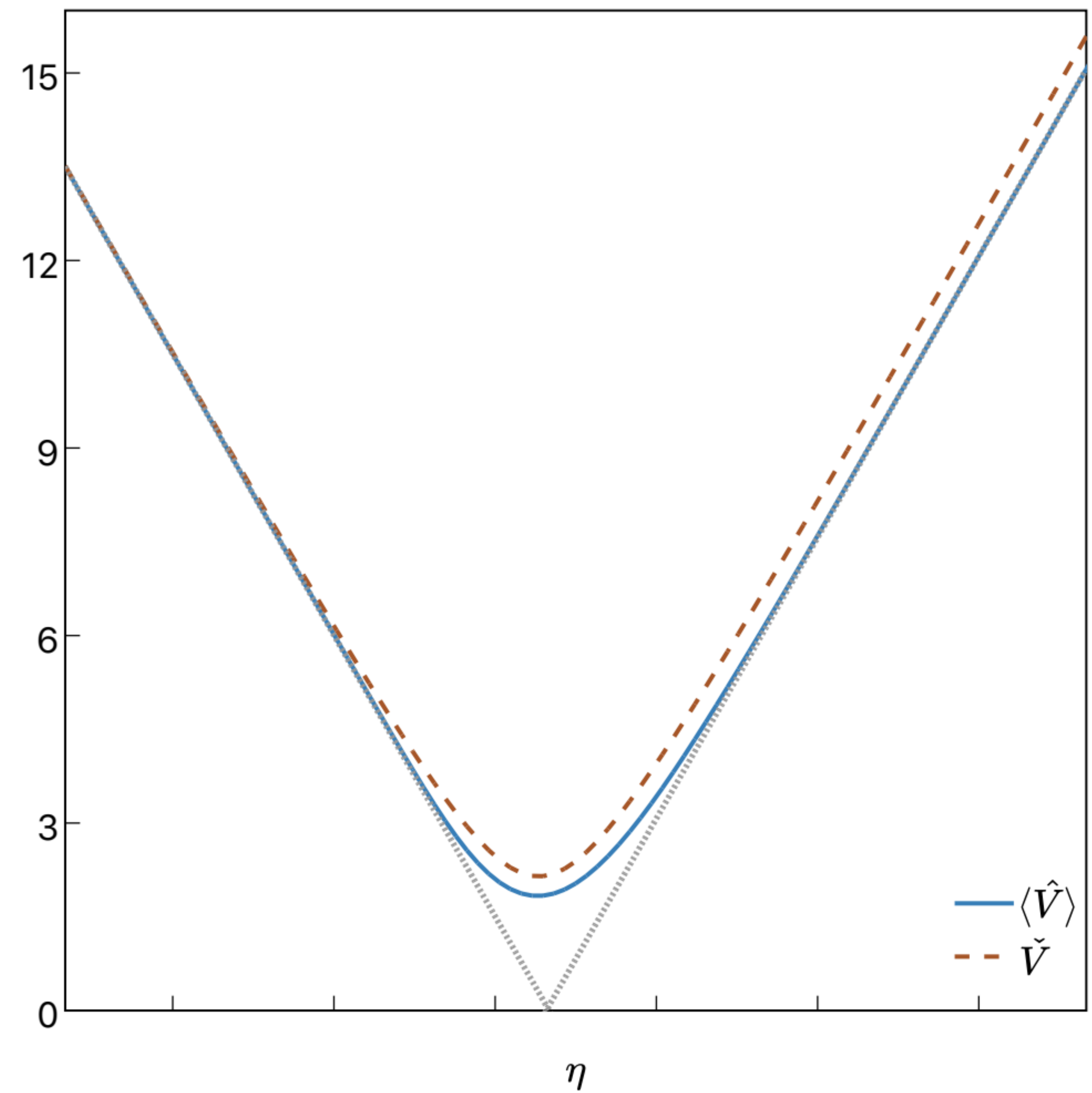
$\Big\downarrow$

NO SINGULARITY





6
3
0
-3
-6
0
2
4
6
8
10
12
$(\langle \hat{V} \rangle, \langle \hat{\pi}_V \rangle)$
$(\check{V}, \check{\pi}_V)$



15
12
9
6
3
0
η
$\langle\hat{V}\rangle$
$\check{V}$

Changing the time variable $\eta' = \eta'(\eta, V, \pi_V)$

redefining the dynamical variables in the process

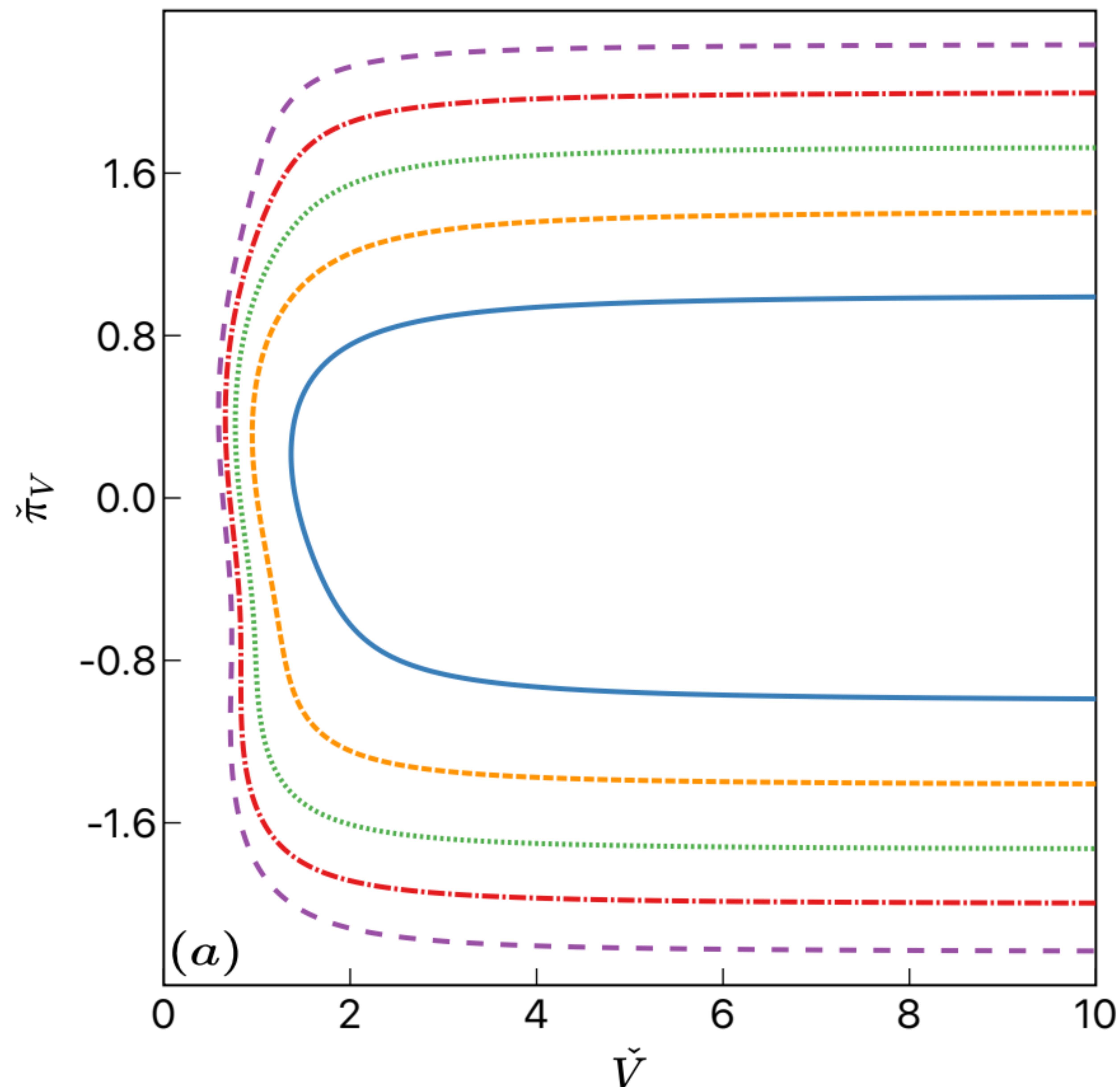
$\pi'_V = \pi_V$ & $V' = V + \pi_V(\eta' - \eta)$ no change of range...

change the canonical one-form

$$\mathrm{d}\theta = \pi_V \mathrm{d}V - \frac{\pi_V^2}{2}\mathrm{d}\eta = \pi'_V \mathrm{d}V' - \frac{\pi_V'^2}{2}\mathrm{d}\eta' + \mathrm{d}\left[(\eta - \eta')\frac{\pi_V'^2}{2}\right]$$

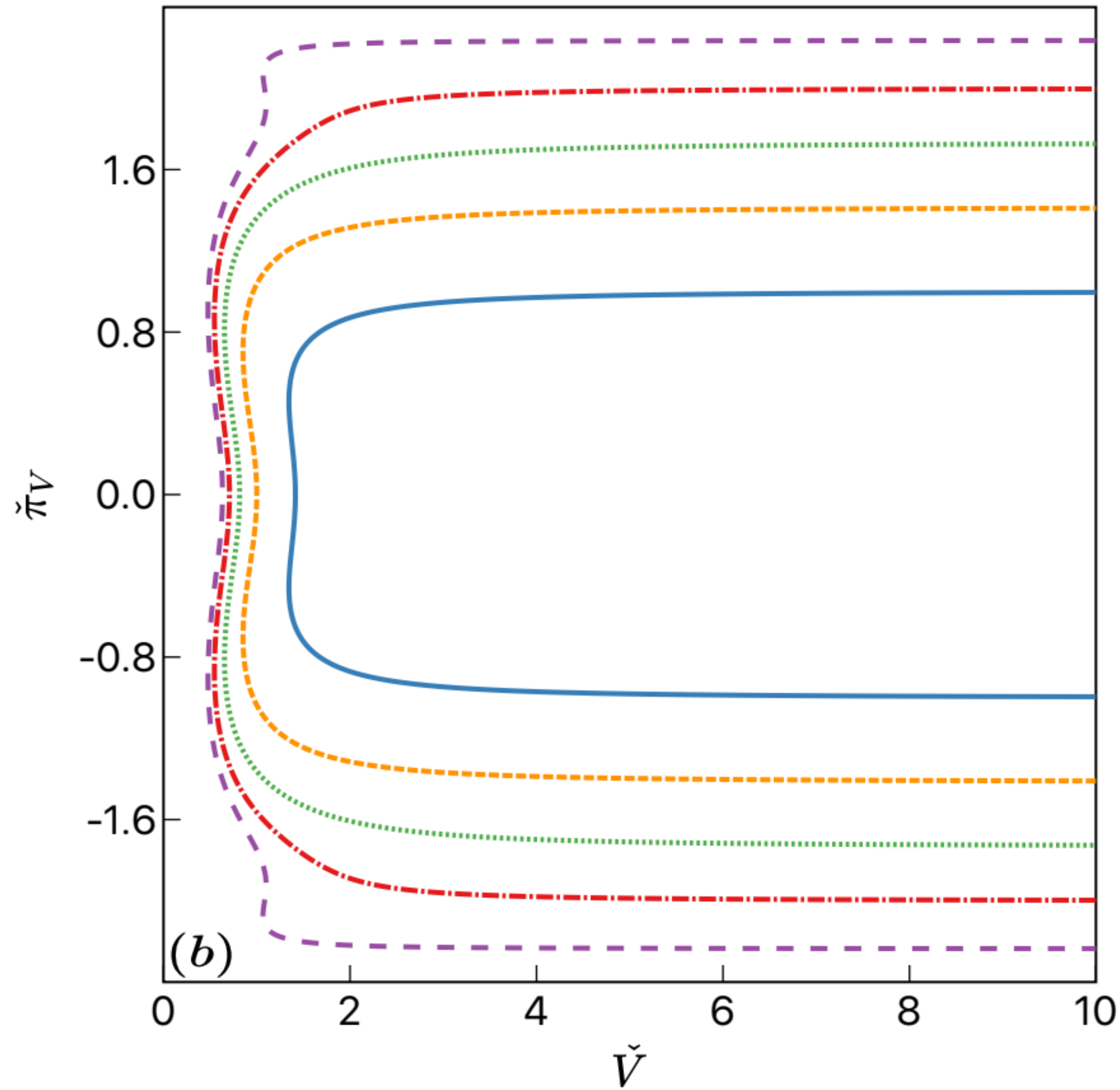
$\longrightarrow$ ***same system!***

delay function $\Delta(V, \pi_V) = \eta' - \eta$ no dependency on time



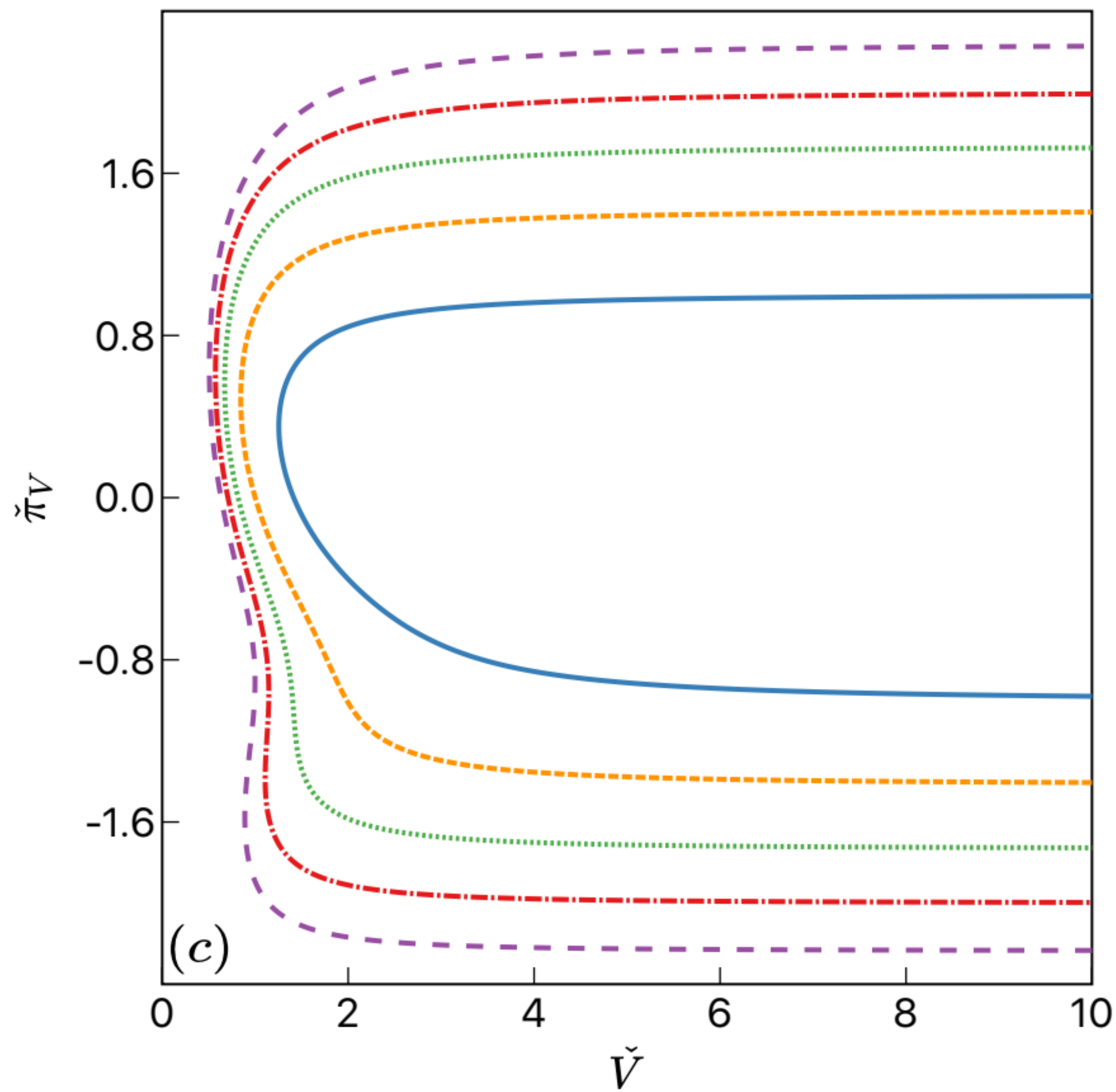
Delay function

$$\Delta = V\mathrm{e}^{-2|\pi_V|/3}\sin(3V\pi_V)/(10\pi_V)$$



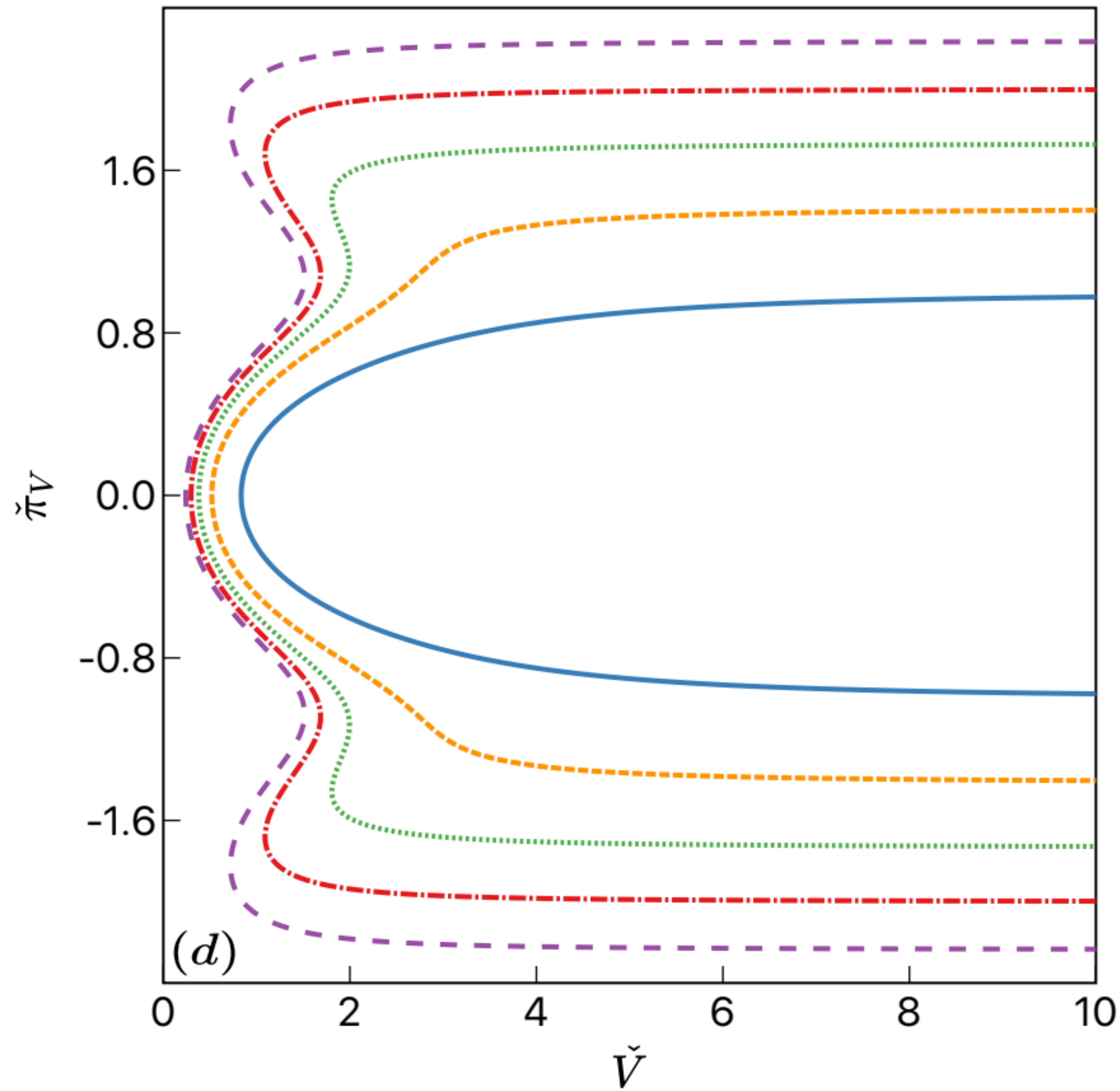
Delay function

$$\Delta = V(\pi_V - 10^{-0.2}\pi_V^3 + \pi_V^5/10)$$



Delay function

$\Delta = 10^{-0.5} V \sin(2\pi_V)/\pi_V$



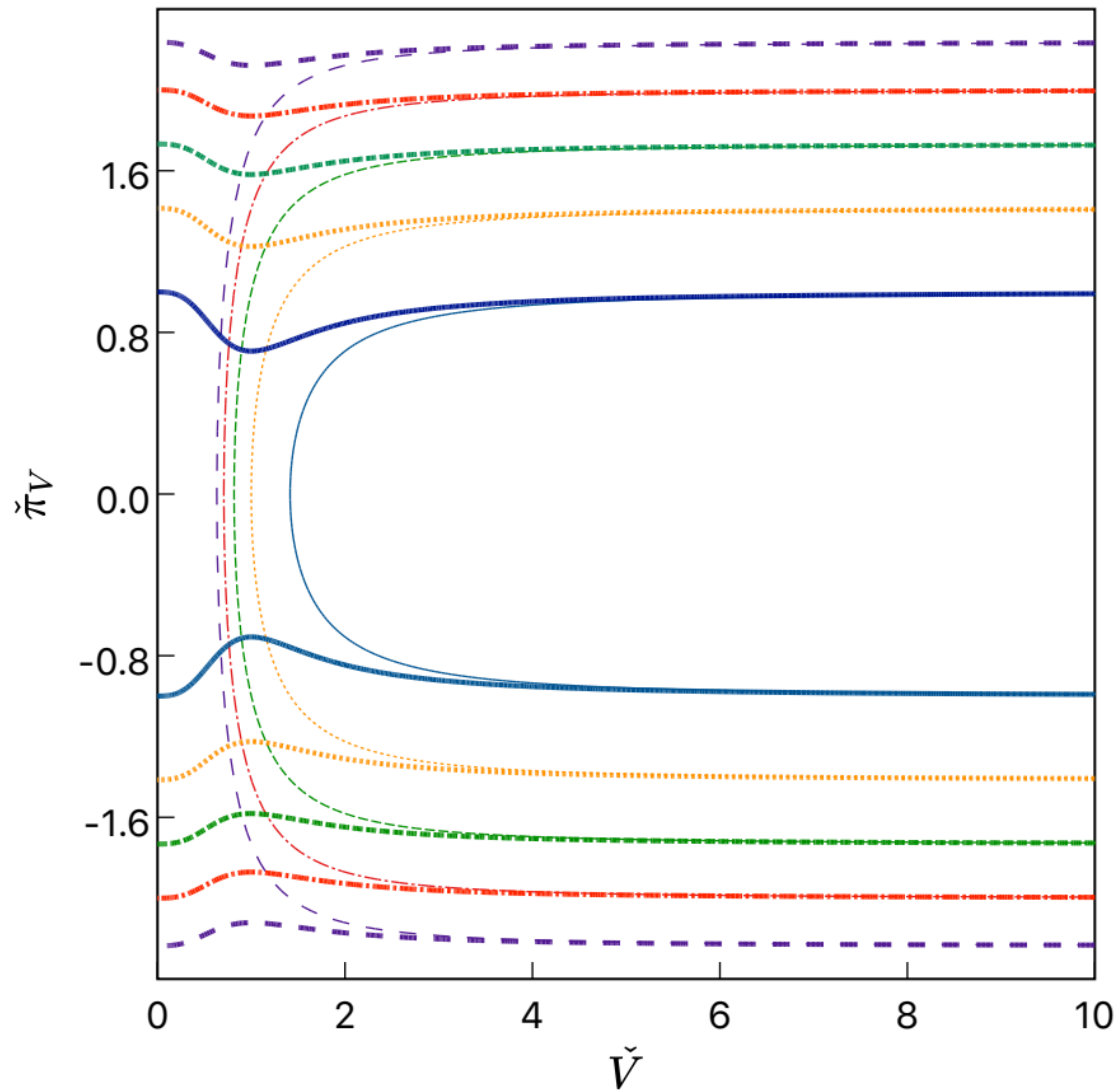
Delay function

$\Delta = 10^{-0.5}(V+1)\cos(3\pi_V)/\pi_V$

Delay function (slow to fast)

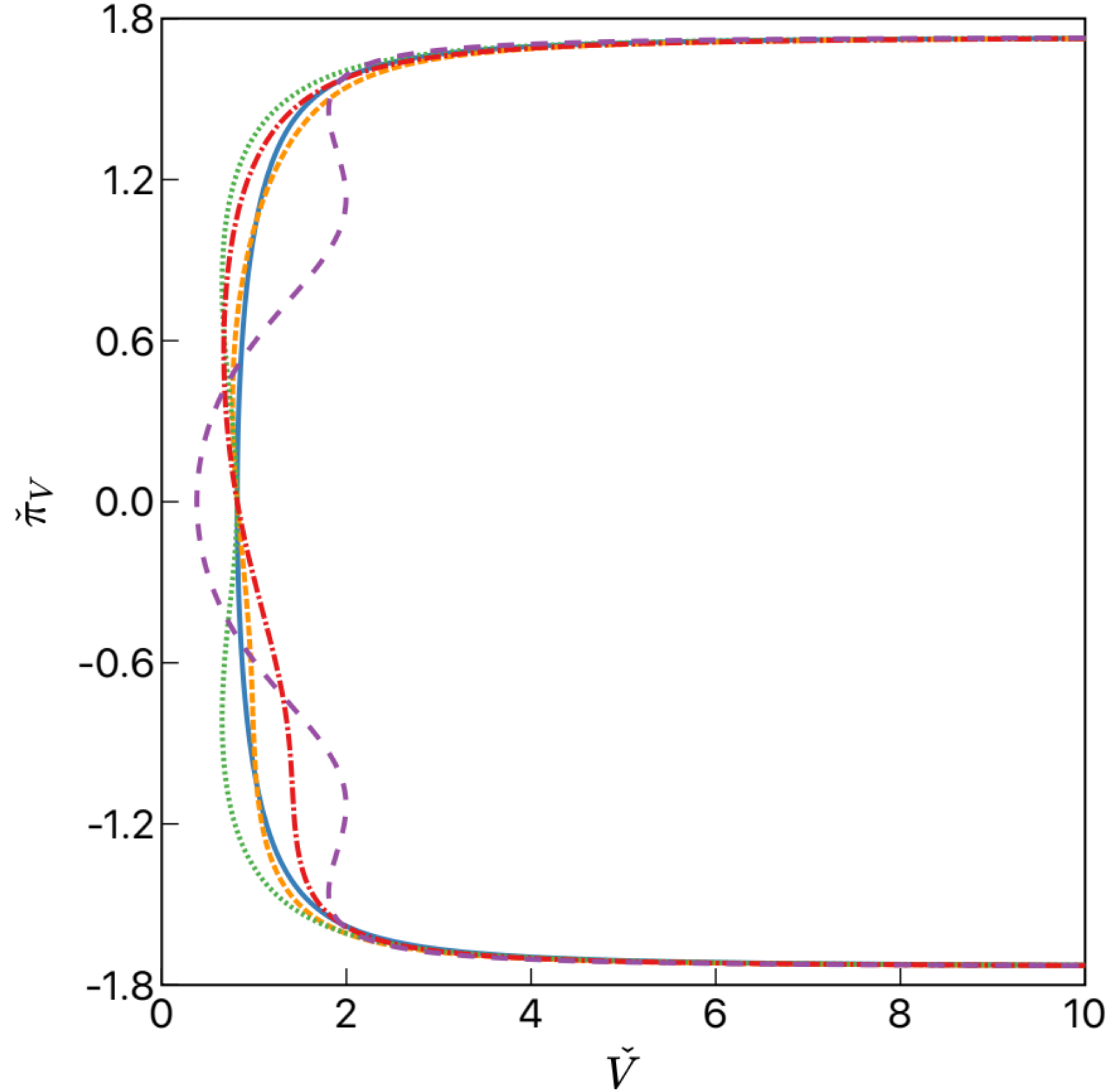
[regular to singular]

$$\Delta_{\text{slow}\to\text{fast}} = \frac{V - \ln V}{V p_V}$$



Comparison between different delay functions

Same asymptotics



Another way to obtain trajectories

Trajectory *formulation* to QM

Schrödinger equation $$i\frac{\partial\psi\left(\boldsymbol{x},t\right)}{\partial t}=\left[-\frac{\boldsymbol{\nabla}^2}{2m}+V\left(\boldsymbol{x}\right)\right]\psi\left(\boldsymbol{x},t\right)$$

polar form of the wave function $\psi\left(\boldsymbol{x},t\right)=A\left(\boldsymbol{x},t\right)\mathrm{e}^{iS(\boldsymbol{x},t)}$

conservation equation $$\frac{\partial|\psi|^2}{\partial t}+\boldsymbol{\nabla}\cdot\left(|\psi|^2\underbrace{\frac{\boldsymbol{\nabla}S}{m}}_{\boldsymbol{v}}\right)=0$$

modified Hamilton-Jacobi equation

$$\boxed{\frac{\partial S}{\partial t}+\underbrace{\frac{\left(\boldsymbol{\nabla}S\right)^2}{2m}}_{\frac{\boldsymbol{p}^2}{2m}}+V\left(\boldsymbol{x}\right)+Q\left(\boldsymbol{x},t\right)=0}$$

Quantum potential $$Q\left(\boldsymbol{x},t\right)\equiv-\frac{1}{2m}\frac{\boldsymbol{\nabla}^2A}{A}$$

Trajectory *formulation* to QM

$\exists\, \boldsymbol{x}(t)$ trajectory satisfying $\quad \boldsymbol{p} = m\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \dfrac{\Im\mathrm{m}\left(\Psi^* \boldsymbol{\nabla} \Psi\right)}{|\Psi(\boldsymbol{x},t)|^2} = \boldsymbol{\nabla} S$

The trajectory *formulation* to QM

$\exists\, \boldsymbol{x}(t)$ trajectory satisfying

$$\dot{\boldsymbol{p}} = m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = -\boldsymbol{\nabla}(V + Q)$$

$$Q = -\frac{1}{2m}\frac{\boldsymbol{\nabla}^2 A}{A}$$

Trajectory *formulation* to QM

$\exists\, \boldsymbol{x}(t)$ trajectory satisfying $\boldsymbol{p} = \boldsymbol{\nabla} S$

properties

- equivalent formulation for QM

  probability distribution $\exists t;\ \rho\,(\boldsymbol{x}, t) = |\psi\,(\boldsymbol{x}, t)\,|^2$
  (attractor)

- classical limit well defined $Q \to 0$

- state dependent

- no need for external classical domain/observer!

S. Vitenti, PP & A. Valentini, *Phys. Rev.* **D 100**, 043506 (2019) [astro-ph 1901.08885]
P. Malkiewicz, PP & S. Vitenti *Phys. Rev.* **D101,** 046012 (2020) [gr-qc 1911.09892]

Conclusions

- Internal clock formulation of QM & QC
- Clock issue in QC can be approached by WDW and set constraints on time
- Asymptotics may solve the problem… perturbations???
- Other trajectory approach = same asymptotics
- Out-of-Quantum-Equilibrium

modified power spectrum

→ ***Planck best-fit…***

$\ell_{\mathrm{NEW}} \simeq 2000\mathrm{Mpc}$