

Spinning black holes fall in Love

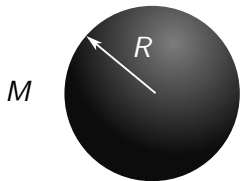
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Collaborators: M. Casals & E. Franzin

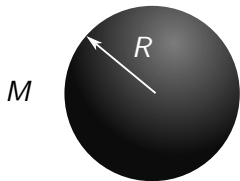
arXiv:2007.00214 and arXiv:2010.15795

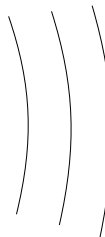
Newtonian theory of Love numbers



$$U = \frac{M}{r}$$

Newtonian theory of Love numbers



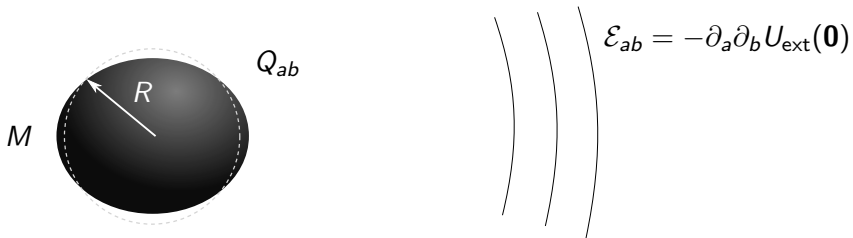


Three vertical, slightly curved lines representing external potential. To their right is the equation $\mathcal{E}_{ab} = -\partial_a \partial_b U_{\text{ext}}(\mathbf{0})$.

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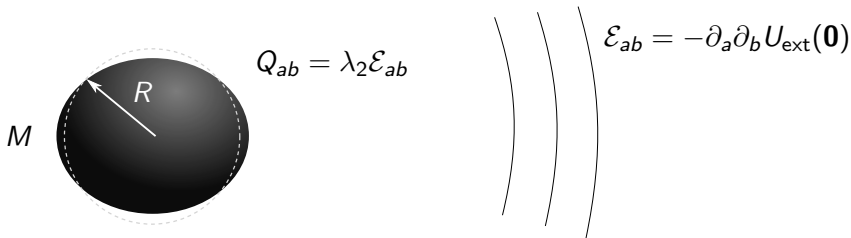
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab}$$

Newtonian theory of Love numbers



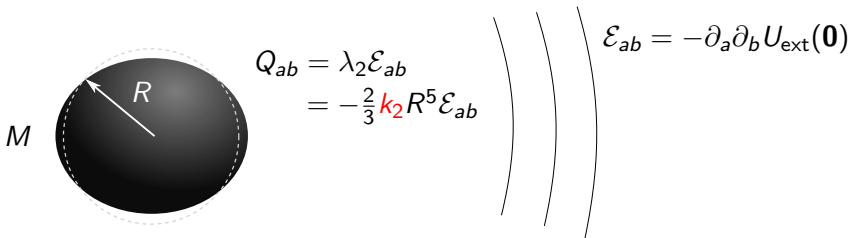
$$U = \frac{M}{r} - \frac{1}{2} x^a x^b \mathcal{E}_{ab} + \frac{3}{2} \frac{x^a x^b Q_{ab}}{r^5}$$

Newtonian theory of Love numbers



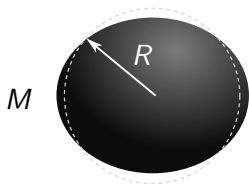
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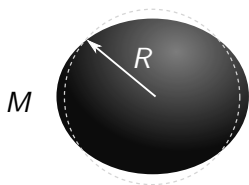


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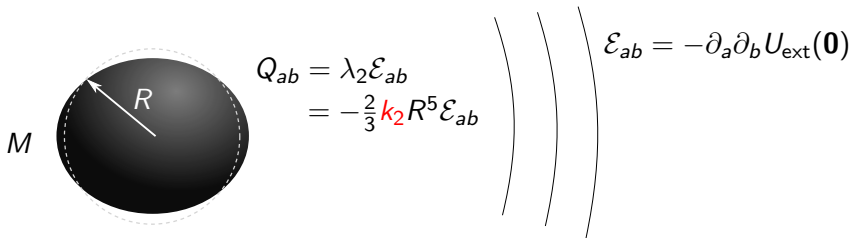


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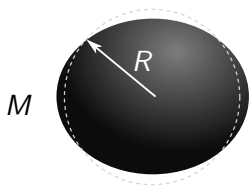
$$U = \frac{M}{r} - \sum_{\ell \geq 2} \frac{(\ell-2)!}{\ell!} x^{a_1} \dots x^{a_\ell} \mathcal{E}_{a_1 \dots a_\ell} \left[1 + 2k_\ell \left(\frac{R}{r} \right)^{2\ell+1} \right]$$

Newtonian theory of Love numbers



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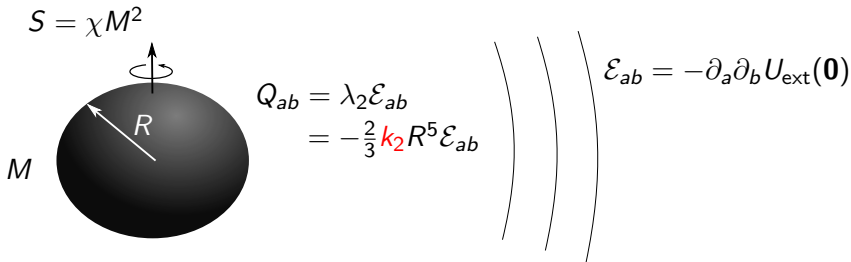


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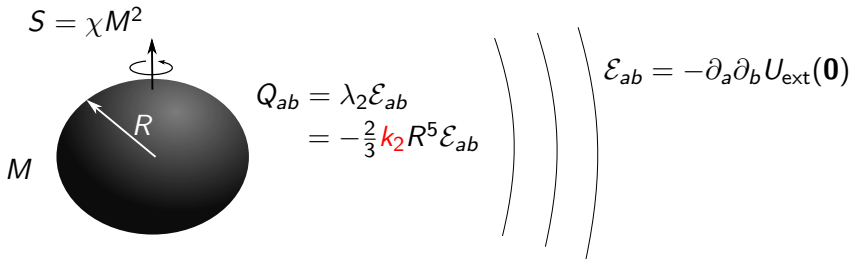
$$\psi_0 = \sum_{l \geq 2} \sum_{|m| \leq l} \sqrt{\frac{(l+2)(l+1)}{l(l-1)}} r^{\ell-2} \mathcal{E}_{\ell m} \left[1 + 2k_\ell \left(\frac{R}{r} \right)^{2\ell+1} \right] {}_2Y_{\ell m}$$

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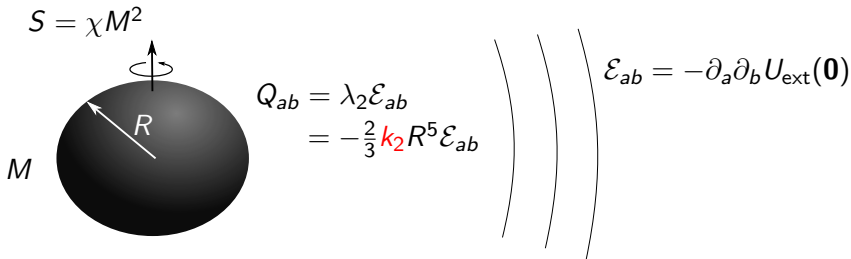
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$$k_{\ell m} = k_{\ell}^{(0)} + im\chi k_{\ell}^{(1)} + O(\chi^2)$$

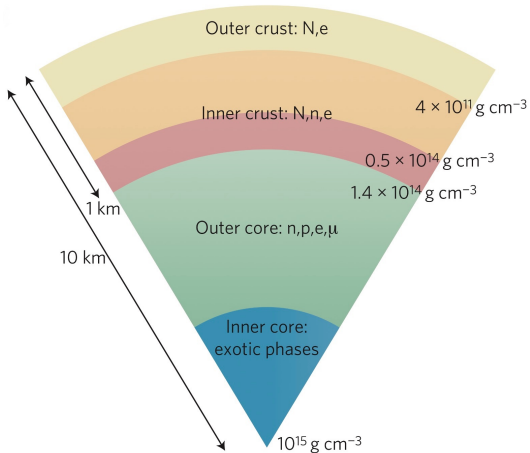
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Tidal Love numbers $k_{\ell m}$ \longleftrightarrow **body's internal structure**

Internal structure of neutron stars

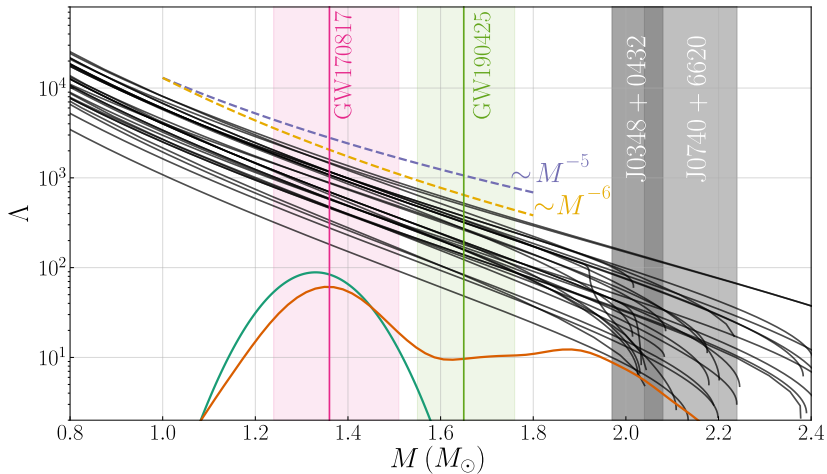


GW observations as probes of **neutron star internal structure**

Gravitational-wave observations

[Chatziioannou, GRG 2020]

$$\Lambda \equiv \lambda_2/M^5 \propto k_2/C^5$$



Relativistic theory of Love numbers

- Electric-type and magnetic-type tidal moments:

$$\mathcal{E}_L \propto [C_{0a_1 0a_2; a_3 \dots a_\ell}]^{\text{STF}} \quad \text{and} \quad \mathcal{B}_L \propto [\varepsilon_{a_1 bc} C_{a_2 0bc; a_3 \dots a_\ell}]^{\text{STF}}$$

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$$\delta M_L = \lambda_\ell^{\text{el}} \mathcal{E}_L \quad \text{and} \quad \delta S_L = \lambda_\ell^{\text{mag}} \mathcal{B}_L$$

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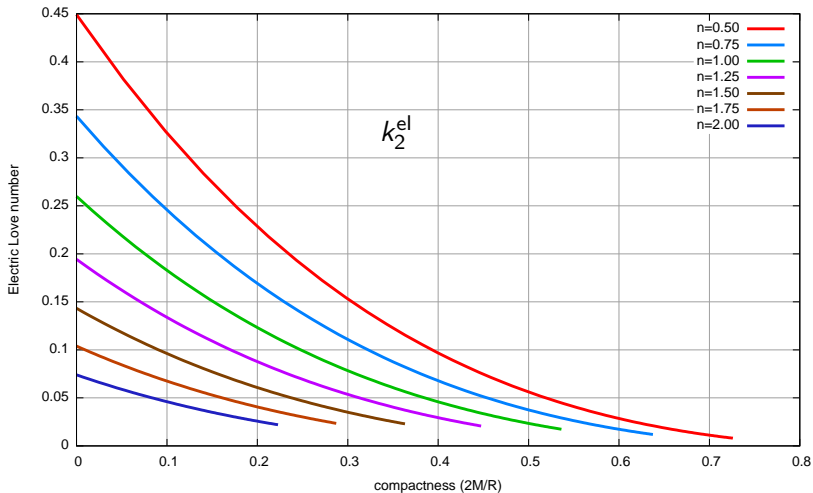
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- Dimensionless tidal Love numbers:

$$k_\ell^{\text{el/mag}} \equiv -\frac{(2\ell-1)!!}{2(\ell-2)!} \frac{\lambda_\ell^{\text{el/mag}}}{R^{2\ell+1}}$$

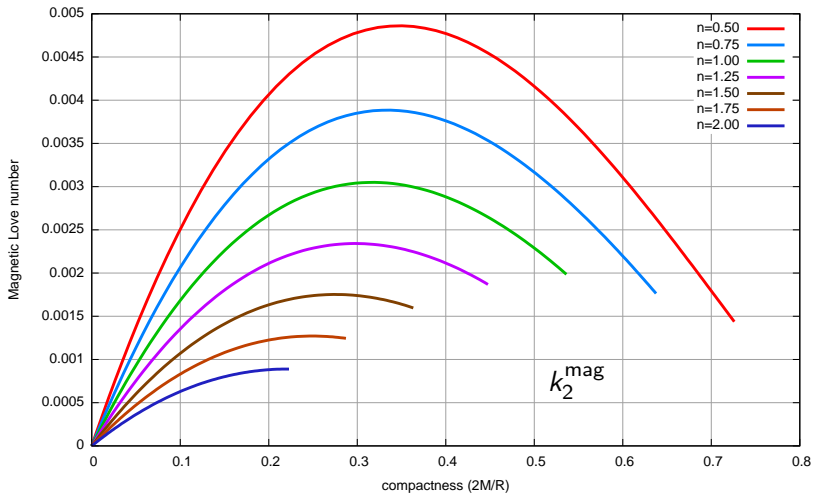
Love numbers of neutron stars

[Binnington & Poisson, PRD 2009]



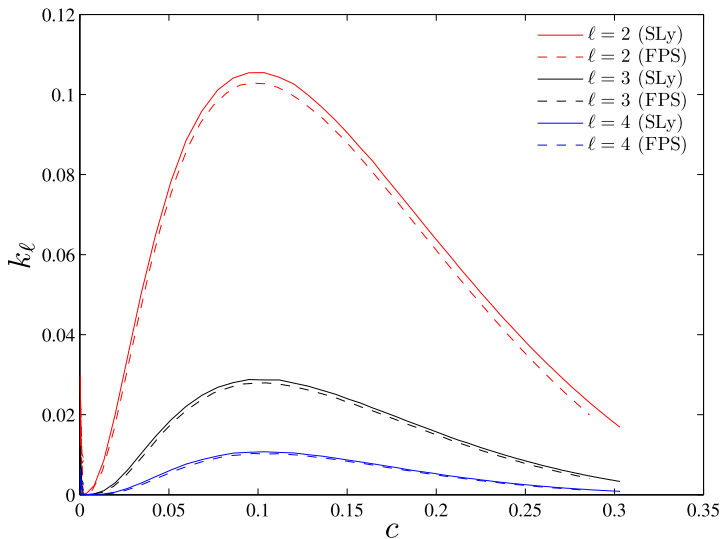
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[Damour & Nagar, PRD 2009]



Love numbers of *spinning* compact objects

- The spin breaks the spherical symmetry of the background
 - No proportionality between $(\delta M_L, \delta S_L)$ and $(\mathcal{E}_L, \mathcal{B}_L)$
 - Degeneracy of the azimuthal number m lifted
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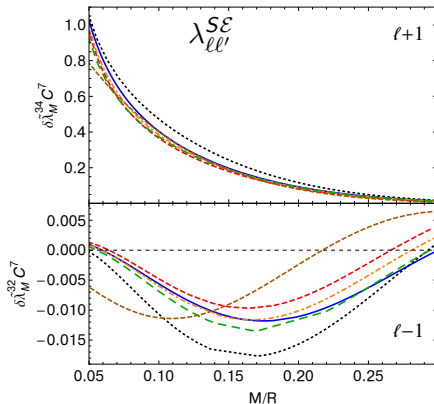
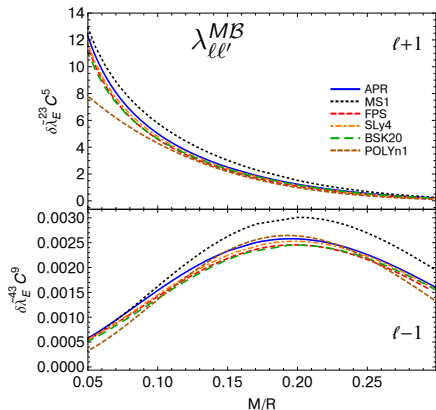
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- *Four* families of tidal deformability parameters:

$$\begin{aligned} \lambda_{\ell\ell' mm'}^{M\mathcal{E}} &\equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{E}_{\ell' m'}} & \lambda_{\ell\ell' mm'}^{S\mathcal{B}} &\equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{B}_{\ell' m'}} \\ \lambda_{\ell\ell' mm'}^{S\mathcal{E}} &\equiv \frac{\partial \delta S_{\ell m}}{\partial \mathcal{E}_{\ell' m'}} & \lambda_{\ell\ell' mm'}^{M\mathcal{B}} &\equiv \frac{\partial \delta M_{\ell m}}{\partial \mathcal{B}_{\ell' m'}} \end{aligned}$$

Love numbers of *spinning* neutron stars

[Pani, Gualtieri & Ferrari, PRD 2015]



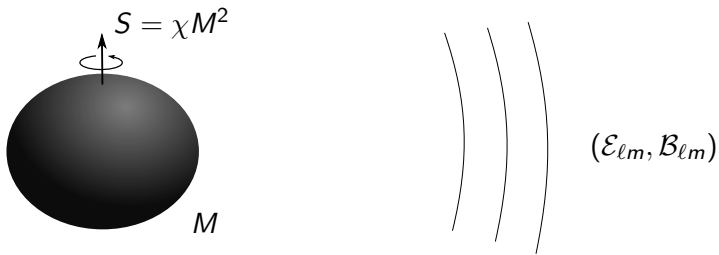
Restricted to an *axisymmetric* tidal perturbation ($m = 0$)

Black holes have zero Love numbers

Reference	Background	Tidal field
[Binnington & Poisson 2009]	Schwarzschild	weak, generic ℓ
[Damour & Nagar 2009]	Schwarzschild	weak, generic ℓ
[Kol & Smolkin 2012]	Schwarzschild	weak, electric-type
[Chakrabarti et al. 2013]	Schwarzschild	weak, electric, $\ell = 2$
[Gürlebeck 2015]	Schwarzschild	strong, axisymmetric
[Landry & Poisson 2015]	Kerr to $O(S)$	weak, quadrupolar
[Pani et al. 2015]	Kerr to $O(S^2)$	weak, $(\ell, m) = (2, 0)$
[Chia 2020]	Exact Kerr	weak, generic ℓ

Problem of **fine-tuning** from an Effective-Field-Theory perspective

Investigating Kerr's Love



$$(\mathcal{E}_{\ell m}, \mathcal{B}_{\ell m}) \rightarrow \psi_0 \rightarrow \Psi \rightarrow h_{\alpha\beta} \rightarrow (M_{\ell m}, S_{\ell m}) \rightarrow \lambda_{\ell m}^{M/S, \mathcal{E}/\mathcal{B}}$$

Metric reconstruction through the Hertz potential Ψ

Perturbed Weyl scalar

- Recall that in the Newtonian limit we established

$$\lim_{c \rightarrow \infty} \psi_0^{\ell m} \propto \mathcal{E}_{\ell m} r^{\ell-2} [1 + 2k_{\ell m} (R/r)^{2\ell+1}] {}_2Y_{\ell m}(\theta, \phi)$$

- For a Kerr black hole the perturbed Weyl scalar reads

$$\psi_0^{\ell m} \propto \left[\mathcal{E}_{\ell m} + \frac{3i}{\ell+1} \mathcal{B}_{\ell m} \right] R_{\ell m}(r) {}_2Y_{\ell m}(\theta, \phi)$$

- Asymptotic behavior of general solution of static radial Teukolsky equation:

$$R_{\ell m}(r) = \underbrace{r^{\ell-2} (1 + \dots)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + \kappa_{\ell m} \underbrace{r^{-\ell-3} (1 + \dots)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Why analytic continuation?

$$R_{\ell m}(r) = \underbrace{r^{\ell-2} (1 + \dots)}_{\text{tidal field } R_{\ell m}^{\text{tidal}}} + \kappa_{\ell m} \underbrace{r^{-\ell-3} (1 + \dots)}_{\text{linear response } R_{\ell m}^{\text{resp}}}$$

Ambiguity in the linear response [Fang & Lovelace 2005; Gralla 2018]

The decaying solution $R_{\ell m}^{\text{resp}}$ is affected by a radial coord. transfo.

Ambiguity in the tidal field [Pani, Gualtieri, Maselli & Ferrari 2015]

The growing solution $R_{\ell m}^{\text{tidal}} + \alpha R_{\ell m}^{\text{resp}}$ still qualifies as a tidal solution

Kerr black hole linear response

$$R_{\ell m}(r) = \underbrace{R_{\ell m}^{\text{tidal}}(r)}_{\sim r^{\ell-2}} + 2k_{\ell m} \underbrace{R_{\ell m}^{\text{resp}}(r)}_{\sim r^{-(\ell+3)}}$$

- The coefficients $k_{\ell m}$ can be interpreted as the Newtonian Love numbers of a Kerr black hole and read

$$k_{\ell m} = -im\chi \frac{(\ell+2)!(\ell-2)!}{4(2\ell+1)!(2\ell)!} \prod_{n=1}^{\ell} [n^2(1-\chi^2) + m^2\chi^2]$$

- The linear response vanishes identically when:
 - the black hole spin vanishes ($\chi = 0$)
 - the tidal field is axisymmetric ($m = 0$)
- Reconstruct the Kerr black hole response $h_{\alpha\beta}^{\text{resp}}$ via Ψ^{resp}

Love numbers of a Kerr black hole

- We compute the Love numbers to **linear** order in $\chi \equiv S/M^2$

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$$\delta M_{2m} \doteq \frac{im\chi}{180} (2M)^5 \mathcal{E}_{2m} \quad \text{and} \quad \delta S_{2m} \doteq \frac{im\chi}{180} (2M)^5 \mathcal{B}_{2m}$$

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- The associated dimensionless tidal Love numbers are

$$k_{2m}^{M\mathcal{E}} = k_{2m}^{S\mathcal{B}} \doteq -\frac{im\chi}{120} \quad \text{and} \quad k_{2m}^{M\mathcal{B}} = k_{2m}^{S\mathcal{E}} \doteq 0$$

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- For a dimensionless black hole spin $\chi = 0.1$ this gives

$$|k_{2,\pm 2}| \simeq 2 \times 10^{-3} \quad \longrightarrow \quad \text{black holes are “rigid”}$$

Love tensor of a Kerr black hole

- For a nonspinning compact body we have the proportionality relations

$$\delta M_{ab} = \lambda_2^{\text{el}} \mathcal{E}_{ab} \quad \text{and} \quad \delta S_{ab} = \lambda_2^{\text{mag}} \mathcal{B}_{ab}$$

- For a **spinning black hole** we have the more general **tensorial** relations

$$\delta M_{ab} = \lambda_{abcd} \mathcal{E}_{cd} \quad \text{and} \quad \delta S_{ab} = \lambda_{abcd} \mathcal{B}_{cd}$$

- To linear order in the black hole spin vector S^a we find

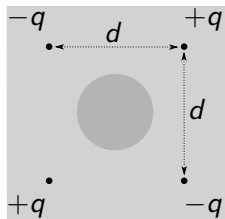
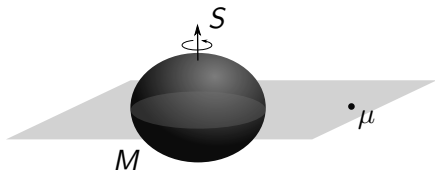
$$\delta M_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{E}^d_{(a} \mathcal{E}_{b)cd}$$
$$\delta S_{ab} \doteq \frac{16}{45} M^3 S^c \mathcal{B}^d_{(a} \mathcal{E}_{b)cd}$$

Love tensor of a Kerr black hole

$$(\lambda_{abcd}) \doteq \frac{\chi}{180} (2M)^5 \begin{pmatrix} \mathbf{l}_{11} & \mathbf{l}_{12} & \mathbf{l}_{13} \\ \mathbf{l}_{12} & -\mathbf{l}_{11} & \mathbf{l}_{23} \\ \mathbf{l}_{13} & \mathbf{l}_{23} & \mathbf{0} \end{pmatrix}$$

$$\mathbf{l}_{11} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{l}_{12} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathbf{l}_{13} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \mathbf{l}_{23} \equiv \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

Newtonian static quadrupolar tide



$$\mathcal{E}_{ab} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

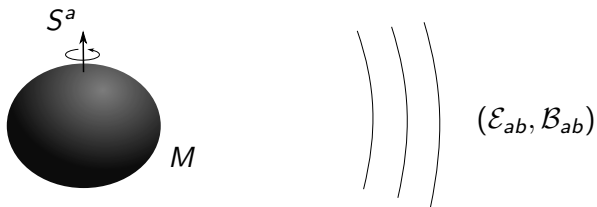
$$\delta M_{ab} \doteq 3Q \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↑

$$\frac{\chi}{180} (2M)^5 \frac{\mu}{r^3} = qd^2$$

Tidal torquing of a spinning black hole

[Thorne & Hartle 1980; Poisson 2004]



- An arbitrary spinning body interacting with a tidal environment suffers a **tidal torquing**:

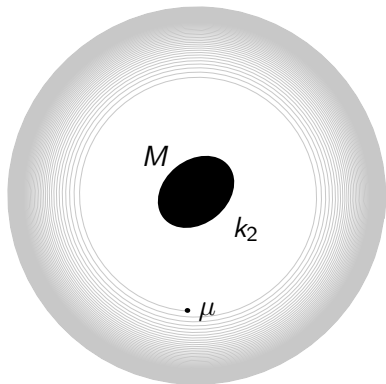
$$\langle \dot{S}^a \rangle = -\epsilon^{abc} \langle M_{bd} \mathcal{E}^d_c + S_{bd} \mathcal{B}^d_c \rangle$$

- Applied to a **spinning black hole** this yields

$$\langle \dot{S} \rangle \doteq -\frac{8}{45} M^5 \chi [2\langle E_1 + B_1 \rangle - 3\langle E_2 + B_2 \rangle]$$

Observing black hole tidal deformability

[Pani & Maselli, IJMPD 2019]



Accumulated **GW phase in LISA**
band during quasi-circular inspiral
down to Schwarzschild ISCO:

$$\Phi_{\text{tidal}} \simeq -2 \times 10^3 \left(\frac{10^{-7}}{\mu/M} \right) \left(\frac{k_2}{0.002} \right)$$

↑

like 1st order dissipative self-force

Summary

- Love numbers of Kerr black holes **do not vanish** in general
- We computed in closed-form the leading (quadrupolar) Love numbers to linear order in the black hole spin
- **Kerr black holes deform** like any other self-gravitating body, despite being particularly “rigid” compact objects
- This is closely related to the phenomenon of **tidal torquing**
- The black hole tidal deformation contribution to the GW phase of EMRIs could be **detectable by LISA**
- **New black hole test** of the Kerr-like nature of the massive compact objects at the center of galaxies

Spinning black holes fall in Love!

Two basis of independent solutions

- Dimensionless radial coordinate and spin parameter:

$$x \equiv \frac{r - r_+}{r_+ - r_-} \quad \text{and} \quad \gamma = \frac{a}{r_+ - r_-}$$

- Smooth and unsmooth solutions:

$$R_{\ell m}^{\text{smooth}} = x^{-2}(1+x)^{-2} \mathbf{F}(-\ell-2, \ell-1, -1+2im\gamma; -x)$$

$$R_{\ell m}^{\text{unsmooth}} = (1+1/x)^{2im\gamma} \mathbf{F}(-\ell+2, \ell+3, 3-2im\gamma; -x)$$

- Tidal and response solutions:

$$R_{\ell m}^{\text{tidal}} = \frac{x^\ell}{(1+x)^2} F(-\ell-2, -\ell-2im\gamma, -2\ell; -1/x) \sim x^{\ell-2}$$

$$R_{\ell m}^{\text{resp}} = \frac{x^{-\ell-1}}{(1+x)^2} F(\ell-1, \ell+1-2im\gamma, 2\ell+2; -1/x) \sim x^{-\ell-3}$$