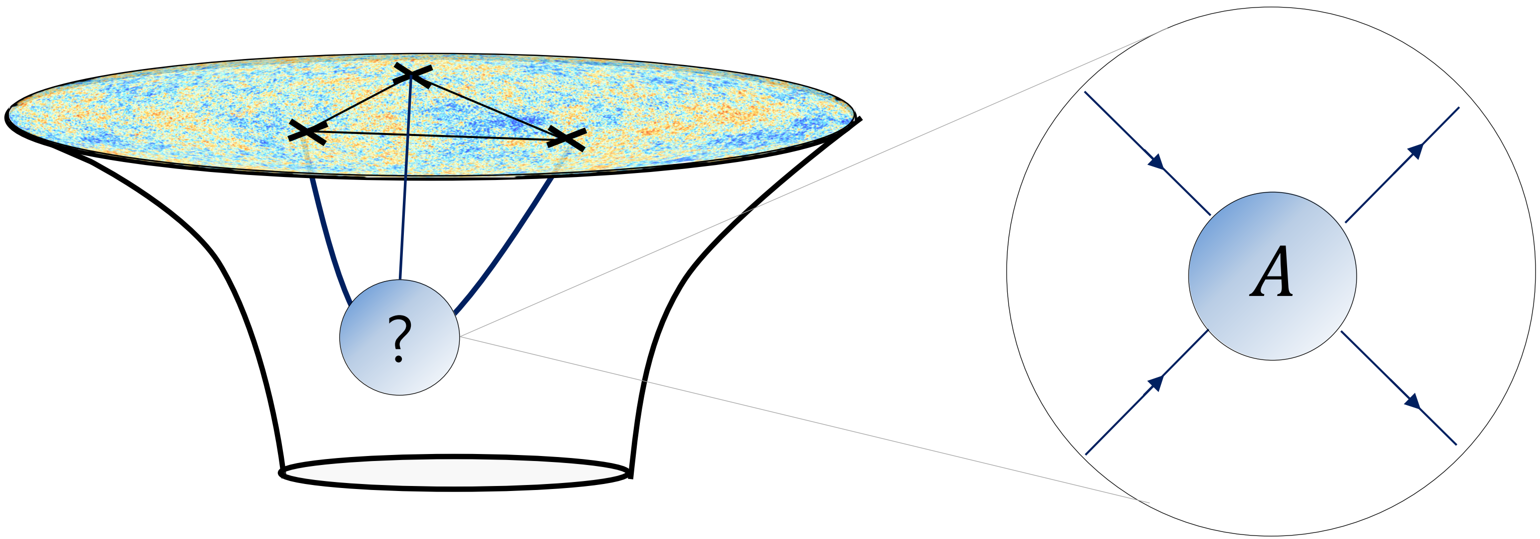
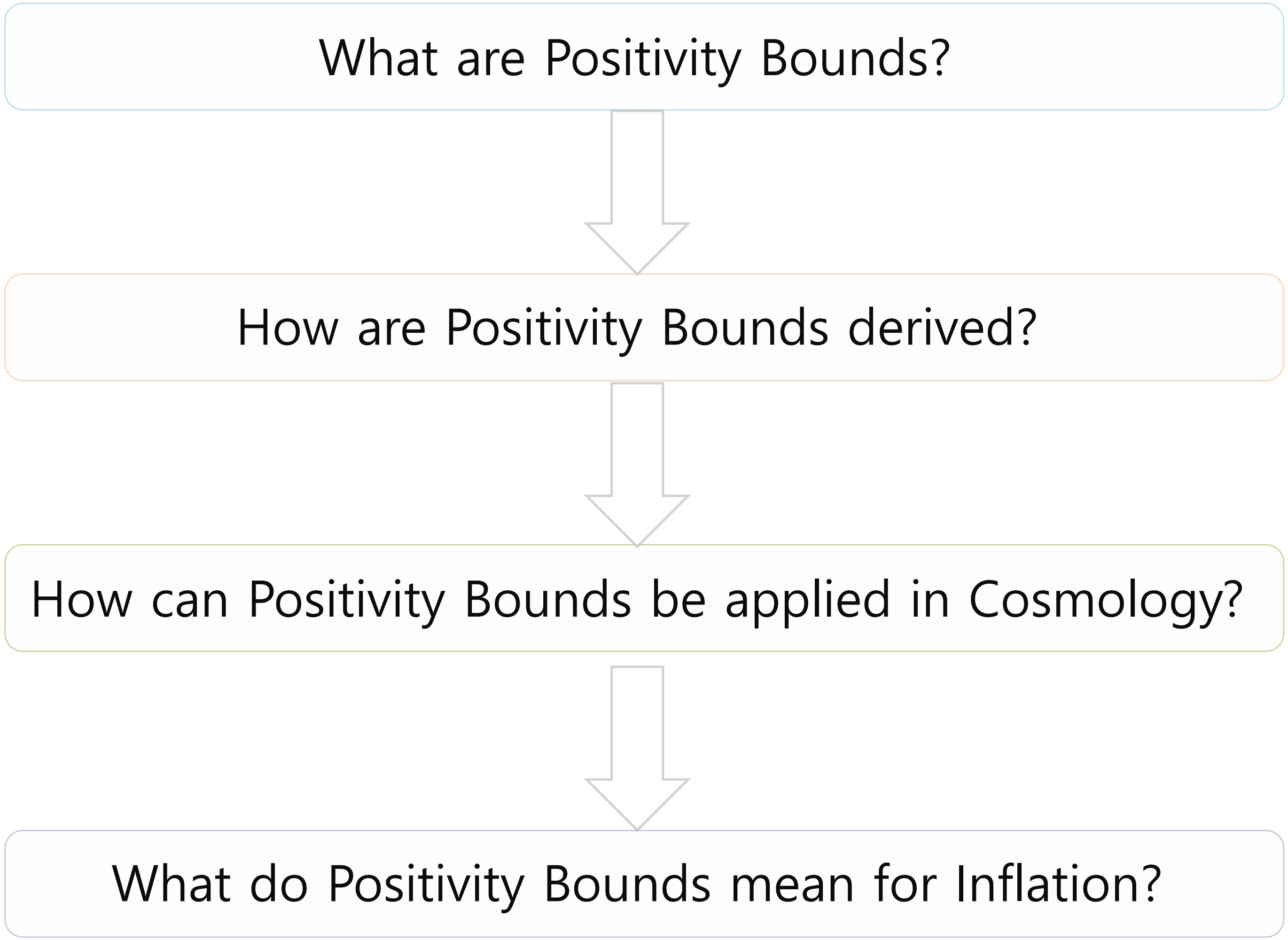
Positivity Bounds for Cosmology

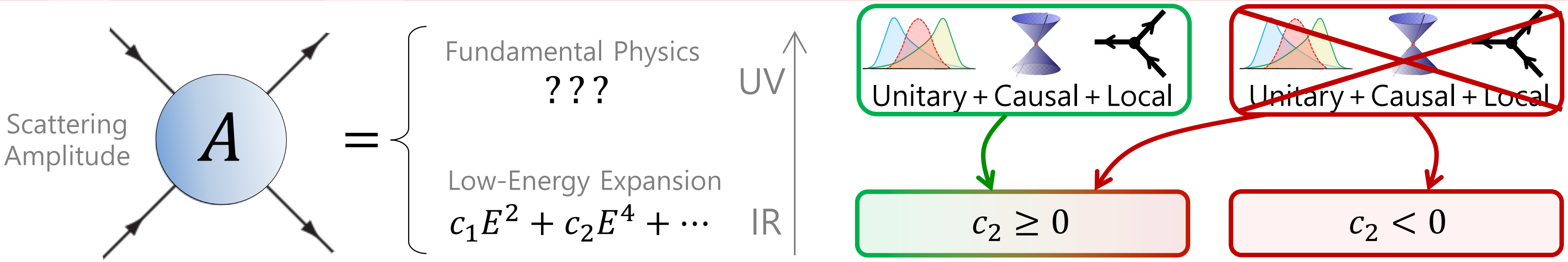


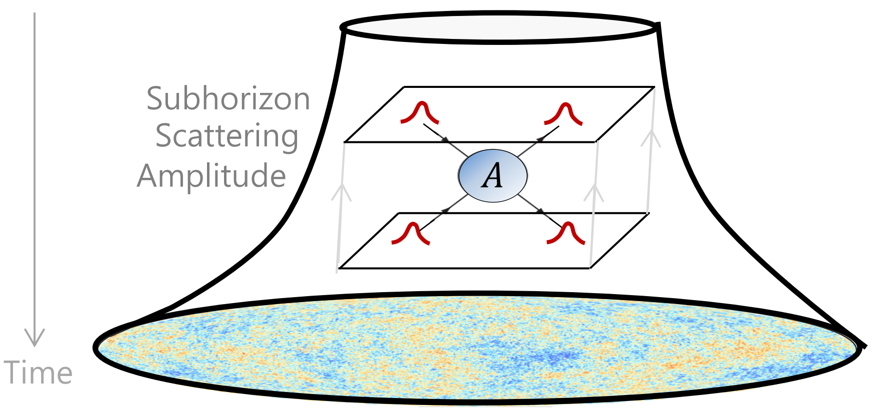
Scott Melville

UNIVERSITY OF CAMBRIDGE


What are Positivity Bounds?
How are Positivity Bounds derived?
How can Positivity Bounds be applied in Cosmology?
What do Positivity Bounds mean for Inflation?

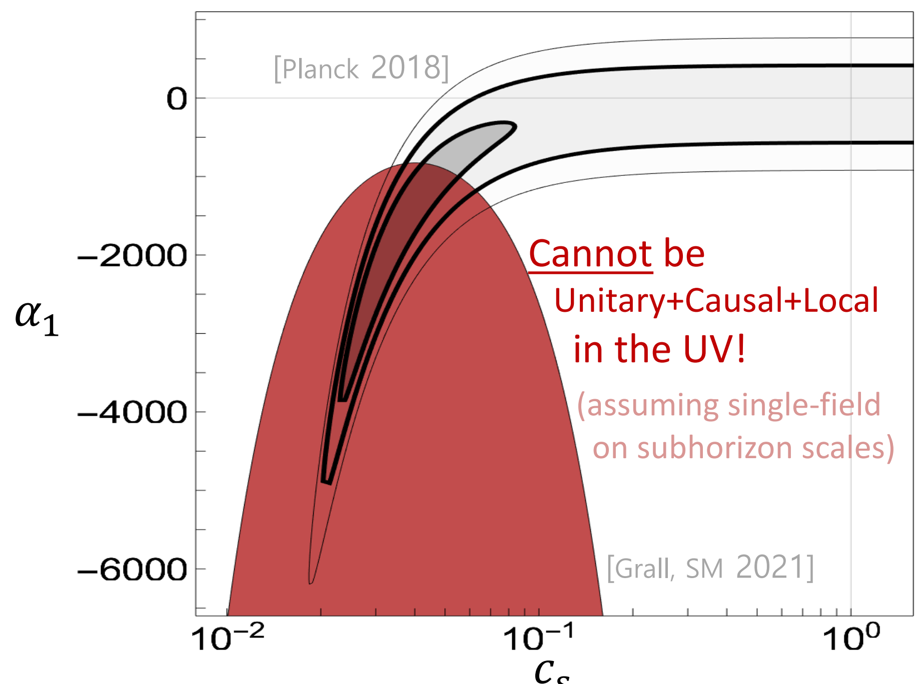
One-Slide Summary



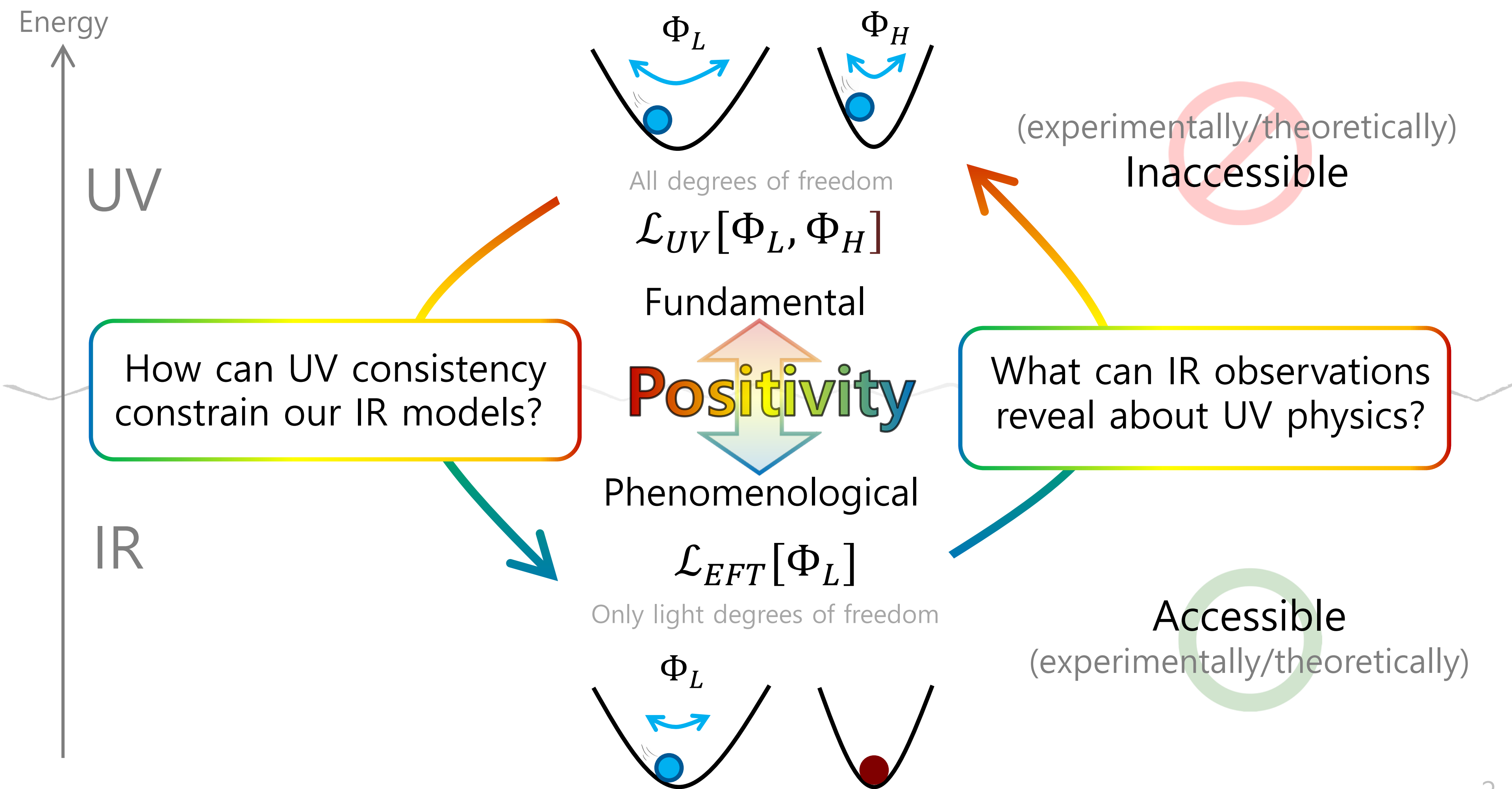


$$\mathcal{L}_{\text{EFT}} = \dot{\pi}^2 - c_s^2 (\partial_i \pi)^2 + \alpha_1 \, \dot{\pi}^3 + \cdots$$

EFT of Single-Field Inflation



What are Positivity Bounds?



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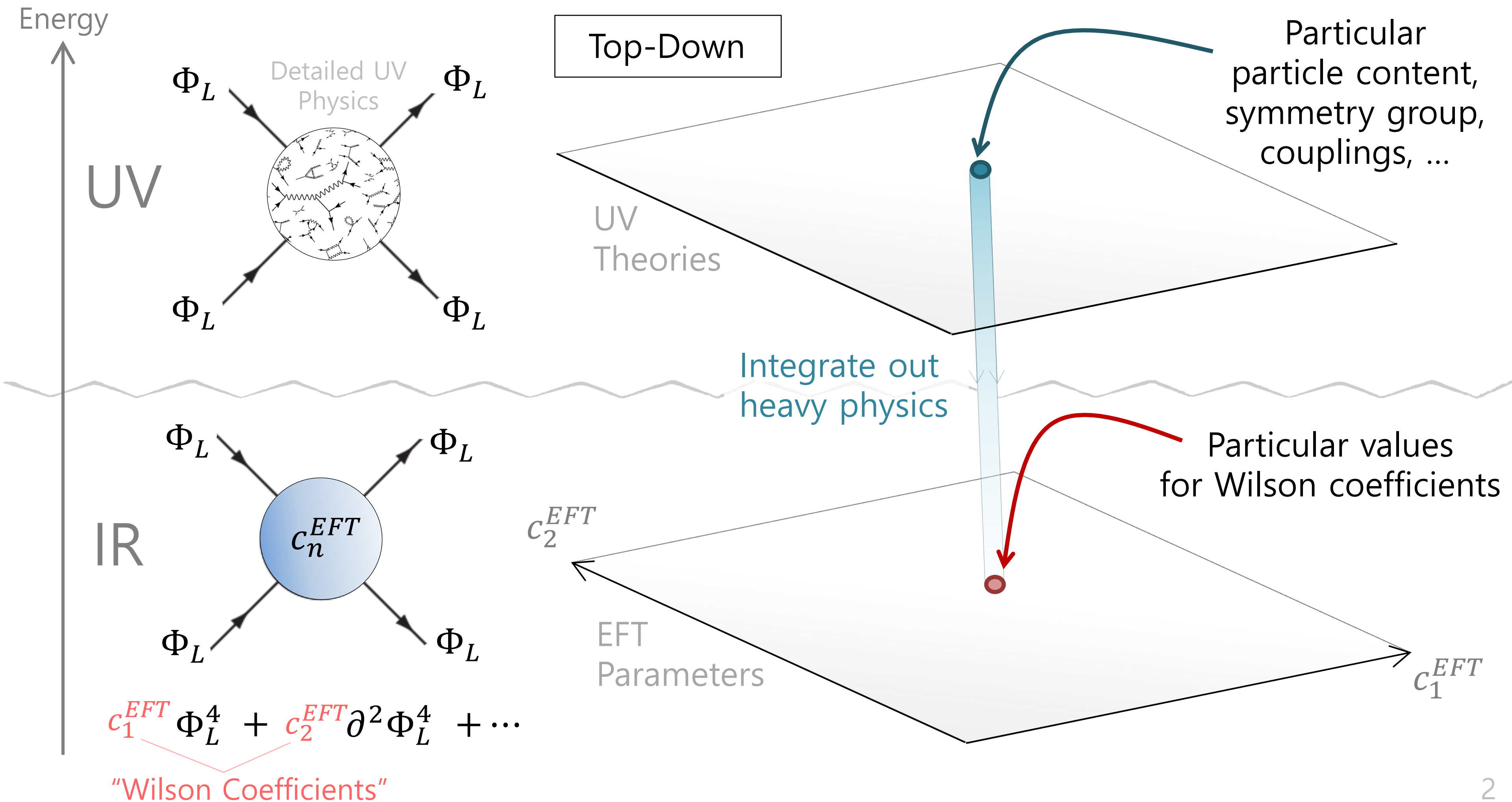

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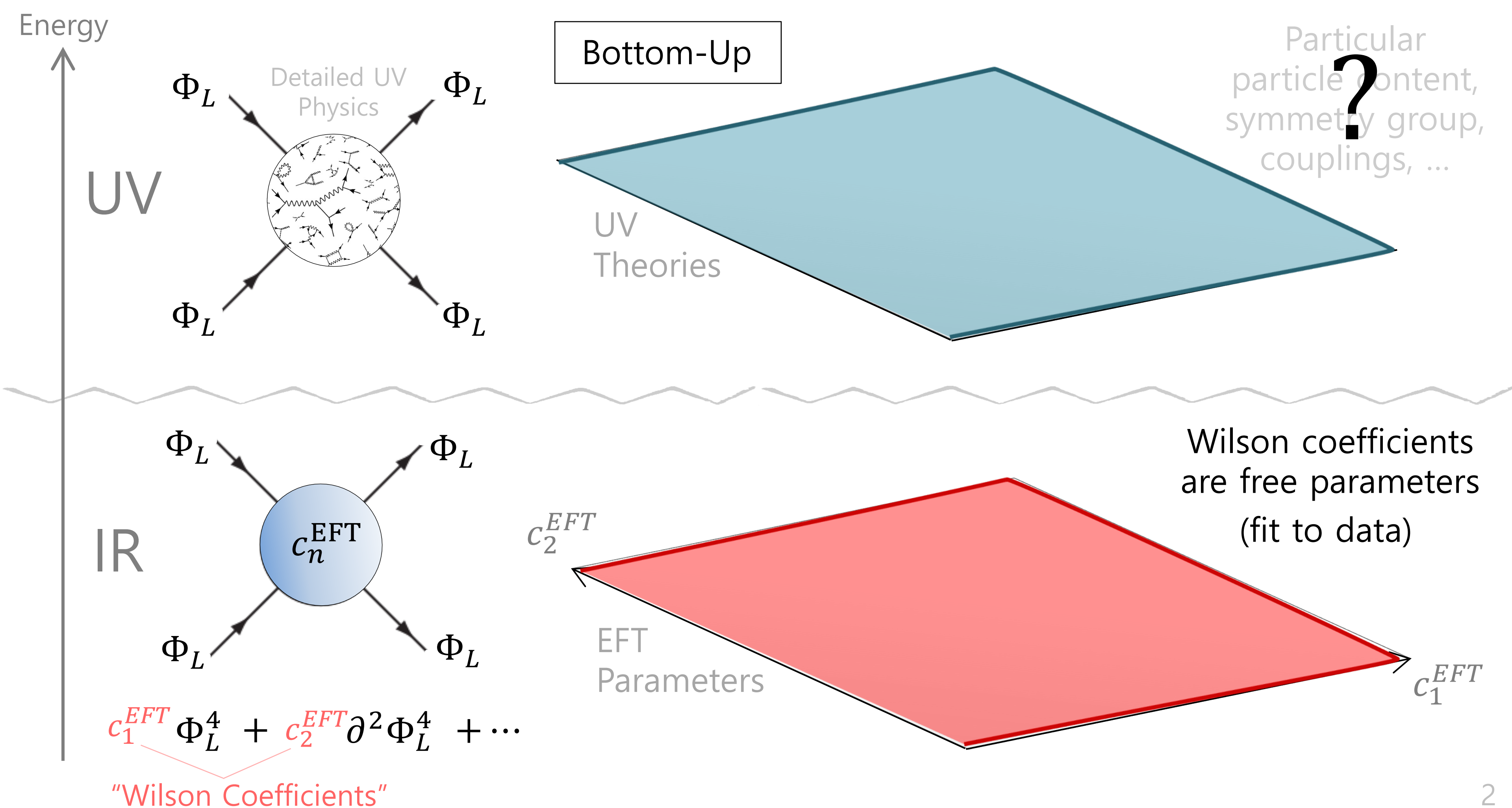

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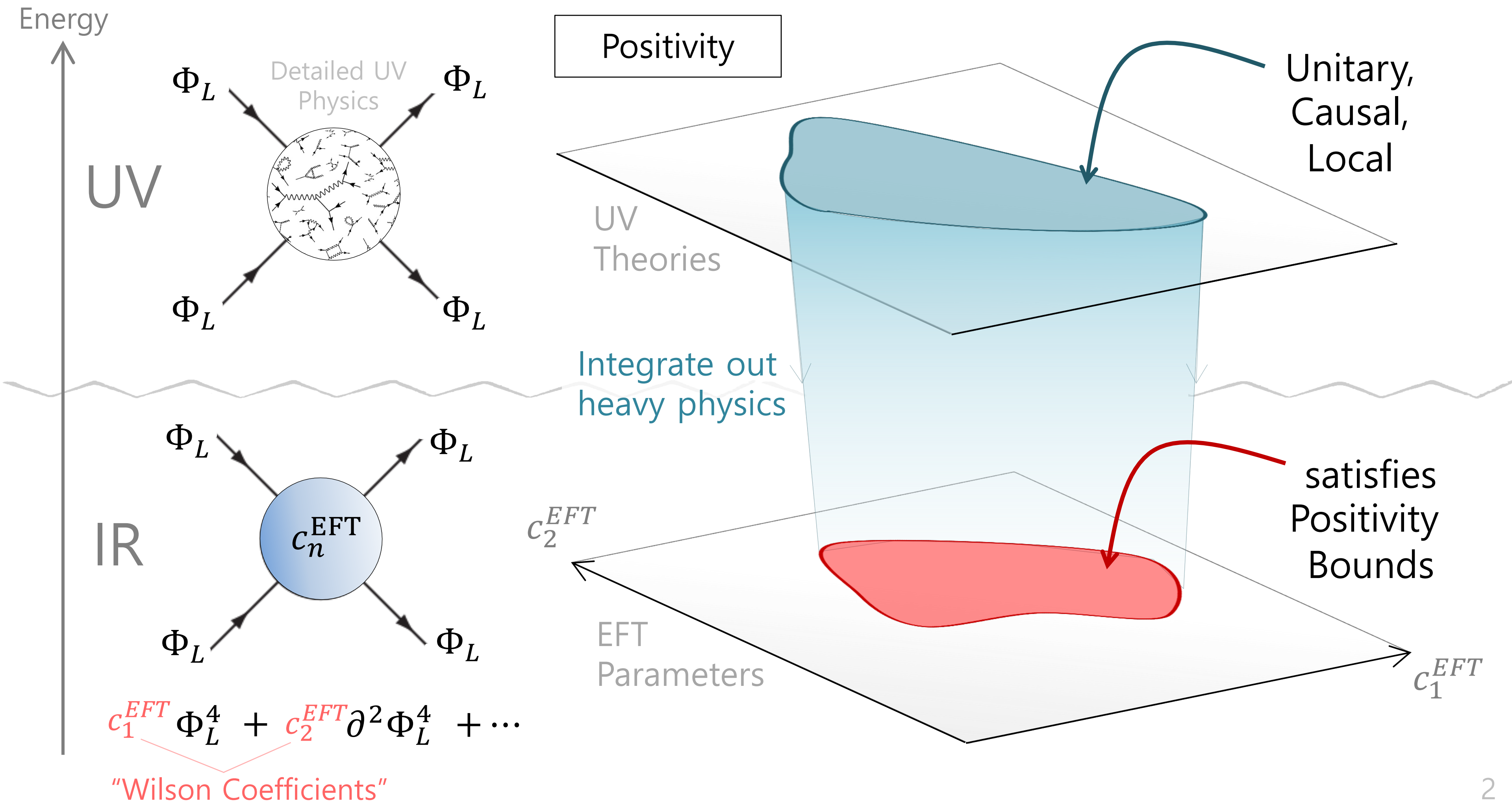
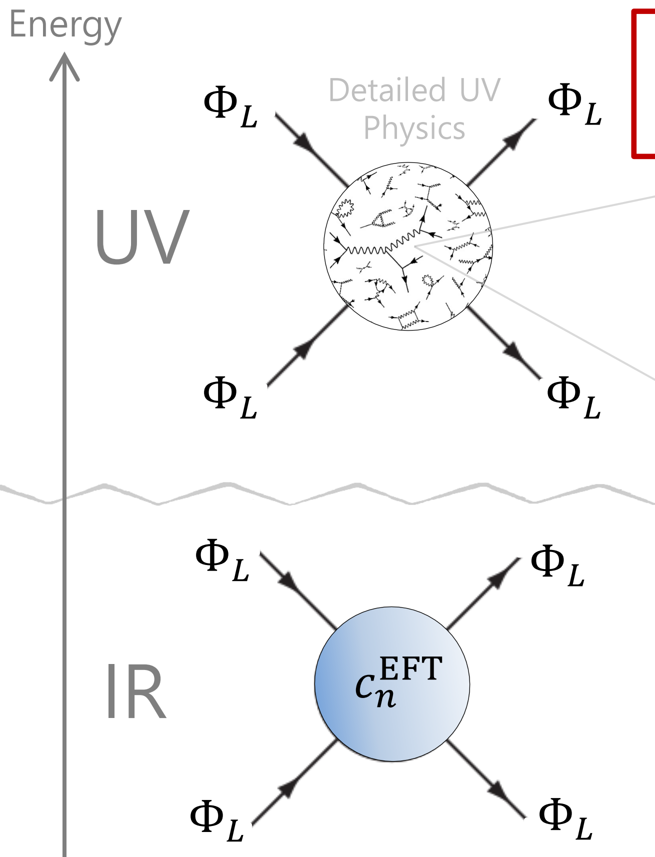


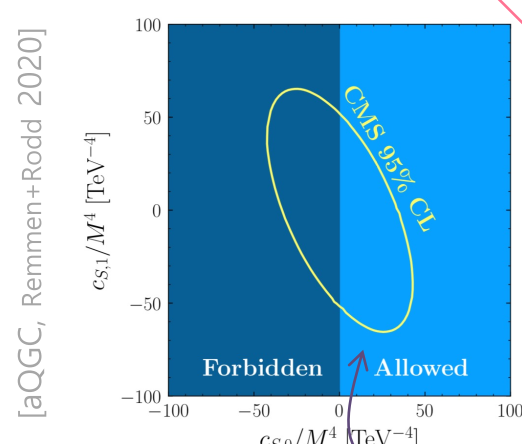
What are Positivity Bounds?

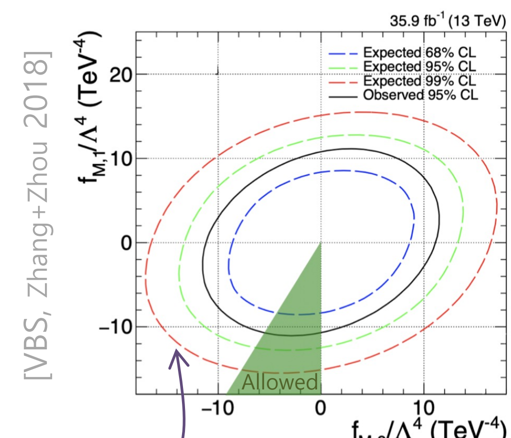
Positivity is increasingly important in the search for new physics
Need UV/IR relations to maximize discovery potential in cosmology



$$c_1^{EFT}\Phi_L^4 + c_2^{EFT}\partial^2\Phi_L^4 + \cdots$$

"Wilson Coefficients"

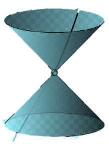




LHC data

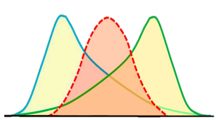
How to derive Positivity Bounds?

(Locality)
+
Causality

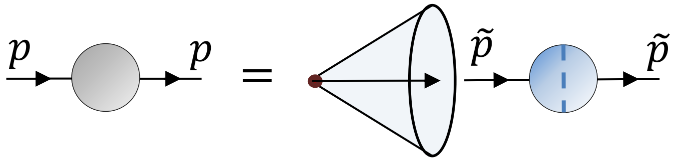


$[\mathcal{O}_x, \mathcal{O}_y] = 0$

Unitarity



$\hat{S}^\dagger \hat{S} = 1$



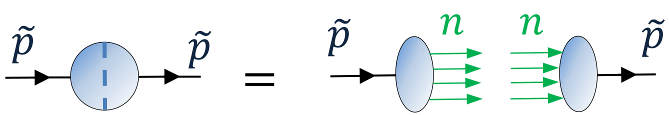
$$A(p) = \int_0^\infty \frac{d\tilde{p}^2}{\tilde{p}^2 - p^2} \operatorname{Im} A(\tilde{p})$$

Relates IR to UV

(via dispersion relation)

UV is positive

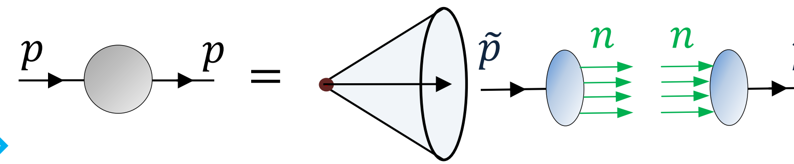
(via optical theorem)



$$\operatorname{Im} A(\tilde{p}) = \frac{1}{2} \sum_n A_{1\to n}(\tilde{p}) A^*_{1\to n}(\tilde{p})$$

Positivity

IR coefficients are positive

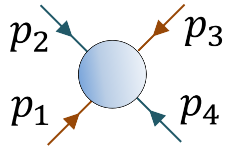


$$A(p) = \sum_j p^{2j} \left(\int_0^\infty \frac{d\tilde{p}^2}{\tilde{p}^{2j+2}} \sum_n |A_{1\to n}(\tilde{p})|^2 \right)$$

$\Rightarrow A_{EFT}(p) = \sum_j p^{2j}\, c_j^{EFT}$ has $c_j^{EFT} > 0$

How to derive Positivity Bounds?

Lorentz Invariance



$A(p_1, p_2, p_3, p_4) = A(s,t)$

function of only two variables

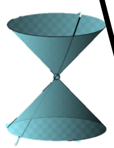
$s = (p_1 + p_2)^2$

$t = (p_1 + p_3)^2$

$u = 4m^2 - s - t$

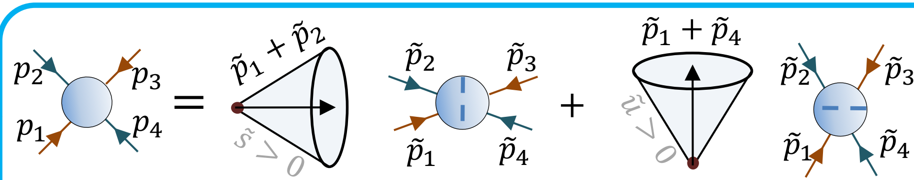
Also translational invariance
(energy/momentum conservation)

(Locality) + Causality



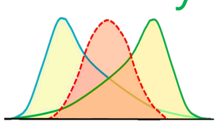
Non-perturbative

[Bremermann, Bros, Epstein, Froissart, Glaser, Gribov, Hepp, Jin, Kallen, Lehmann, Mandelstam, Martin, Taylor, 1960s]



$$A(s,t) = \int_0^\infty \frac{d\tilde{s}}{\tilde{s} - s} \, \mathrm{Im}\, A(\tilde{s}, t) + \int_0^\infty \frac{d\tilde{u}}{\tilde{u} - u} \, \mathrm{Im}\, A(\tilde{u}, t)$$

Unitarity



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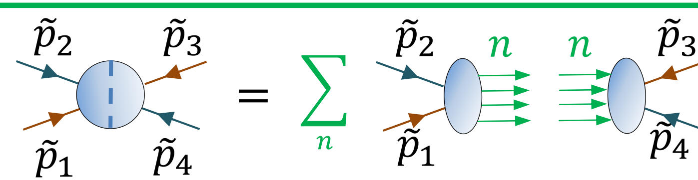
[Bohr++, 1939]
[...]



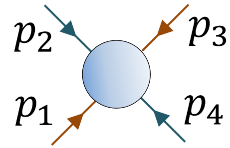
$$\mathrm{Im}\, A_{12\to34} = \sum_n A_{12\to n} \, A^*_{34\to n}$$

Positivity

$A_{EFT}(s,t) = \sum_{a,b} s^a t^b c_{ab}$ has <u>bounded</u> c_{ab}

How to derive Positivity Bounds?

Lorentz Invariance



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$s = (p_1+p_2)^2$
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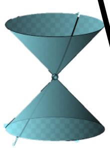
e.g. in Forward Limit

(Locality) + Causality



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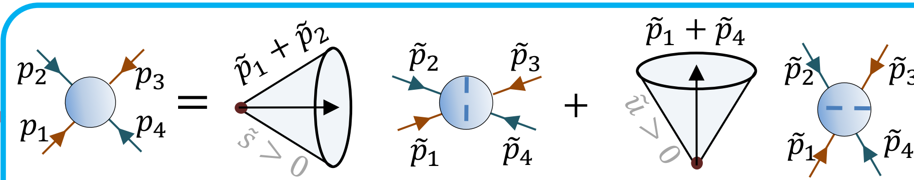
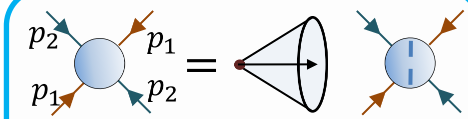


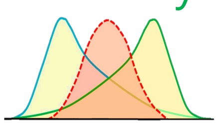
$$A(s,t) = \int_0^\infty \frac{d\tilde s}{\tilde s - s}\,\mathrm{Im}\,A(\tilde s,t) + \int_0^\infty \frac{d\tilde u}{\tilde u - u}\,\mathrm{Im}\,A(\tilde u,t)$$



$$\partial_s^2 A_{\rm EFT} \propto \int_0^\infty \frac{d\tilde s}{\tilde s^3}\,\mathrm{Im}\,A_{\rm UV}$$

(with $t=0$ held fixed)

Unitarity



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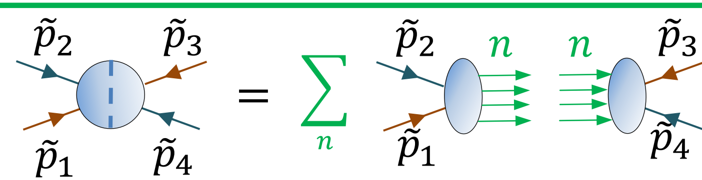


$$\mathrm{Im}\,A_{12\to34} = \sum_n A_{12\to n}\,A^*_{34\to n}$$

$$\mathrm{Im}\,A_{\rm UV}(s,0) > 0$$

Positivity

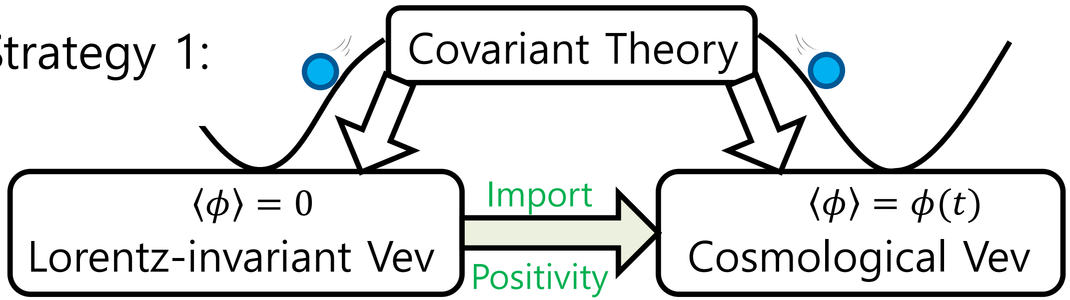
$$A_{EFT}(s,t) = \sum_{a,b} s^a t^b\, c_{ab}$$ has bounded c_{ab}

$$\partial_s^2 A_{\rm EFT}(s,0) > 0$$

[Adams++, 2006]

How to apply Positivity Bounds in Cosmology?

Strategy 1:



e.g. Dark Energy

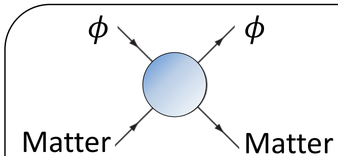
[Horndeski 1974]

$P(X)$ — Scalar-Tensor

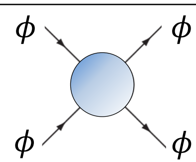
$$\int d^4x\,\sqrt{-g}\,\left[G_2(X) + G_4(X)R + G_4'(X)\left((\Box\phi)^2 - (\nabla\nabla\phi)^2\right)\right]$$

$\langle\phi\rangle = 0$, $g_{\mu\nu}$ = Minkowski

$\langle\phi\rangle = \phi(t)$, $g_{\mu\nu}$ = FLRW



$G_4' \geq 0$



${G_4'}^2 \leq G_4 G_4''$

Tensor Speed: $\alpha_T = 4XG_4'$

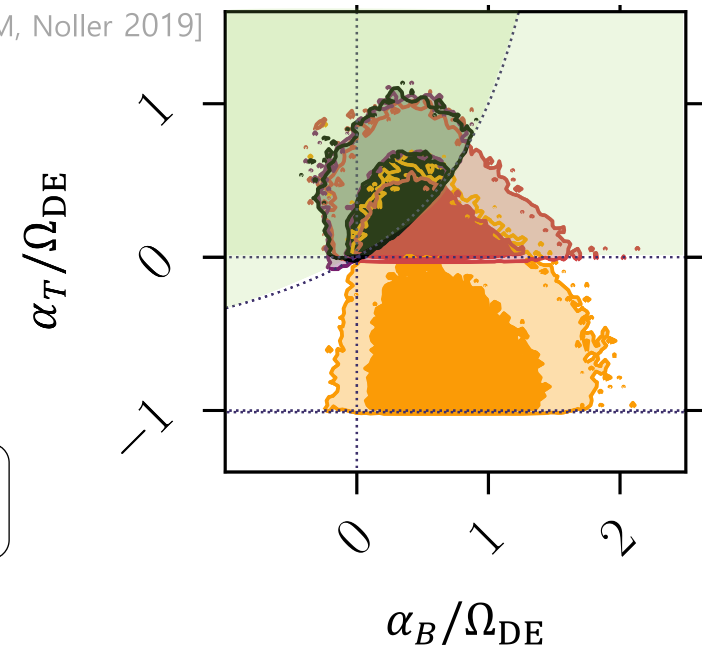
$\alpha_T \geq 0$

Braiding (DE Clustering): $\alpha_B = 8X(G_4' + 2XG_4'')$

$\alpha_B \leq \frac{2\alpha_T}{1+\alpha_T}$

CMB (Planck)
BAO (SDSS/BOSS)
Matter (SDSS)
RSD (BOSS/6dF)

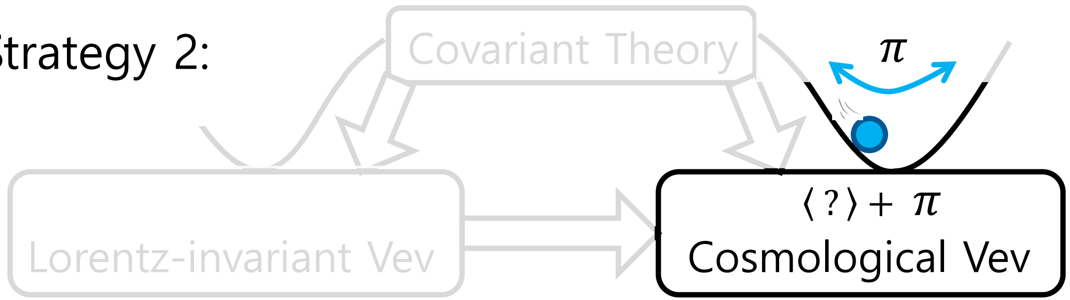
[SM, Noller 2019]

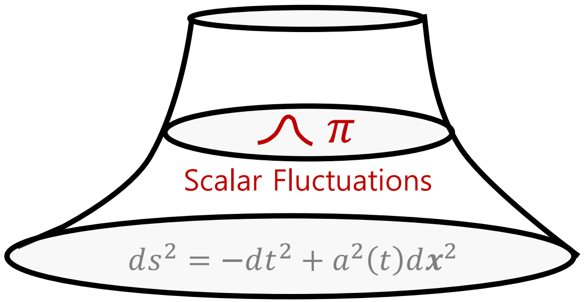


[de Rham, SM, Noller 2021] [SM, Noller 2019]

How to apply Positivity Bounds in Cosmology?

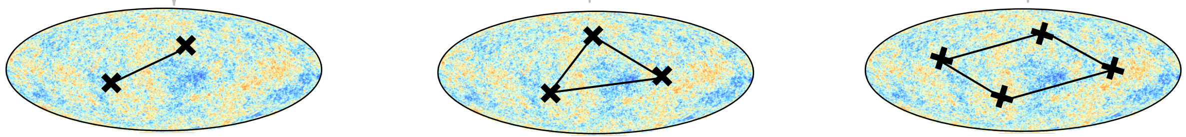
Strategy 2:





$$\mathcal{L}_{\mathrm{EFT}} = \dot{\pi}^2 - c_s^2(\partial_i\pi)^2 + \alpha_1\,\dot{\pi}^3 + \alpha_2\dot{\pi}(\partial\pi)^2 + \beta_1\,\dot{\pi}^4 + \beta_2\dot{\pi}^2(\partial\pi)^2 + \beta_3(\partial\pi)^4$$

[Cheung++ 2008]



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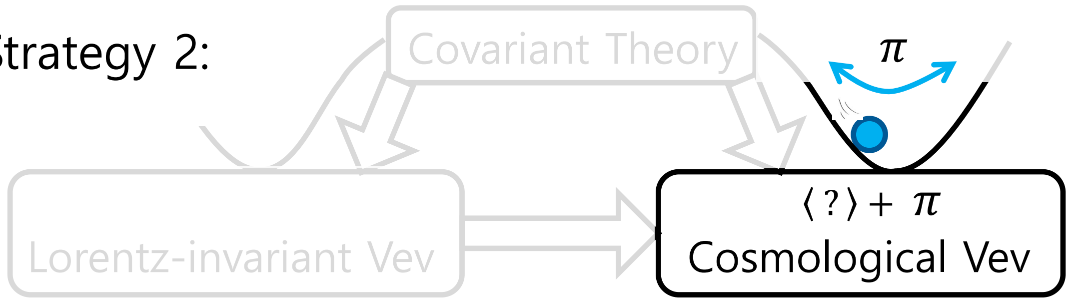

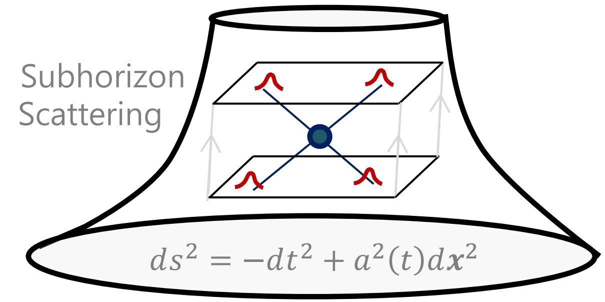

$$\mathcal{L}_{\rm EFT} = \dot{\pi}^2 - c_s^2(\partial_i \pi)^2 + \alpha_1 \, \dot{\pi}^3 + \alpha_2 \dot{\pi}(\partial\pi)^2 + \beta_1 \, \dot{\pi}^4 + \beta_2 \dot{\pi}^2 (\partial \pi)^2 + \beta_3 (\partial\pi)^4$$

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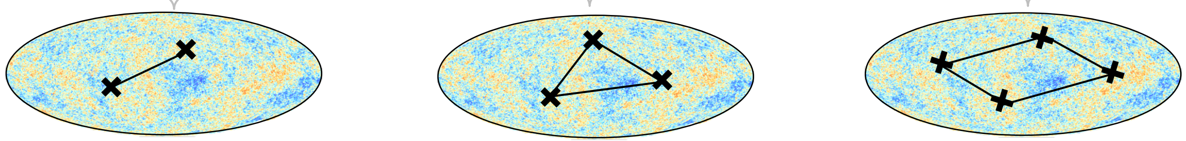


(Only c_s , α_1 , β_1 are independent)

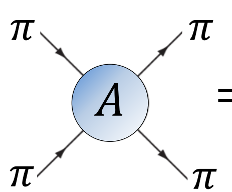
$\alpha_2 = \frac{1-c_s^2}{c_s^2}$, $\beta_2 = \frac{3\alpha_1}{2c_s^2} + \frac{(1-c_s^2)^2}{2c_s^4}$, $\beta_3 = \frac{1-c_s^2}{8c_s^4}$

Work in limit $M_P \to \infty$, $\dot{H} \to 0$ $\Rightarrow$ gravity decouples (no tensor modes)

(with $f_\pi^4 = M_P^2 \dot{H}$ fixed) $\Rightarrow$ time translations ($\alpha, \beta \approx$ constant on subhorizon scales)

However... <u>boosts are broken</u> $\Rightarrow$ Need new positivity bounds

$\mathcal{L}_{\rm EFT} \supset \beta_1 \dot{\pi}^4 \Rightarrow$



$= \omega_1 \omega_2 \omega_3 \omega_4$ (Explicit energy dependence)

$$A(s,t) \to A(s,t,\omega_1,\omega_2,\omega_3)$$

$$\partial_s^2 \Big|_t A(s,t) \to \partial_s^2 \Big|_{t,?,?,?} A(s,t,\omega_1,\omega_2,\omega_3)$$

How to apply Positivity Bounds in Cosmology?

~~Lorentz~~ Rotations

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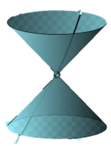

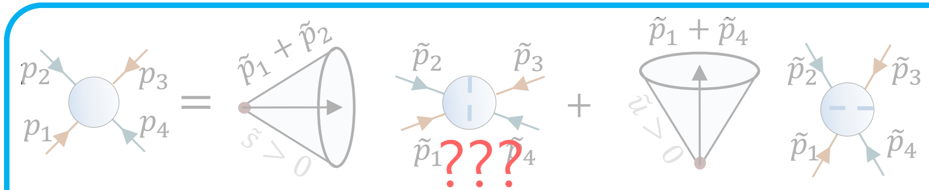

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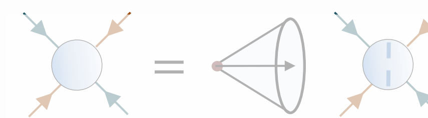
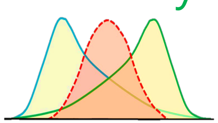


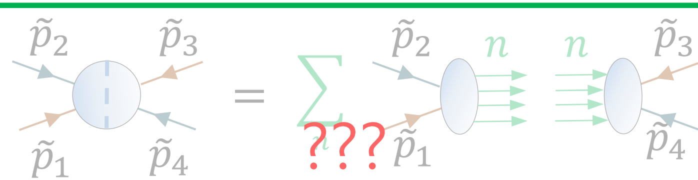
$$\partial_s^2 A_{\mathrm{EFT}} \propto \int_0^\infty \frac{d\tilde{s}}{\tilde{s}^3} \, \mathrm{Im}\, A_{\mathrm{UV}}$$

???

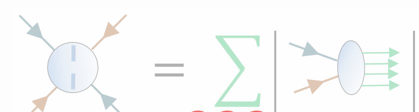
(with $t = 0$ held fixed)

Unitarity





$$\mathrm{Im}\, A_{12\to34} = \sum_n A_{12\to n} \, A^*_{34\to n}$$



???

$$\mathrm{Im}\, A_{\mathrm{UV}}(s,0) > 0$$

Positivity

???

???

How to apply Positivity Bounds in Cosmology?

~~Lorentz~~ Rotations

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Amplitude depends explicitly on three energies

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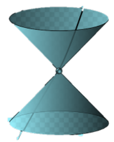


[SM+Grall, 2021]

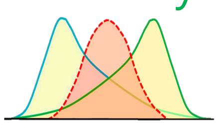
$$\partial_s^2 A_{\text{EFT}} \propto \int_0^\infty \frac{d\tilde{s}}{\tilde{s}^3} \, \text{Im}\, A_{\text{UV}}$$

with

$$\{ \, t = 0 \, , \, \omega_3 = -\omega_1 \, , \, \frac{s-u}{\omega_2} \, \}$$

held fixed

Unitarity



Non-perturbative

[SM+Grall, 2020]

In forward limit
($t = 0, \ \omega_3 = -\omega_1$)

$$\text{Im}\, A_{\text{fwd}}(s,\omega_1,\omega_2) > 0$$

Positivity

$$\partial_s^2 A_{\text{EFT}} > 0$$

with $t = 0 \, , \, \omega_1 = -\omega_3$
and $\frac{s-u}{\omega_2}$ held fixed

[SM+Grall, 2021]

($A_{\text{EFT}} = \sum_{a,b} (\omega_2 + \omega_4)^a t^b C_{ab}$
has <u>bounded</u> C_{ab})

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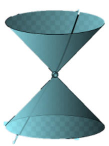


[SM+Grall, 2021]

$$\partial_s^2 A_{\rm EFT} \propto \int_0^\infty \frac{d\tilde{s}}{\tilde{s}^3}\, {\rm Im}\, A_{\rm UV}$$

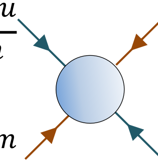
with

$\{\, t=0\ ,\ \omega_3=-\omega_1\ ,\ \frac{s-u}{\omega_2}\,\}$

held fixed

[Bremermann++, 1958]

$\omega_2=\frac{s-u}{8m}$, $\omega_3=-m$, $\omega_1=m$, $\omega_4=\frac{u-s}{8m}$

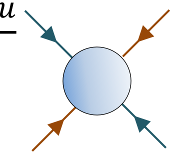


$\boldsymbol{p}_1+\boldsymbol{p}_3=0$

→

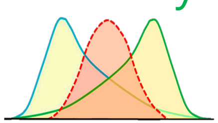
$A_{\rm fwd}(s,\gamma,M)$

$\omega_2=\frac{s-u}{M}$, $\omega_3=-\gamma M$, $\omega_1=\gamma M$, $\omega_4=\frac{u-s}{M}$



$\boldsymbol{p}_1+\boldsymbol{p}_3=\boldsymbol{p}_{\rm Breit}$

Unitarity



Non-perturbative

[SM+Grall, 2020]

In forward limit ($t=0,\ \omega_3=-\omega_1$)

$${\rm Im}\, A_{\rm fwd}(s,\omega_1,\omega_2)>0$$

$\boldsymbol{p}_1+\boldsymbol{p}_2=0$

$A(s,\theta)=\sum_\ell P_\ell(\theta)a_\ell(s)$ →

${\rm Im}\, a_\ell(s)>0$

$\boldsymbol{p}_1+\boldsymbol{p}_2=\boldsymbol{p}_{\rm CM}$

$$A(s,|\boldsymbol{p}_{\rm CM}|,\theta_1,\varphi_1,\theta_3,\varphi_3) = \sum_{\ell_j m_j} Y_{\ell_1}^{m_1}(\theta_1,\varphi_1)Y_{\ell_3}^{m_3 *}(\theta_3,\varphi_3)a_{\ell_1\ell_3}^{m_1m_3}(s,|\boldsymbol{p}_{\rm CM}|)$$

${\rm Im}\, a_{\ell_1\ell_3}^{m_1m_3}(s,|\boldsymbol{p}_{\rm CM}|)$ positive matrix

Positivity

$$\partial_s^2 A_{\rm EFT}>0$$

with $t=0\ ,\ \omega_1=-\omega_3$ and $\frac{s-u}{\omega_2}$ held fixed

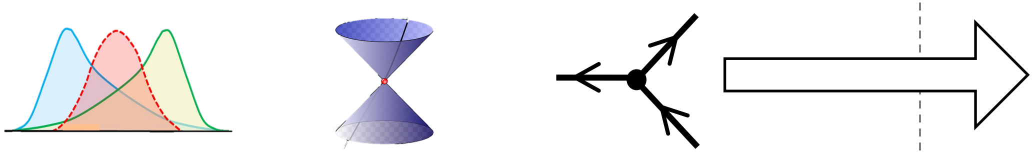
[SM+Grall, 2021]

$\left(A_{\rm EFT}=\sum_{a,b}(\omega_2+\omega_4)^a t^b C_{ab} \right.$ has <u>bounded</u> $\left. C_{ab} \right)$

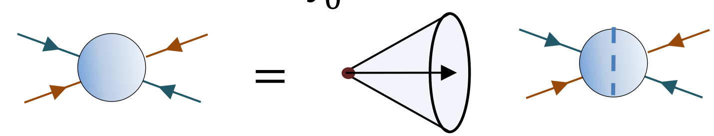
What do Positivity Bounds mean for Inflation?

Small-scale (UV) | Large-scale (IR)

Unitary + Causal + Local



$$\partial_s^2 A_{EFT} = \int_0^\infty \frac{d\tilde{s}}{\tilde{s}^3} \, \mathrm{Im}\, A_{UV} \geq 0$$



Big impact on simple models

$$\mathcal{L}_{\text{int}} = \beta_1 \dot{\pi}^4 \Rightarrow \partial_s^2 A = \frac{3}{2}\beta_1 \Rightarrow \beta_1 > 0$$

$$\mathcal{L}_{\text{int}} = \alpha_1 \dot{\pi}^3 \Rightarrow \partial_s^2 A = -\frac{9}{4}\alpha_1^2 \Rightarrow \alpha_1 = 0$$

[Grall, SM 2021]

Trispectrum is bounded by the bispectrum

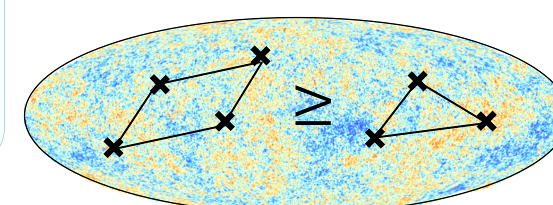
$$\mathcal{L}_{\text{EFT}}[\pi] \Rightarrow \partial_s^2 A = f(c_s, \alpha_1, \beta_1) \Rightarrow \beta_1 \geq \frac{3}{2}\alpha_1^2 - 2\alpha_1 - \frac{1}{3}\frac{1-c_s^2}{c_s^4}$$

$$\beta_1 \geq \beta_{\min}(c_s, \alpha_1)$$

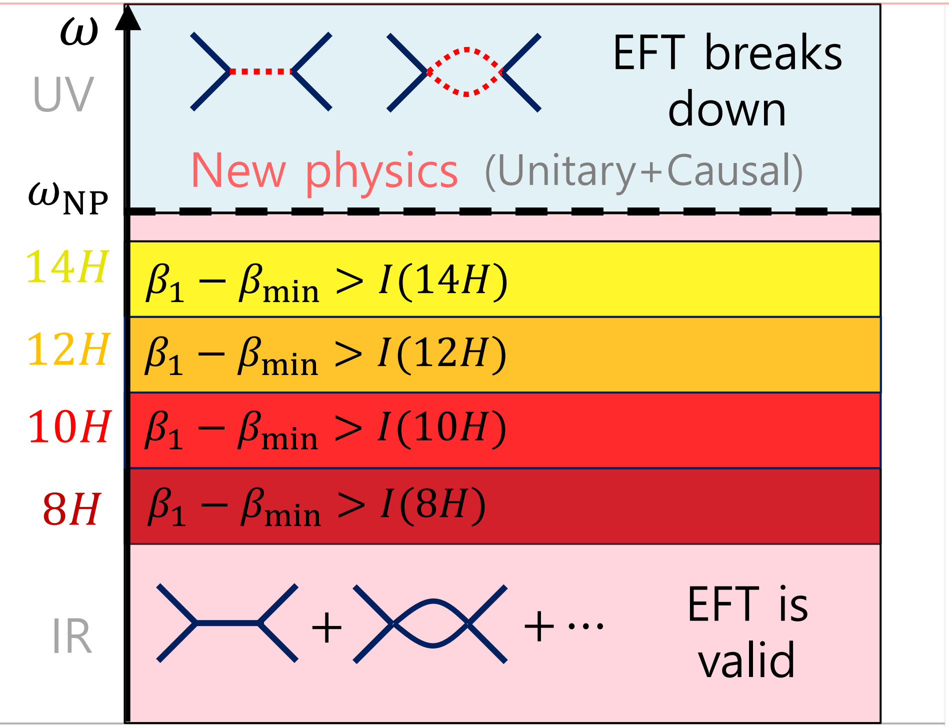
$$\underbrace{\partial_s^2 A_{\text{EFT}}}_{\beta_1 - \beta_{\min}} = \underbrace{\int^{\omega} \mathrm{Im}\, A_{\text{EFT}}}_{I(\omega) > 0} + \underbrace{\int_{\omega}^{\infty} \mathrm{Im}\, A_{\text{UV}}}_{? > 0}$$

$I(\omega) > 0$ (Known) | $? > 0$ (Not known)

Can strengthen bound by subtracting IR information

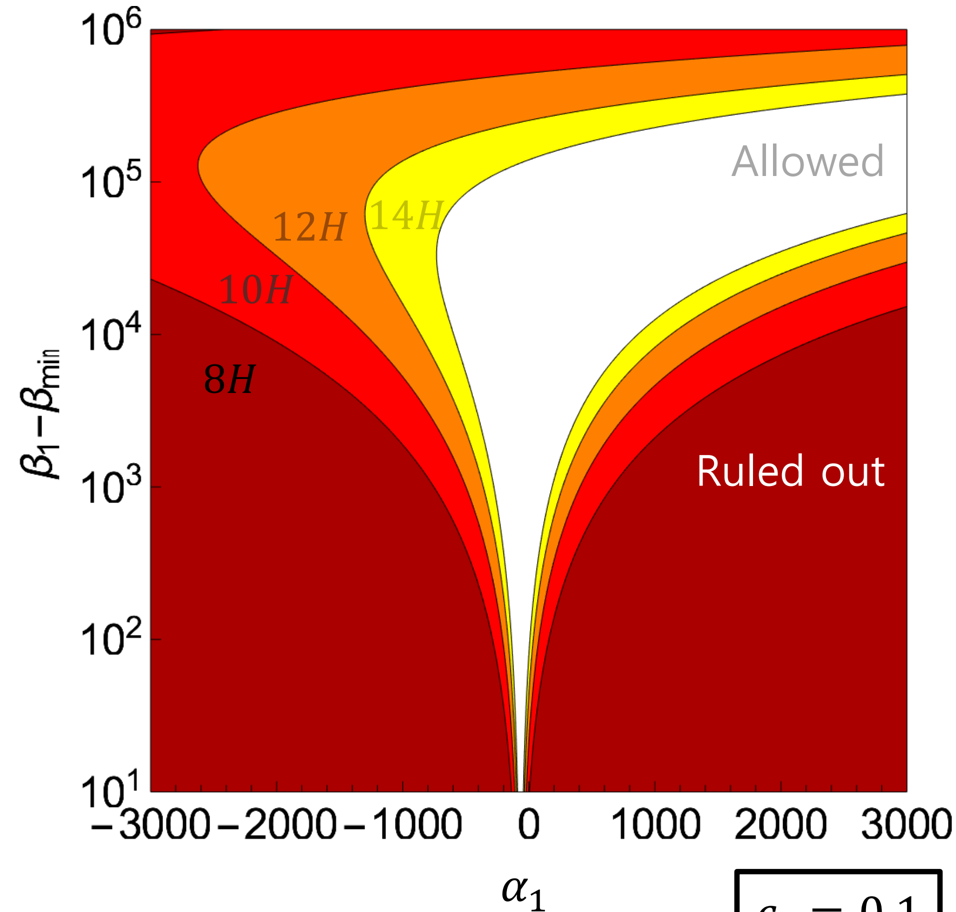


What do Positivity Bounds mean for Inflation?



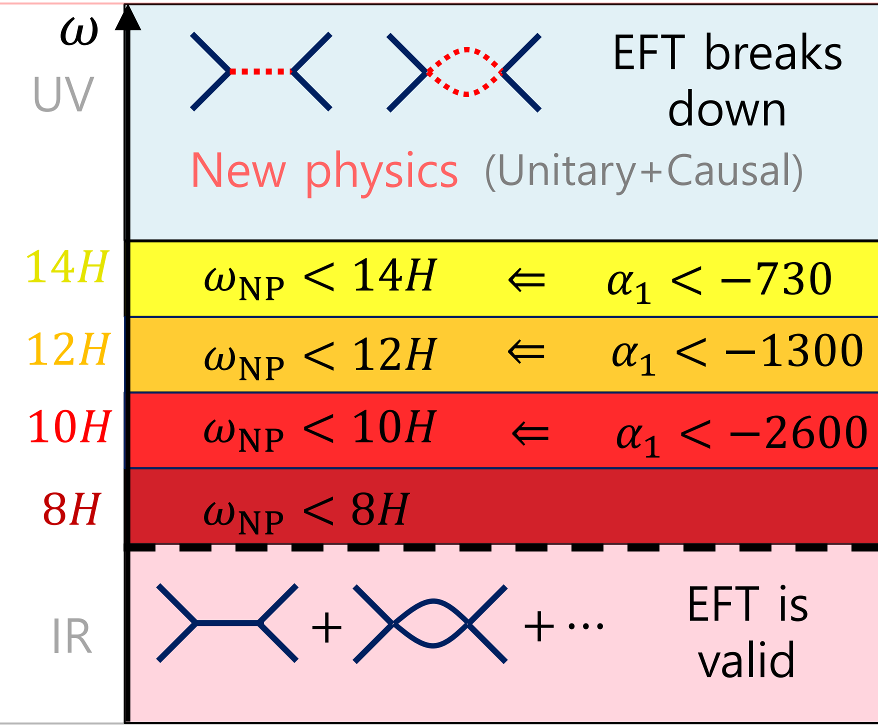
$$\partial_s^2 A_{\mathrm{EFT}} = \int^{\omega} \mathrm{Im}\, A_{\mathrm{EFT}} + \int_{\omega}^{\infty} \mathrm{Im}\, A_{\mathrm{UV}}$$

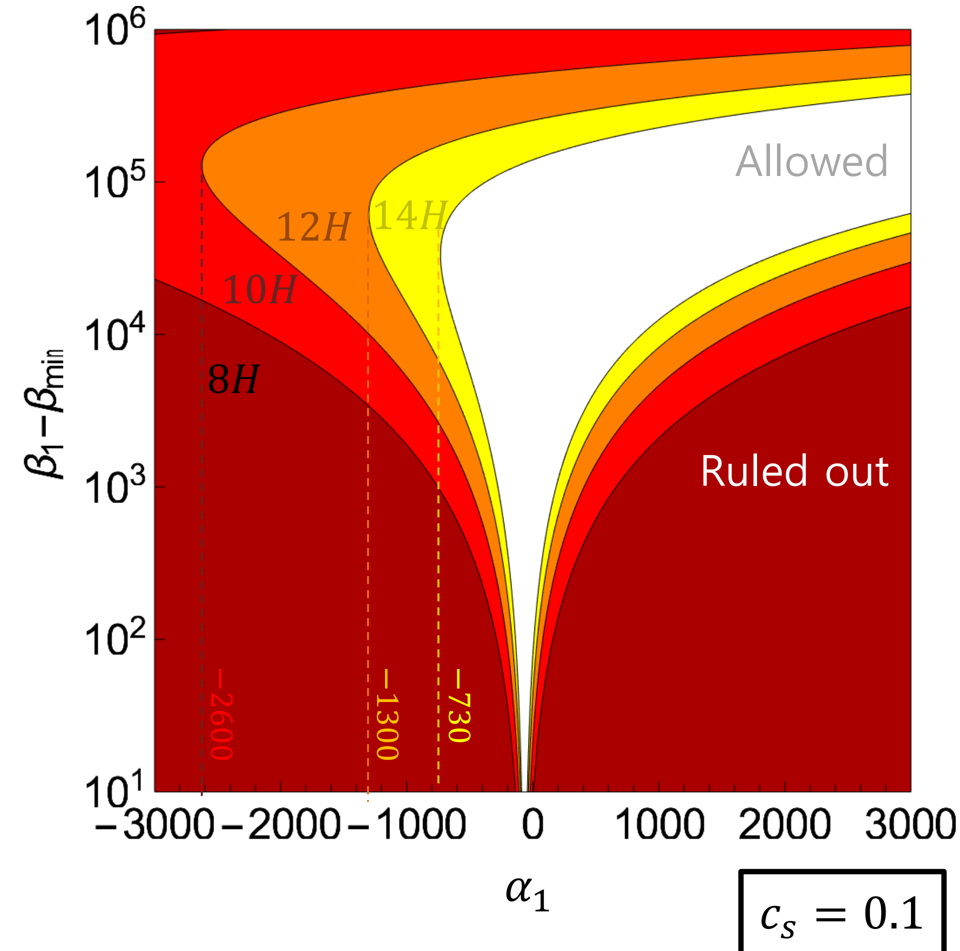
$\beta_1 - \beta_{\min}$ | $I(\omega) > 0$ (Known) | $? > 0$ (Not known)



$c_s = 0.1$

What do Positivity Bounds mean for Inflation?



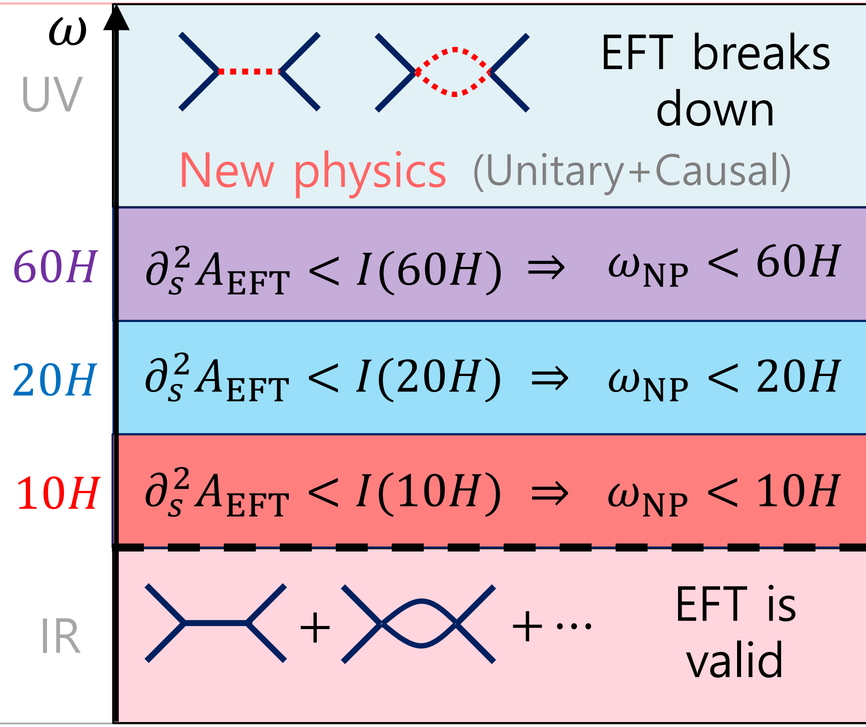


New physics (beyond single-field weakly-coupled inflation) must complete EFT before:

$$\frac{\omega_{\rm NP}^4}{f_\pi^4} \leq \frac{30\pi^2 c_s^4}{\left|1 - c_s^2 + \frac{3}{2}\alpha_1 c_s^2\right|} \qquad (f_\pi \approx 60H)$$

… independently of trispectrum (β_1)!

What do Positivity Bounds mean for Inflation?



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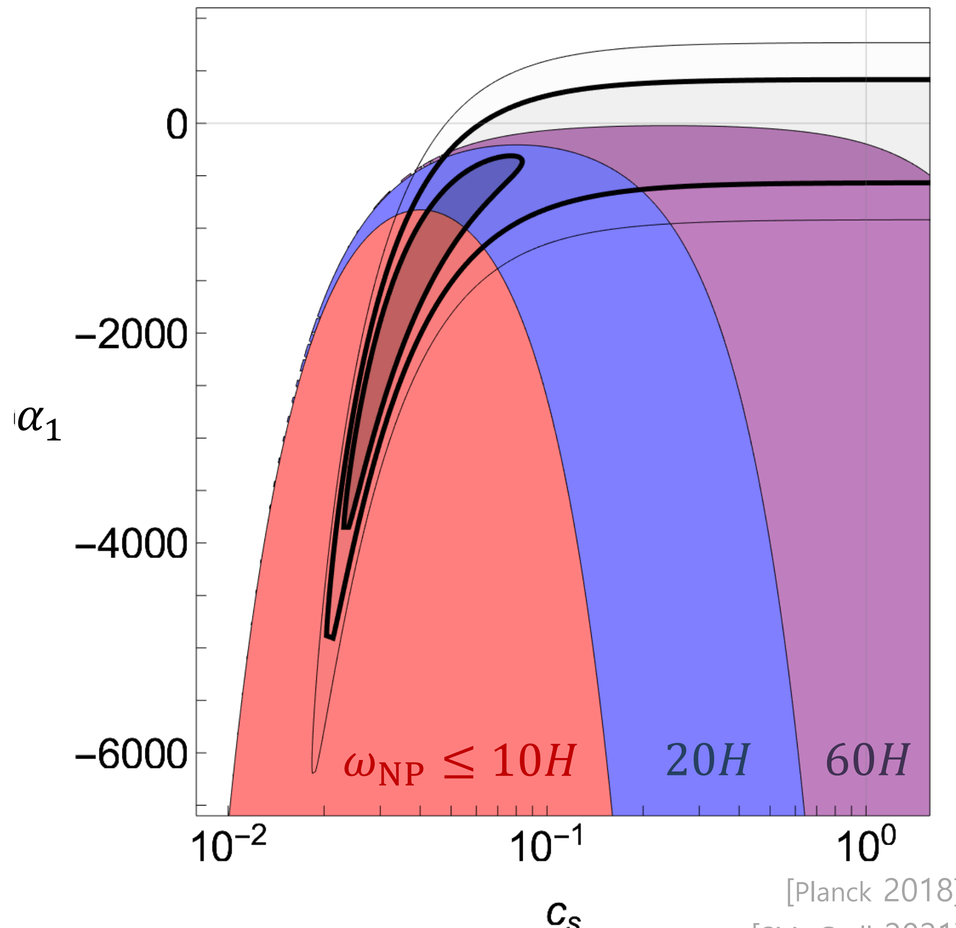
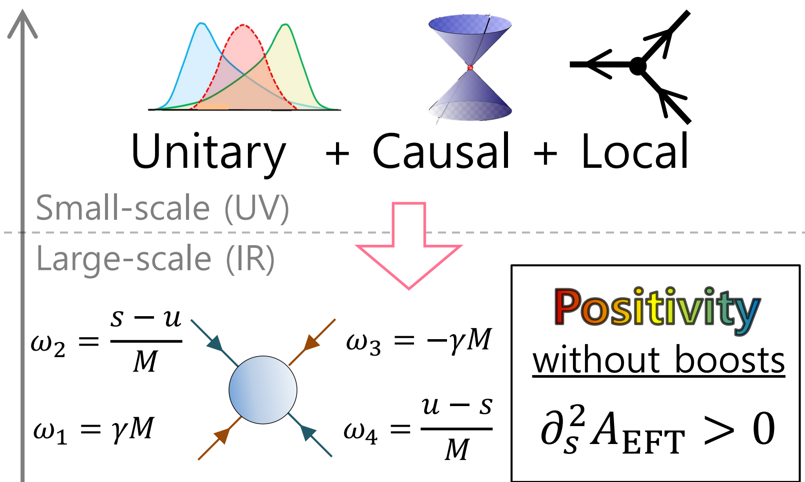
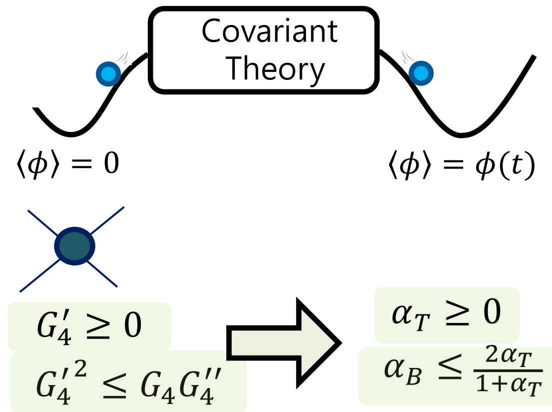


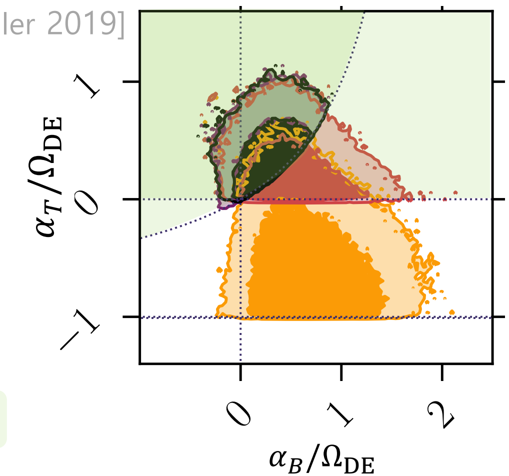
[Planck 2018]
[SM+Grall 2021]

Positive Outlook



[SM, Noller 2019]





Next Steps

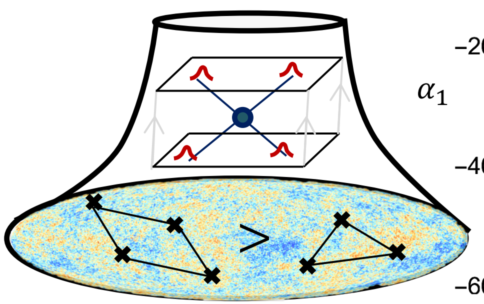
Stronger bounds? (crossing, arcs, EFThedron)

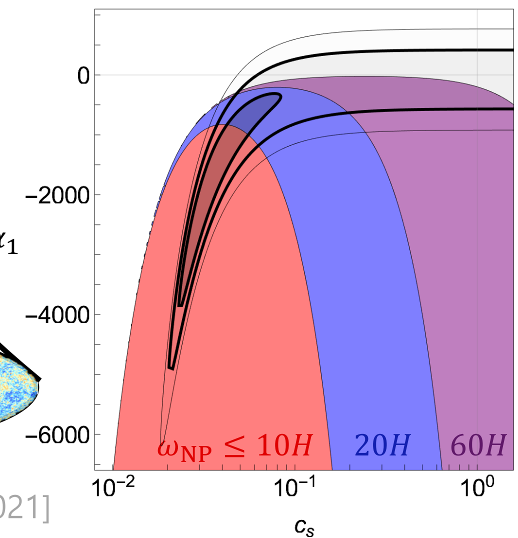
Other systems? (dark energy, condensed matter)

Beyond subhorizon scattering? (time translations)

Beyond decoupling limit? (include gravity/tensor modes)

EFT of Inflation





[SM, Grall 2021]