# Unitarily inequivalent quantum cosmological bouncing models


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dépasser les frontières


arXiv 2111.02963

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# CONTEXT

- ***Cosmological perturbation theory***

$\longrightarrow$ **Classical / semiclassical background**

$\longrightarrow$ **Quantum vacuum initial state**

$\longrightarrow$ **Unique / unambiguous predictions** $\mathcal{P}_k,\ \frac{\Delta T}{T},\ c_\ell,\ \cdots$

but see J. Grain & V. Vennin, *JCAP* **02**, 022 (2020)

- ***Inflation***

$\longrightarrow$ **Monotonic evolution of the scale factor** $a(t);\, \dot{a} > 0\ \forall t$

$\longrightarrow$ $\exists$ **singularity**

$\longrightarrow$ **resolve by quantum effects?**

- ***Quantum background***

$\longrightarrow$ **Phase space / coherent state quantization** $a > 0\ (a \neq 0)\ \forall t$

$\longrightarrow$ **Semiclassical approximation**

$\oplus$ **Canonical quantization of perturbations**

$\Longrightarrow$ *Ambiguities!*

# Motivations: (quantum) cosmology

*Homogeneous & isotropic metric (FLRW):* $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right]$

→ *spatial curvature*

*Hubble rate* $H \equiv \frac{\dot{a}}{a}$

*Matter component: perfect fluid* $T_{\mu\nu} = p g_{\mu\nu} + (\rho + p)\, u_\mu u_\nu$

*equation of state* $p = w\rho$ → $\begin{cases} w = 0 & \text{dust} \\ w = \frac{1}{3} & \text{radiation} \end{cases}$

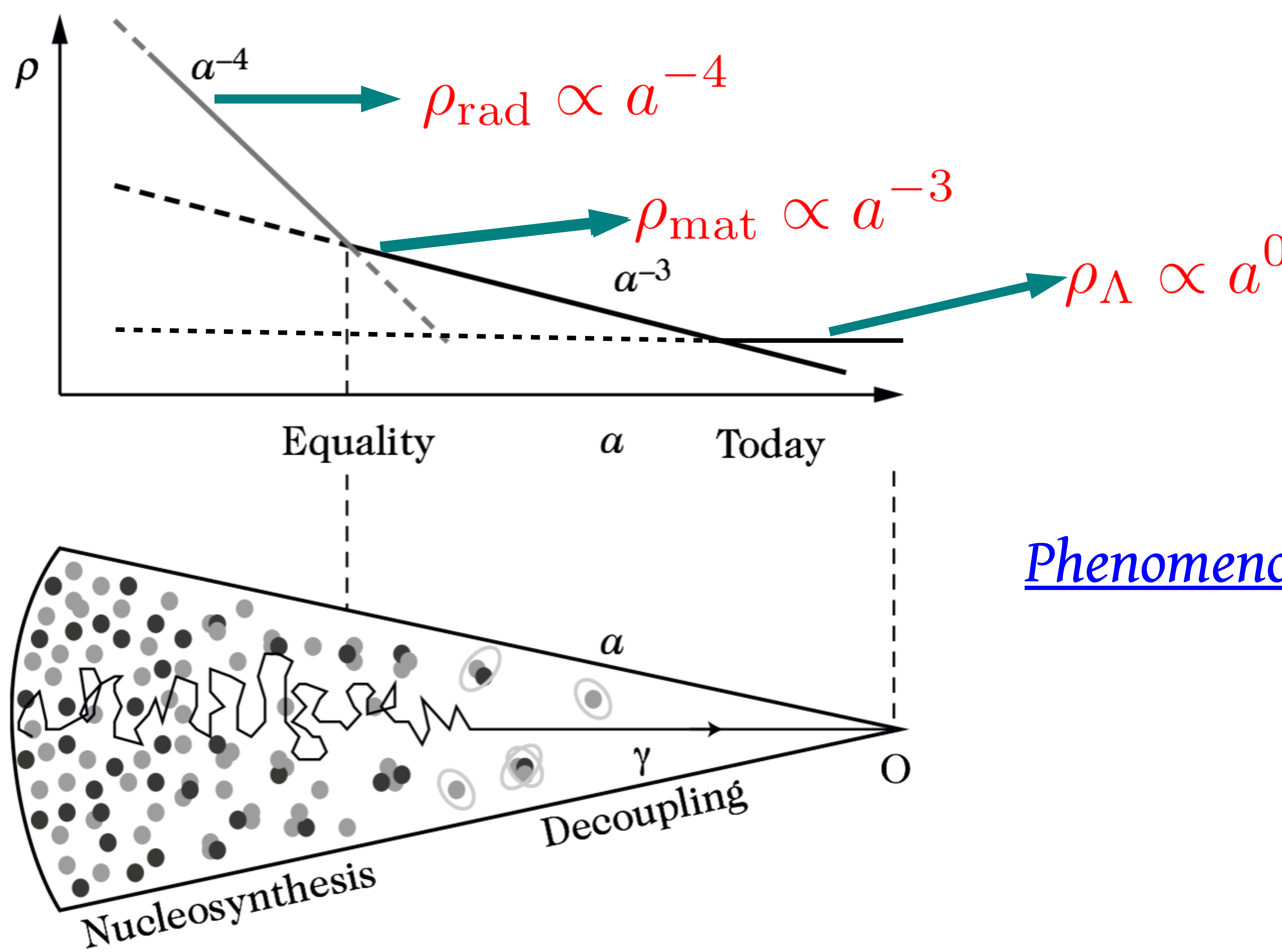
*+ cosmological constant = Einstein equations*

$$\begin{cases} H^2 + \dfrac{\mathcal{K}}{a^2} = \dfrac{1}{3}\left(8\pi G_{\mathrm{N}}\rho + \Lambda\right) \\ \dfrac{\ddot{a}}{a} = \dfrac{1}{3}\left[\Lambda - 4\pi G_{\mathrm{N}}\left(\rho + 3p\right)\right] \end{cases}$$

integrate conservation equation

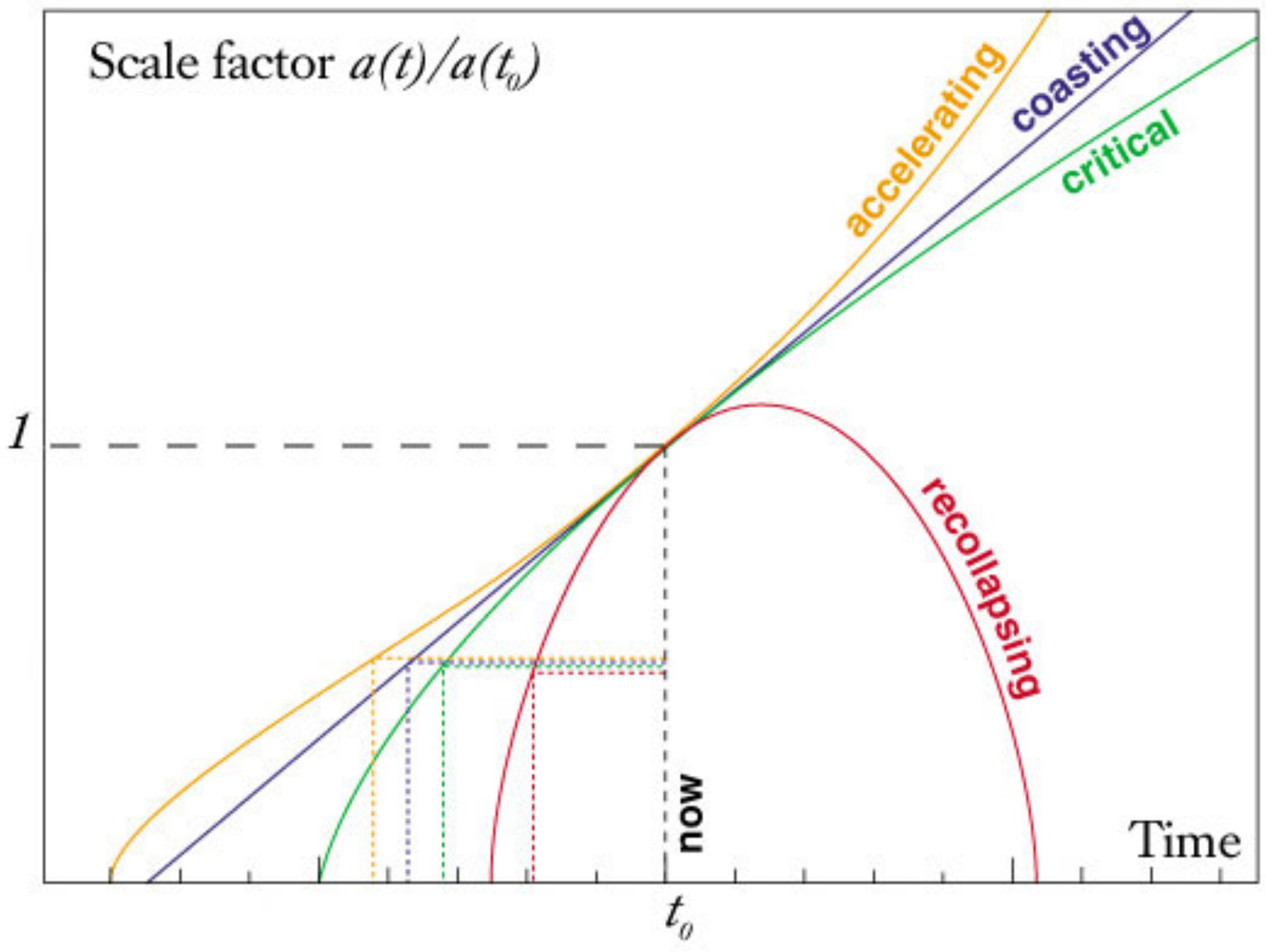
$$\rho\left[a(t)\right] = \rho_{\text{ini}} \exp\left\{-3\int\left[1+w(a)\right]\mathrm{d}\ln a\right\} \underset{w\to\text{cst}}{=} \rho_{\text{ini}}\left(\frac{a}{a_{\text{ini}}}\right)^{-3(1+w)}$$

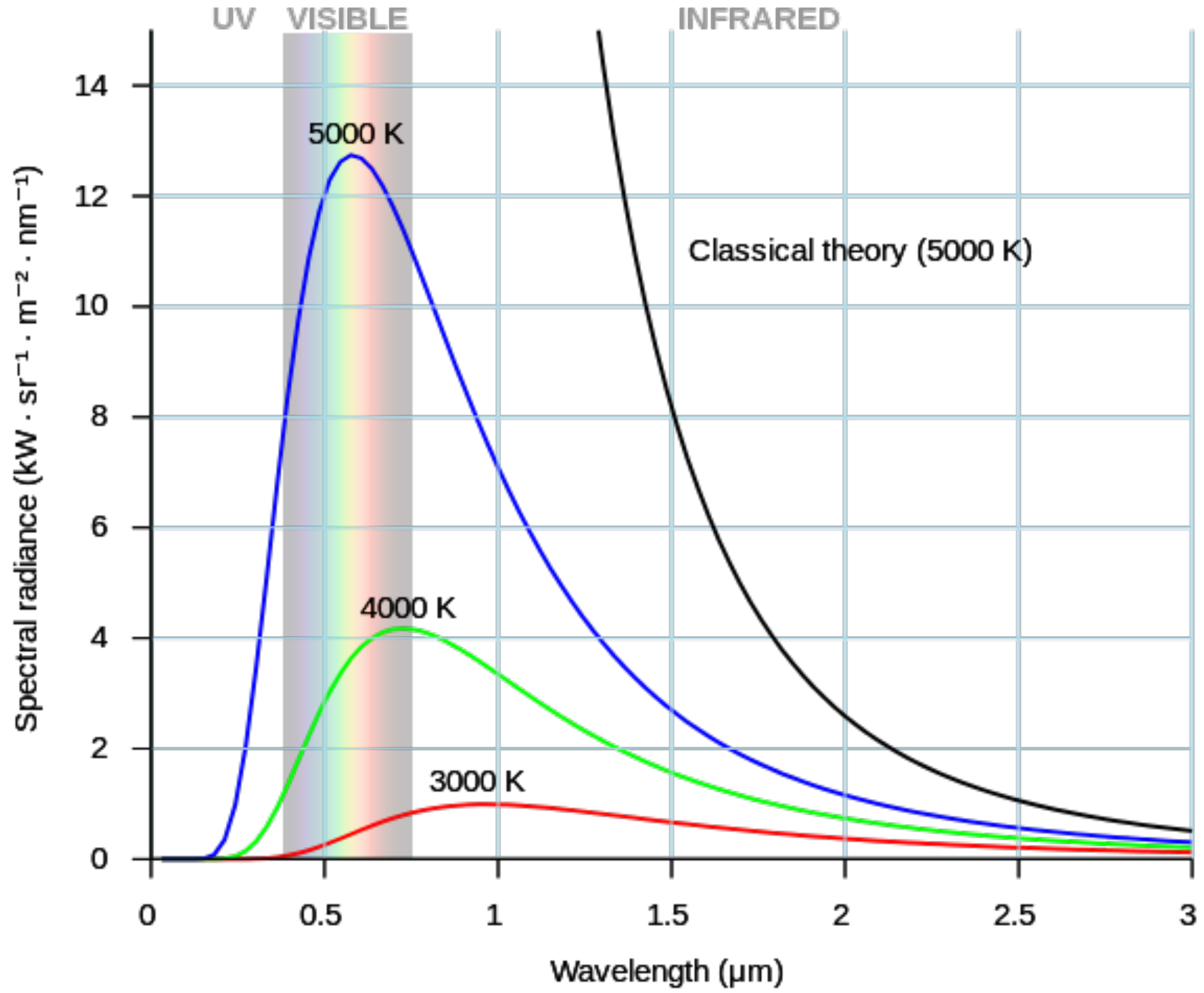
## Particular solution: dust and radiation

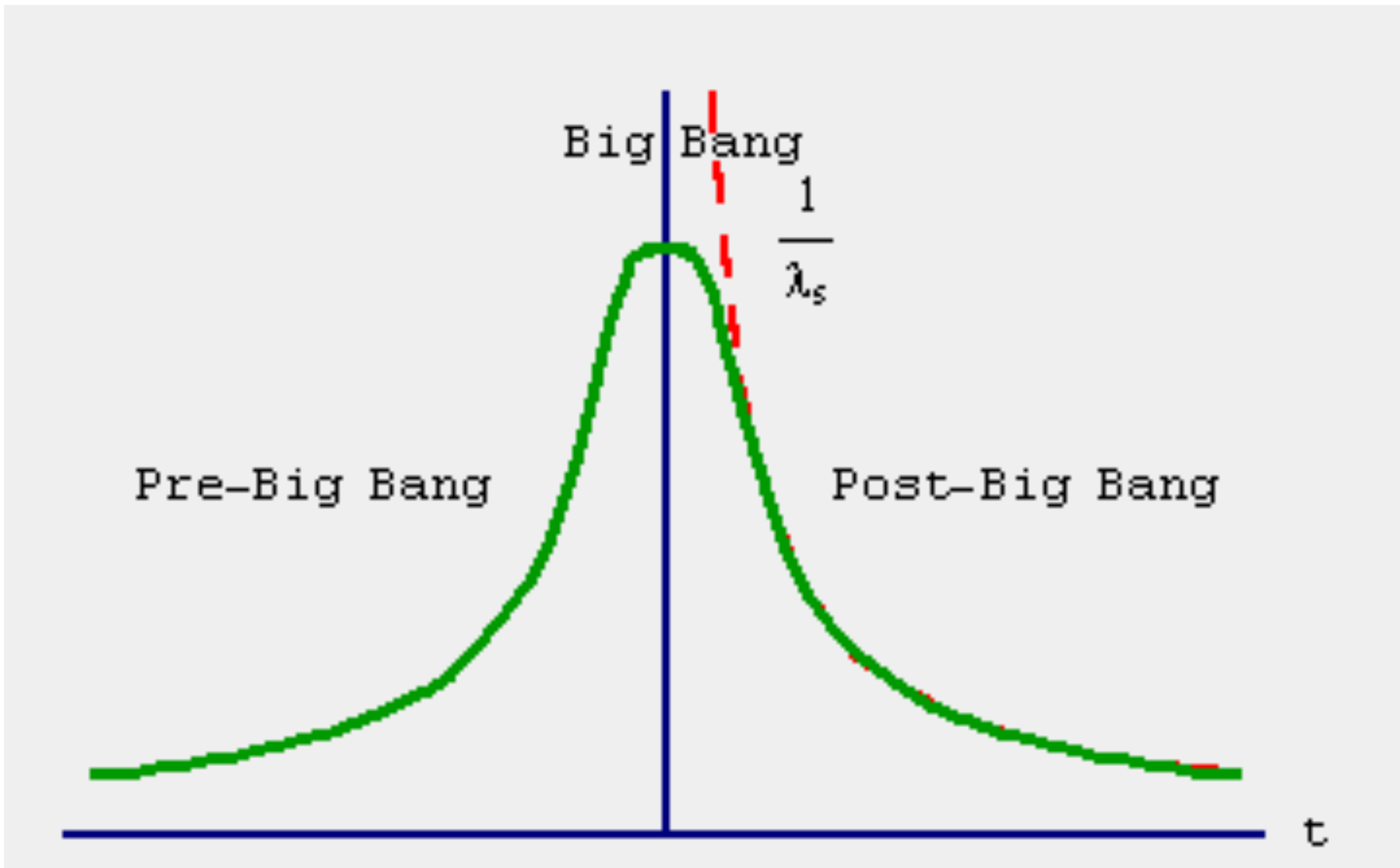



*Phenomenologically valid description for 14 Gyrs!!!!*

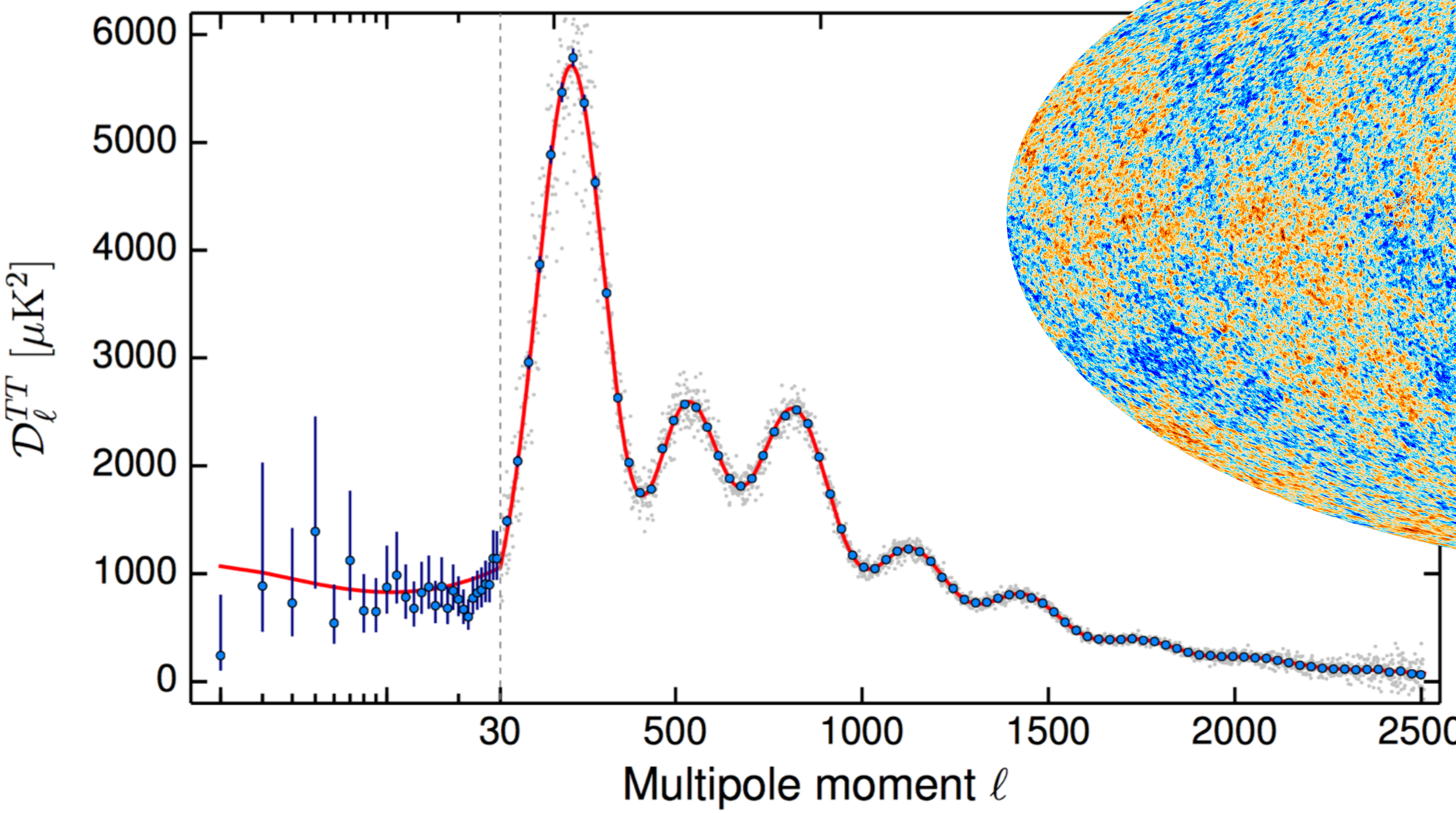
# Singularity problem... a quantum effect?

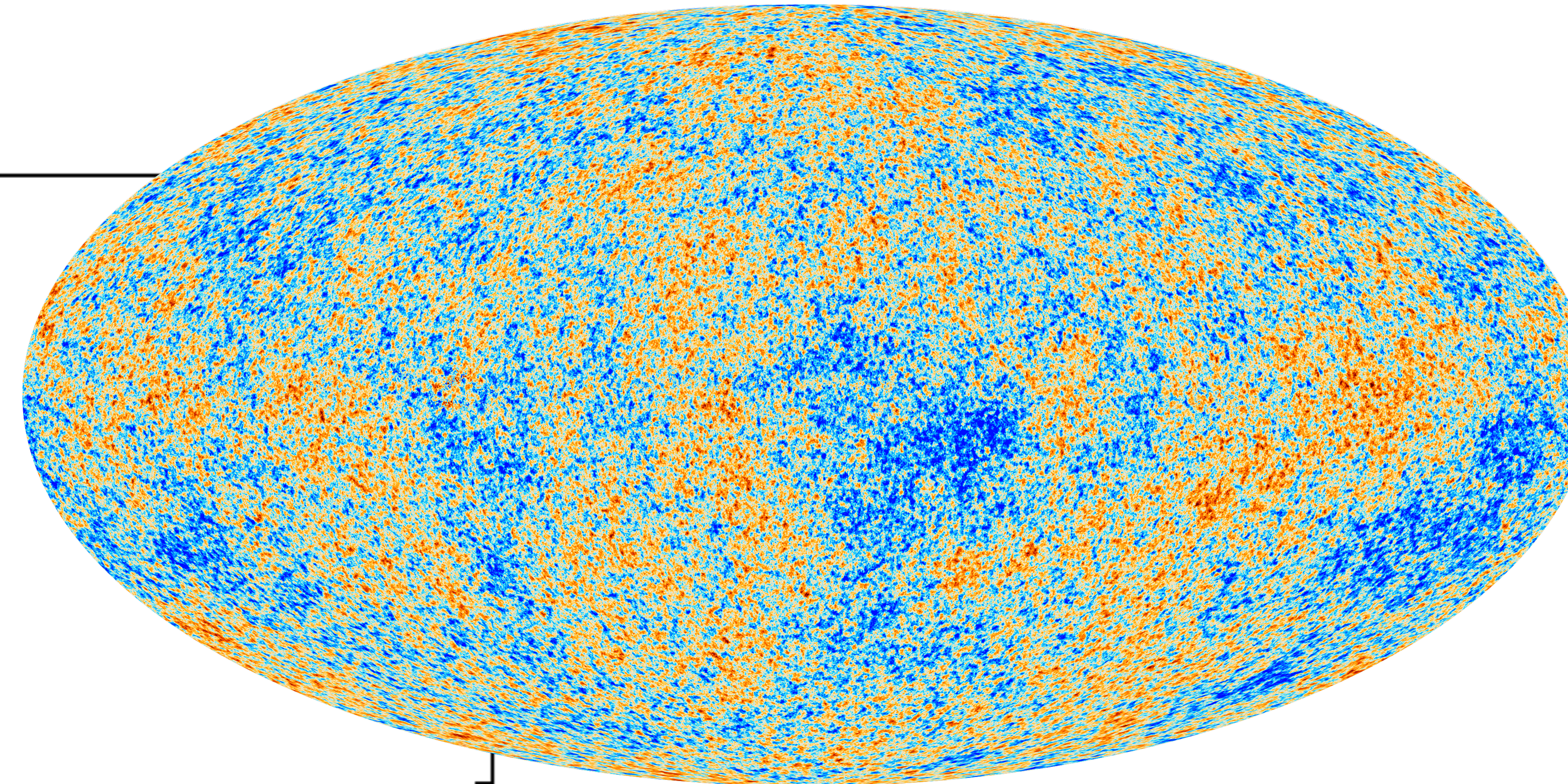


## ***Planck***



$$\Omega_{\mathcal{K}} = 0.000 \pm 0.005$$

$n_{\rm s} = 0.9639 \pm 0.0047$ almost scale invariant

excluded

quantum vacuum fluctuations of a single scalar d.o.f

$f_{\rm NL}^{\rm loc} = 0.8 \pm 5$

$f_{\rm NL}^{\rm eq} = -4 \pm 43$

$f_{\rm NL}^{\rm ort} = -26 \pm 21$

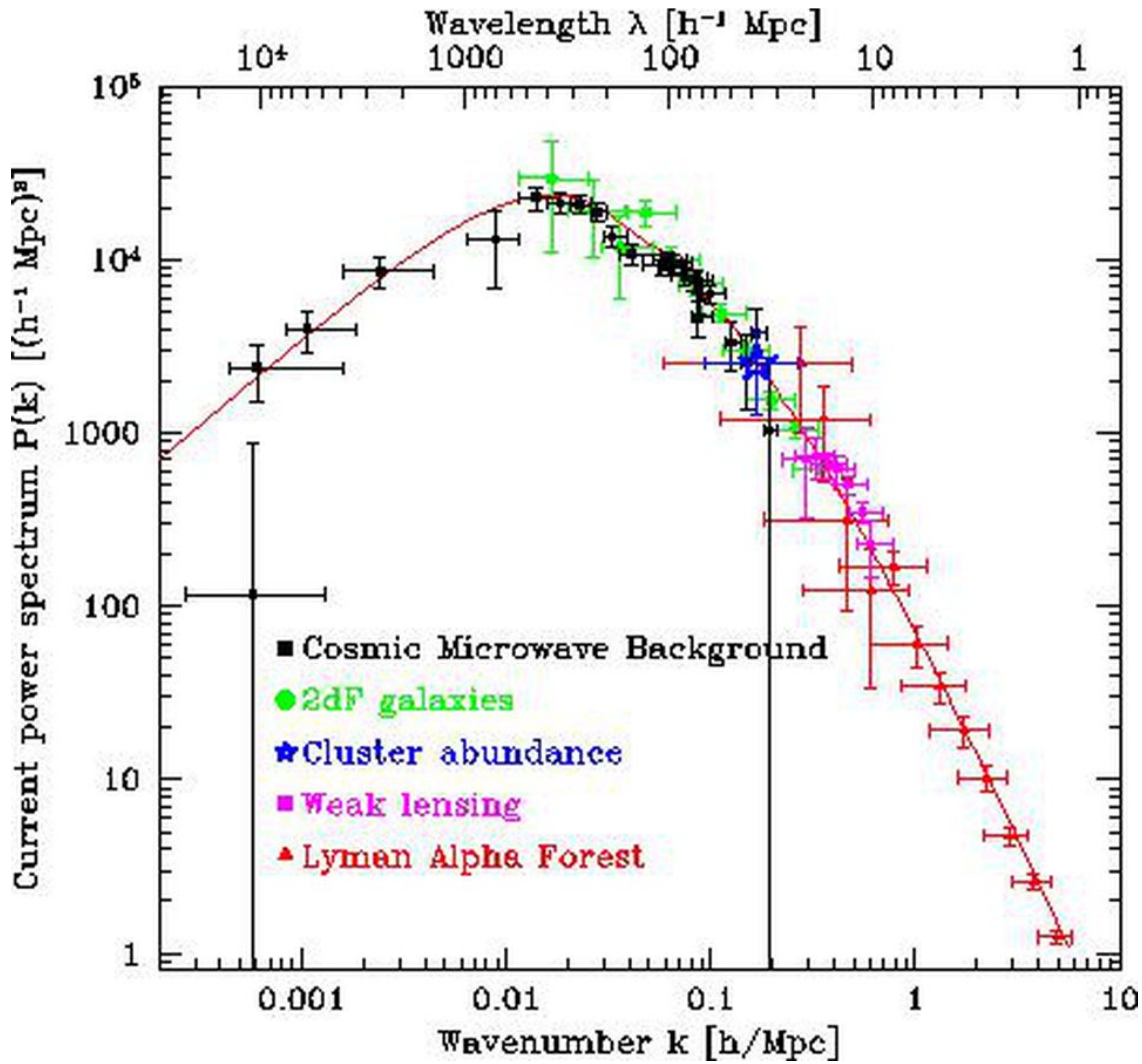
gaussian signal

compatible with ***INFLATION***

isocurvature $\lesssim 1\%$

$r < 0.11$

# Numerical simulation for large scale structure formation...



## ***Quantum mechanics***

Physical system = Hilbert space of configurations
state vector
observables = self-adjoint operators
measurement = eigenvalue $\hat{O}|n\rangle = \omega_n|n\rangle$

Evolution = Schrödinger equation (time translation invariance) $i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

***Born rule...*** $\mathrm{Prob}\left[\omega_n;t\right] = \left|\langle n|\psi(t)\rangle\right|^2$

Collapse of the wave function: $|\psi(t)\rangle$ before measurement, $|n\rangle$ after measurement.

Schrödinger equation = linear (superposition principle) / unitary evolution
Wavepacket reduction = nonlinear / stochastic

} mutually incompatible

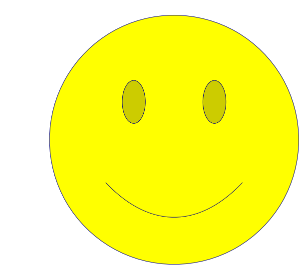
+ external observer...!

# ***Predictions for a quantum theory***

quantum average of observable: $\langle \Psi | \hat{\mathcal{O}} | \Psi \rangle$

laboratory: repeat experiments

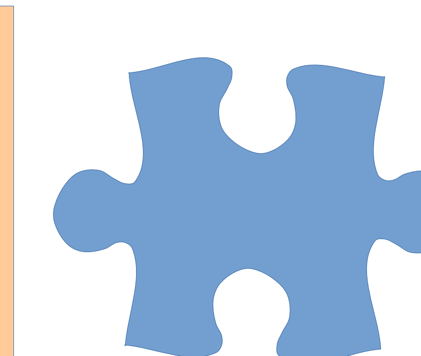
ensemble average ⟷ quantum average



cosmology: a single experiment

→ ergodicity

spatial average (directions in sky) ⟷ quantum average



## ***Quantum cosmology***

Hamiltonian GR (3+1)

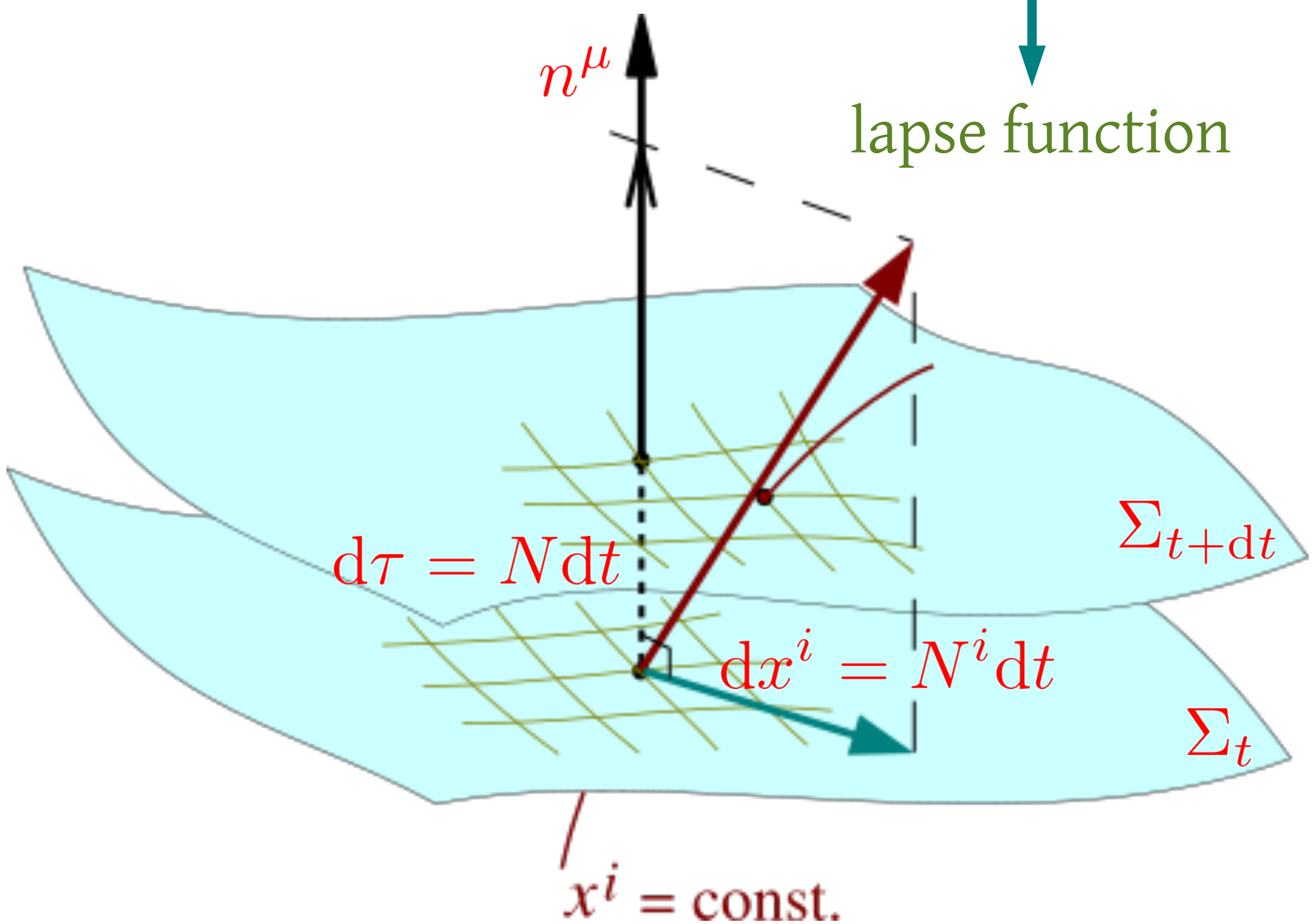
$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2\mathrm{d}t^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right)$$

lapse function

intrinsic metric =
first fundamental form

shift vector

intrinsic curvature tensor
$^3R^i{}_{jkl}(h)$




extrinsic curvature =
second fundamental form:

$$K_{ij} = -\nabla_j^{(h)} n_i = \frac{1}{2N}\left(\nabla_j^{(h)} N_i + \nabla_i^{(h)} N_j - \frac{\partial h_{ij}}{\partial t}\right)$$

Action (Einstein-Hilbert, compact space):

$$\mathcal{S} = \frac{1}{16\pi G_{\rm N}} \left[ \int_{\mathcal{M}} \sqrt{-g}\ (R - 2\Lambda)\, \mathrm{d}^4x + 2 \int_{\partial\mathcal{M}} \sqrt{h} K^i{}_i \mathrm{d}^3x \right] + \mathcal{S}_{\rm matter}\left[\Phi\left(x\right)\right]$$

$$\longrightarrow \mathcal{S} = \int L \mathrm{d}t = \frac{1}{16\pi G_{\rm N}} \int \mathrm{d}t \left[ \int \mathrm{d}^3x\, N\sqrt{h} \left(K_{ij}K^{ij} - K^2 + {}^3R - 2\Lambda\right) + L_{\rm matter} \right]$$

Canonical momenta

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{h}}{16\pi G_{\rm N}} \left(K^{ij} - h^{ij} K\right)$$

$$\pi^{\Phi} \equiv \frac{\delta L}{\delta \dot{\Phi}} = -\frac{\sqrt{h}}{N} \left(\dot{\Phi} - N \frac{\partial \Phi}{\partial x^i}\right)$$

$$\pi^0 \equiv \frac{\delta L}{\delta \dot{N}} = 0$$

$$\pi^i \equiv \frac{\delta L}{\delta \dot{N}^i} = 0$$

} primary constraints

Hamiltonian $$H \equiv \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + \pi^{ij} \dot{h}_{ij} + \pi^\Phi \dot{\Phi}\right) - L$$

$$= \int \mathrm{d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + N\mathcal{H} + N_i \mathcal{H}^i\right)$$

variation wrt lapse: $\mathcal{H} = 0 \rightarrow$ Hamiltonian constraint

variation wrt shift: $\mathcal{H}^i = 0 \rightarrow$ momentum constraint

$\Longrightarrow$ classical description complete

variation wrt 3D metric $h_{ij}$: dynamical equations

Superspace & canonical quantization

relevant configuration space $\text{Riem}\,(\Sigma) \equiv \left\{ h_{ij}\,(\boldsymbol{x})\,, \Phi\,(\boldsymbol{x})\,|\boldsymbol{x} \in \Sigma \right\}$

↳ parameters

GR $\Longrightarrow$ invariance/diffeomorphisms $\Longrightarrow$ $\text{Conf} = \dfrac{\text{Riem}\,(\Sigma)}{\text{Diff}\,(\Sigma)}$ : superspace

Wave functional $\Psi\left[h_{ij}\,(\boldsymbol{x})\,, \Phi\,(\boldsymbol{x})\right]$ $\quad (\equiv \text{“}\langle h_{ij}, \Phi|\Psi\rangle\text{”})$

+ Dirac canonical quantization procedure

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}} \qquad \pi^{\Phi} \to -i\frac{\delta}{\delta \Phi} \qquad \pi^{0} \to -i\frac{\delta}{\delta N} \qquad \pi^{i} \to -i\frac{\delta}{\delta N_i}$$

primary constraints
$$\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\rm N}\hat{T}^{0i}\Psi$

same $\Psi$ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = \left[-16\pi G_{\rm N}\mathcal{G}_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} + \frac{\sqrt{h}}{16\pi G_{\rm N}}\left(-{}^3R + 2\Lambda + 16\pi G_{\rm N}\hat{T}^{00}\right)\right]\Psi = 0$$

***Wheeler - DeWitt equation***

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$$

De Witt metric

primary constraints
$$\begin{cases} \hat{\pi}^0 = -i\dfrac{\delta\Psi}{\delta N} = 0 \\ \hat{\pi}^i = -i\dfrac{\delta\Psi}{\delta N_i} = 0 \end{cases}$$

momentum $\hat{\mathcal{H}}^i\Psi = 0 \quad \Longrightarrow \quad i\nabla_j^{(h)}\left(\dfrac{\delta\Psi}{\delta h_{ij}}\right) = 8\pi G_{\rm N}\hat{T}^{0i}\Psi$

same $\Psi$ for configurations related by a coordinate transformation

Hamiltonian

$$\hat{\mathcal{H}}\Psi = 0$$

***TIMELESS Schrödinger equation***

## mini-superspace

*restrict attention from an infinite dimensional configuration space to a 2 dimensional space = mini-superspace*

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j \quad \mapsto \quad a^2(t)\left[\frac{\mathrm{d}r^2}{1-\mathcal{K}r^2}+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)\right]$$

WDW equation becomes Schrödinger like for $\Psi\left[a(t),\phi(t)\right]$

## Conceptual & technical issues

infinite # d.o.f. to a few: mathematical consistency?

freeze momenta... Heisenberg uncertainties?

A simple example: perfect fluid matter

$$\mathrm{d}s^2 = -N^2(\tau)\mathrm{d}\tau + a^2(\tau)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

perfect fluid formalism: B. Schutz (1970)

→ velocity potentials + canonical transformations

$$H = N\left(-\frac{p_a^2}{4a} - \mathcal{K}a + \frac{p_\tau}{a^{3w}}\right)$$

→ defines time...

choose wisely... $a^{3w}$

WDW $H\Psi = 0$ & $\mathcal{K} \to 0$

$$i\frac{\partial\Psi}{\partial\tau} = \frac{1}{4}\frac{\partial^2\Psi}{\partial x^2}$$

*free particle Schrödinger!*

$$\left[x = \frac{2a^{3(1-w)/2}}{3(1-w)}\right]$$

$$i\frac{\partial\Psi}{\partial\tau} = \frac{1}{4}\frac{\partial^2\Psi}{\partial x^2}$$

$\Longleftrightarrow$

$$\Psi = \sqrt[4]{\frac{8\tau_0}{\pi\left(\tau_0^2+\tau^2\right)^2}}\exp\left(-\frac{\tau_0 x^2}{\tau_0^2+\tau^2}\right)\mathrm{e}^{-iS(x,\tau)}$$

*phase* $$S(x,\tau) = \frac{\tau x^2}{\tau_0^2+\tau^2} + \frac{1}{2}\arctan\left(\frac{\tau_0}{\tau}\right) - \frac{\pi}{4}$$

Gaussian wave packet

# Basic model: GR + perfect fluid

$$\mathcal{S}_{\text{EHS}} = \underbrace{\frac{1}{16\pi G_{\text{N}}}\int \mathrm{d}^4x\sqrt{-g}R}_{\mathcal{S}_{\text{EH}}} + \underbrace{\int \mathrm{d}^4x\sqrt{-g}P(w,\phi)}_{\mathcal{S}_{\text{M}}}$$

FLRW metric

$$\mathrm{d}s^2 = -N^2(\tau)\mathrm{d}\tau^2 + a^2(\tau)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$-\frac{1}{2\kappa}\int \mathrm{d}\tau N a^3 \underbrace{\int \sqrt{\gamma}\mathrm{d}^3x}_{\mathcal{V}_0}\underbrace{\frac{6\dot{a}^2}{a^2N^2}}_{R}$$

$$P = w\rho$$

$$N = (1+w)a^{3w} \implies H_{\text{M}}^{(0)} = p_\tau$$

Use as clock

$$q = \frac{4\sqrt{6}}{3(1-w)\sqrt{1+w}}a^{\frac{3}{2}(1-w)} \equiv \gamma a^{\frac{3}{2}(1-w)}$$

$$p = \frac{\sqrt{6(1+w)}}{2\kappa_0}a^{\frac{3}{2}(1+w)}\frac{\dot{a}}{Na}$$

$\frac{\dot{a}}{Na} \rightarrow H$ Hubble rate

New canonical variables $\quad H^{(0)} = 2\kappa_0 p^2$, $\quad \kappa_0 \rightarrow \kappa/\mathcal{V}_0$

Hamilton equations $\frac{\mathrm{d}q}{\mathrm{d}\tau} = 4\kappa_0 p$ and $\frac{\mathrm{d}p}{\mathrm{d}\tau} = 0$

singular solutions

$$q(\tau) = \sqrt{8\kappa_0 H^{(0)}}(\tau - \tau_{\mathrm{s}}) \to q_{\mathrm{B}}\omega\tau \qquad p(\tau) \to \sqrt{\frac{H^{(0)}}{2\kappa_0}} = \frac{q_{\mathrm{B}}\omega}{4\kappa_0}$$

***Perturbations:*** $H_{\mathrm{full}} = H^{(0)} + \sum_{\boldsymbol{k}} H^{(2)}_{\boldsymbol{k}}$

Fluid perturbation (intrinsic + curvature) $\phi_{\boldsymbol{k}} = \frac{p^{\frac{1-w}{1+w}}\delta\phi_{\boldsymbol{k}}}{\sqrt{2w(1+w)\kappa_0}} + \sqrt{\frac{3}{w\kappa_0}}\frac{a^{-\frac{3w-7}{2}}}{4k^2}\delta R_{\boldsymbol{k}}$

P. Małkiewicz, *CQG* **36**, 215003 (2019)

$$\Longrightarrow H^{(2)}_{\boldsymbol{k}} = \frac{1}{2}|\pi_{\phi,\boldsymbol{k}}|^2 + \frac{1}{2}w(1+w)^2\left(\frac{q}{\gamma}\right)^{\frac{4(3w-1)}{3(1-w)}} k^2|\phi_{\boldsymbol{k}}|^2$$

Hamilton equations

$$\phi''_{\boldsymbol{k}} + \left(\frac{q}{\gamma}\right)^{\frac{4(3w-1)}{3(1-w)}} w(1+w)^2 k^2 \phi_{\boldsymbol{k}} = 0$$

Fluid (F-)parameterization: $(\phi_{\boldsymbol{k}}, \pi_{\phi,\boldsymbol{k}})$

Let's begin the fun… changing the variables! $Z(\tau) = \sqrt{1+w}\left(\frac{q}{\gamma}\right)^{\frac{3w-1}{3(1-w)}}$

$v_{\boldsymbol{k}} = Z\phi_{\boldsymbol{k}}$ and $\pi_{v,\boldsymbol{k}} = Z^{-1}\pi_{\phi,\boldsymbol{k}} + \frac{\dot{Z}}{Z^2}\phi_{\boldsymbol{k}}$ Conformal (C-)parameterization: $(v_{\boldsymbol{k}}, \pi_{v,\boldsymbol{k}})$

$$H_{\boldsymbol{k}}^{(2)} = \frac{1}{2}Z^2\left\{|\pi_{v,\boldsymbol{k}}|^2 + \left[wk^2 - \mathcal{V}_{\rm cl}(\tau)\right]|v_{\boldsymbol{k}}|^2\right\}$$

$$\mathcal{V}_{\rm cl}(\tau) \to \frac{1}{Z^4}\left[\frac{\ddot{Z}}{Z} - 2\left(\frac{\dot{Z}}{Z}\right)^2\right]$$

Explicitly…

$$H_{\boldsymbol{k}}^{(2)} = \frac{1}{2}(1+w)\left(\frac{q}{\gamma}\right)^{\frac{2(3w-1)}{3(1-w)}}\left\{|\pi_{v,\boldsymbol{k}}|^2 + \left[wk^2 - \frac{8}{9q^2}\left(\frac{q}{\gamma}\right)^{\frac{4(1-3w)}{3(1-w)}}\frac{(2\kappa_0)^2(1-3w)p^2}{(1+w)^2(1-w)^2}\right]|v_{\boldsymbol{k}}|^2\right\}$$

upon using the background e.o.m.

Switch to conformal time $\mathrm{d}\eta = Z^2 \mathrm{d}\tau = (1+w)\left(\frac{q}{\gamma}\right)^{\frac{2(3w-1)}{3(1-w)}} \mathrm{d}\tau$ yields $\mathcal{V}_{\rm cl} = \frac{Z''}{Z}$

$$v_{\boldsymbol{k}}'' + \left(wk^2 - \frac{z''}{z}\right) v_{\boldsymbol{k}} = 0$$

$$z = q^r$$

$$r_1 = \frac{3w-1}{3(1-w)} \qquad r_2 = \frac{2}{3(1-w)}$$

$$\eta = \frac{1+w}{2r_1+1}\left(\frac{q_{\rm B}\omega}{\gamma}\right)^{2r_1} \tau^{2r_1+1} \longrightarrow q(\eta) = q_{\rm B}\omega \left[\frac{2r_1+1}{1+w}\left(\frac{q_{\rm B}\omega}{\gamma}\right)^{-2r_1} \eta\right]^{1/(2r_1+1)} \propto \eta^{\frac{3(1-w)}{1+3w}}$$

$$\mathcal{V}_{\rm cl} = \frac{(q^{r_1})''}{q^{r_1}} = \frac{(q^{r_2})''}{q^{r_2}} = \frac{2(1-3w)}{(1+3w)^2\eta^2}$$

Solutions = Hankel functions, divergent perturbations at the singularity

# Coherent state quantization

Background phase space $(q,p) \in \mathbb{R}^{*+} \times \mathbb{R} = \{(q,p) | q > 0, p \in \mathbb{R}\}$

Natural choice = 2 parameter affine group of the real line $t \in \mathbb{R} \mapsto (q,p) \cdot t = \frac{t}{q} + p$

$$\{(q_0, p_0), (q,p)\} \mapsto (q', p') = (q_0, p_0) \circ (q,p) = \left( q' q, \frac{p}{q'} + p' \right)$$

left-invariant measure $\mathrm{d}q' \wedge \mathrm{d}p' = \mathrm{d}q \wedge \mathrm{d}p$

# Coherent state quantization

Background phase space $(q,p) \in \mathbb{R}^{*+} \times \mathbb{R}$

Natural choice = 2 parameter affine group of the real line $t \in \mathbb{R} \;\mapsto (q,p)\cdot t = \frac{t}{q} + p$

unitary, irreducible and square-integrable representation in the Hilbert space $\mathcal{H} = L^2\left(\mathbb{R}^{*+}, \frac{\mathrm{d}x}{x}\right)$

$$\langle x|U(q,p)|\psi\rangle = \langle x|q,p\rangle = \frac{\mathrm{e}^{ipx/\hbar}}{\sqrt{q}}\psi\left(\frac{x}{q}\right)$$

$$\psi(x) = \langle x|\psi\rangle$$

covariant integral quantization on (affine) coherent states

$$\mathbb{R}^{+} \times \mathbb{R} \ni (q,p) \mapsto |q,p\rangle := U(q,p)|\xi\rangle \in \mathcal{H}$$

normalized fiducial state

$$U(q,p) = \mathrm{e}^{ip\widehat{Q}/\hbar}\mathrm{e}^{-i\ln q\widehat{D}/\hbar}$$

dilation $\widehat{D} := \frac{1}{2}(\widehat{Q}\widehat{P} + \widehat{P}\widehat{Q})$

algebra

$$\left[\widehat{Q}, \widehat{D}\right] = i\hbar\widehat{Q}$$

define $\rho(\alpha) = \int \frac{\langle \xi | x \rangle \langle x | \xi \rangle}{x^{\alpha+1}} \, \mathrm{d}x = \int \frac{|\xi(x)|^2}{x^{\alpha+1}} \, \mathrm{d}x < \infty$

and $\sigma(\alpha) = \int \left| \frac{\mathrm{d}\xi(x)}{\mathrm{d}x} \right|^2 \frac{\mathrm{d}x}{x^{\alpha+1}}$

$$\int \frac{\mathrm{d}q \mathrm{d}p}{2\pi\hbar\rho_0} |q,p\rangle\langle q,p| = \mathbb{1}$$ (resolution of unity)

affine coherent state quantization: $$f(q,p) \mapsto A_f := \int_{\mathbb{R}^+ \times \mathbb{R}} \frac{\mathrm{d}q \mathrm{d}p}{2\pi\hbar\rho_0} |q,p\rangle \, f(q,p) \, \langle q,p|$$

$\longrightarrow \quad A_1 = \mathbb{1}$

$\longrightarrow \quad A_{q^\alpha} = \mathfrak{l}(\alpha) \widehat{Q}^\alpha$

impose $\mathfrak{l}(1) = 1$ to ensure $[A_q, A_p] = i\hbar$

$\longrightarrow \quad A_p = \widehat{P}$

$\longrightarrow \quad A_{q^\alpha p^2} = \mathfrak{a}(\alpha) \widehat{Q}^\alpha \widehat{P}^2 - i\alpha\hbar\mathfrak{a}(\alpha) \widehat{Q}^{\alpha-1} \widehat{P} + \mathfrak{c}(\alpha)\hbar^2 \widehat{Q}^{\alpha-2} = \mathfrak{a}(\alpha) \widehat{P} \widehat{Q}^\alpha \widehat{P} + \hbar^2 \mathfrak{c}(\alpha) \widehat{Q}^{\alpha-2}$

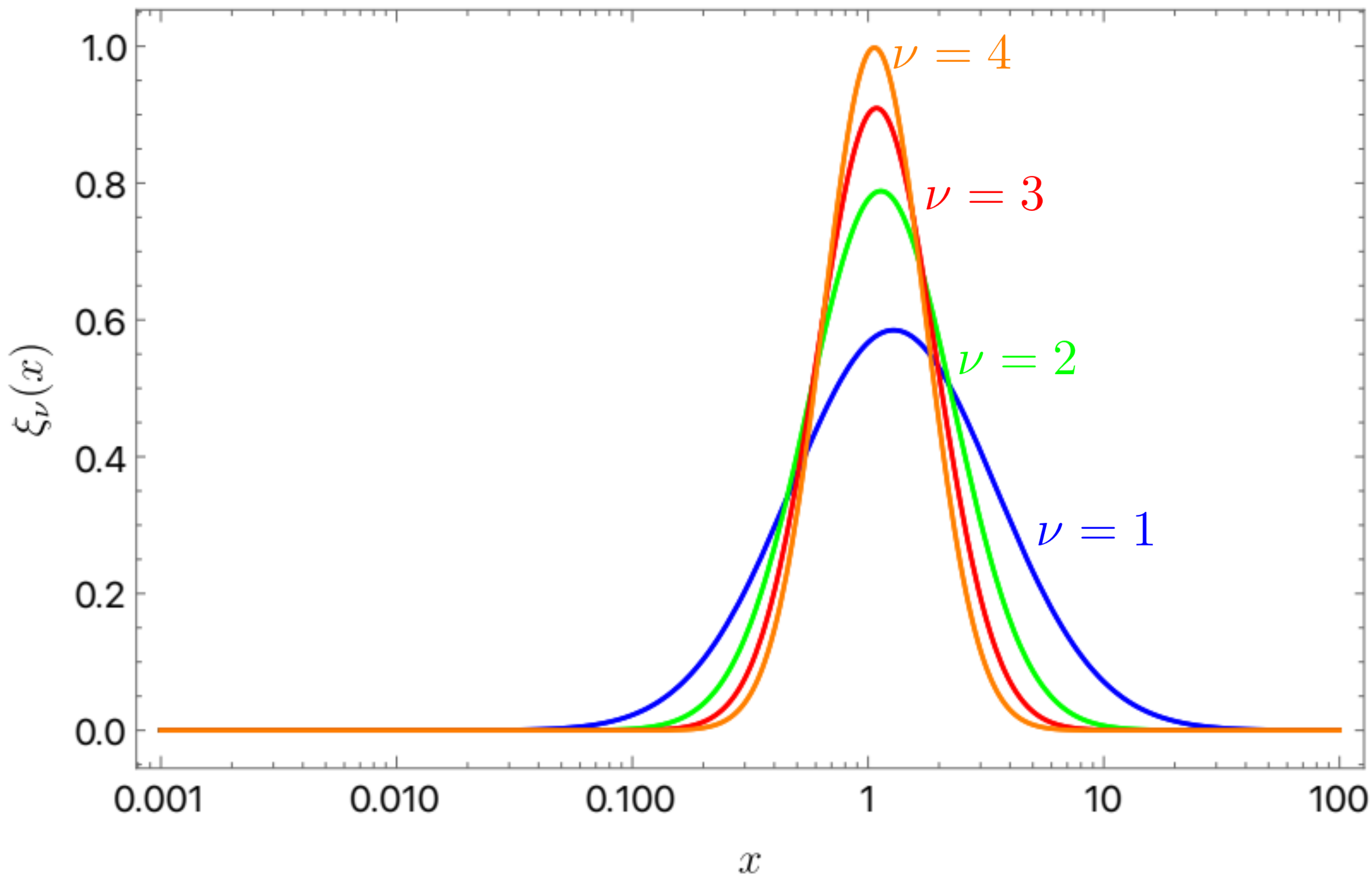
$$\mathfrak{l}(\alpha) = \mathfrak{a}(\alpha) = \frac{\rho(\alpha)}{\rho_0}$$

$$\mathfrak{c}(\alpha) = \frac{1}{2}\alpha(1-\alpha)\mathfrak{a}(\alpha) + \frac{\sigma(\alpha-2)}{\rho_0}$$

Example of fiducial vector family $\xi_\nu(x) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{x}} \exp\left[-\frac{\nu}{2}\left(\ln x - \frac{3}{4\nu}\right)^2\right]$

$$\sigma_\nu(\alpha) = \left[\frac{\nu}{2} + \left(\frac{\alpha+2}{2}\right)^2\right] \exp\left[\frac{\alpha(\alpha+3)}{4\nu}\right]$$

$$\rho_\nu(\alpha) = \exp\left[\frac{(\alpha-2)(\alpha+1)}{4\nu}\right]$$




$$\rho_\nu(1) = \rho_\nu(0) = \mathrm{e}^{-1/(2\nu)}$$

$$\mathfrak{l}(\alpha) = \frac{\rho(\alpha)}{\rho_0} \Longrightarrow \mathfrak{l}(1) = 1 \quad \checkmark$$

## Quantum Background Hamiltonian

$$H^{(0)} \mapsto \widehat{H}^{(0)} = 2\kappa_0 \left(\widehat{P}^2 + \hbar^2 \mathfrak{c}_0 \widehat{Q}^{-2}\right)$$

$$\mathfrak{c}_0 = \mathfrak{c}(0) = \frac{\sigma(-1)}{\rho_0} \rightarrow \frac{1}{2}\left(\nu + \frac{1}{2}\right)$$

$$\mathfrak{c}_0 \geqslant \frac{3}{4} \Leftrightarrow \nu \geqslant 1$$

$\widehat{H}^{(0)}$ essentially self-adjoint

$\mathfrak{c}_0 \rightarrow 0$ canonical quantization

plug back in perturbation

$$H_{\boldsymbol{k}}^{(2)} = \frac{1}{2}|\pi_{\phi,\boldsymbol{k}}|^2 + \frac{1}{2}w(1+w)^2\left(\frac{q}{\gamma}\right)^{4r_1} k^2|\phi_{\boldsymbol{k}}|^2$$

### ⊕ **Canonical quantization of perturbations**

F-***parameterization quantization***

$$\phi_{\boldsymbol{k}} \mapsto \widehat{\phi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}\phi_k^*(\tau) + a_{-\boldsymbol{k}}^{\dagger}\phi_k(\tau)\right]$$

C-***parameterization quantization***

$$v_{\boldsymbol{k}} \mapsto \widehat{v}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}v_k^*(\tau) + a_{-\boldsymbol{k}}^{\dagger}v_k(\tau)\right]$$

F-*parameterization quantization*

$$\phi_{\boldsymbol{k}} \mapsto \widehat{\phi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}\phi_k^*(\tau) + a_{-\boldsymbol{k}}^\dagger \phi_k(\tau)\right]$$

One free parameter

$$\mathfrak{L}_{\mathrm{Q}} = \mathfrak{l}(4r_1)$$

$$\widehat{H}_{\boldsymbol{k}}^{(2)} = \frac{1}{2}|\widehat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\mathrm{Q}}}{2} w(1+w)^2 \left(\frac{\widehat{Q}}{\gamma}\right)^{4r_1} k^2 |\widehat{\phi}_{\boldsymbol{k}}|^2$$

C-*parameterization quantization*

$$v_{\boldsymbol{k}} \mapsto \widehat{v}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}} v_k^*(\tau) + a_{-\boldsymbol{k}}^\dagger v_k(\tau)\right]$$

$$\widehat{H}_{\boldsymbol{k}}^{(2)} = \frac{1}{2}(1+w)\left(\frac{\widehat{Q}}{\gamma}\right)^{2r_1} \mathfrak{M}_{\mathrm{Q}} \left\{ |\widehat{\pi}_{v,\boldsymbol{k}}|^2 + \left[ wk^2 - \frac{8\mathfrak{M}_{\mathrm{Q}}^{-1}}{9\widehat{Q}^2} \frac{(2\kappa_0)^2(1-3w)}{(1-w)^2(1+w)^2} \left(\frac{\widehat{Q}}{\gamma}\right)^{-4r_1} \left( \mathfrak{N}_{\mathrm{Q}} \widehat{P}^2 + i\hbar \mathfrak{R}_{\mathrm{Q}} \widehat{Q}^{-1}\widehat{P} + \hbar^2 \mathfrak{T}_{\mathrm{Q}} \widehat{Q}^{-2} \right) \right] |\widehat{v}_{\boldsymbol{k}}|^2 \right\}$$

$$\mathfrak{M}_{\mathrm{Q}} = \mathfrak{l}(2r_1)$$

$$\mathfrak{N}_{\mathrm{Q}} = \mathfrak{a}(-2r_2)$$

$$\mathfrak{T}_{\mathrm{Q}} = \mathfrak{c}(-2r_2)$$

$$\mathfrak{R}_{\mathrm{Q}} = \mathfrak{b}(-2r_2) = 2r_2 \mathfrak{N}_{\mathrm{Q}}$$

Three free parameters

# Coherent state semiclassical approximation

new fiducial state $|\tilde{\xi}\rangle$ satisfying $\langle\tilde{\xi}|\widehat{Q}|\tilde{\xi}\rangle = 1$ (physical centering condition)

time-dependent functions $q(\tau)$ and $p(\tau)$

define time-dependent coherent states $|q(\tau), p(\tau)\rangle = \mathrm{e}^{ip(\tau)\widehat{Q}/\hbar}\mathrm{e}^{-i\ln q(\tau)\widehat{D}/\hbar}|\tilde{\xi}\rangle$

$$\left.\begin{aligned}\langle\widehat{Q}\rangle &= q(\tau)\\ \langle\widehat{P}\rangle &= p(\tau)\end{aligned}\right\}$$

$$\mathcal{S}_{\mathrm{Sch}} = \int \langle q(\tau), p(\tau)| \left(i\hbar\frac{\partial}{\partial\tau} - \widehat{H}\right)|q(\tau), p(\tau)\rangle \mathrm{d}\tau = \int \left\{\dot{q}(\tau)p(\tau) - H_{\mathrm{sem}}\left[q(\tau), p(\tau)\right]\right\}\mathrm{d}\tau$$

$H_{\mathrm{sem}}$: $\langle q(\tau), p(\tau)|\widehat{H}|q(\tau), p(\tau)\rangle$

$$\left\{\begin{aligned}\dot{q} &= \frac{\partial H_{\mathrm{sem}}}{\partial p}\\ \dot{p} &= -\frac{\partial H_{\mathrm{sem}}}{\partial q}\end{aligned}\right.$$

define $\tilde{\rho}(\alpha) = \int \frac{\langle \tilde{\xi}|x\rangle\langle x|\tilde{\xi}\rangle}{x^{\alpha+1}}\,\mathrm{d}x = \int \frac{|\tilde{\xi}(x)|^2}{x^{\alpha+1}}\,\mathrm{d}x < \infty$

and $\tilde{\sigma}(\alpha) = \int \left|\frac{\mathrm{d}\tilde{\xi}(x)}{\mathrm{d}x}\right|^2 \frac{\mathrm{d}x}{x^{\alpha+1}}$

expectation values of operators

$\longrightarrow \langle q,p|\widehat{Q}^\alpha|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha$

$\longrightarrow \langle q,p|\widehat{Q}^\alpha\widehat{P}|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha p + i\frac{\alpha}{2}\tilde{\rho}(-\alpha)q^{\alpha-1}$

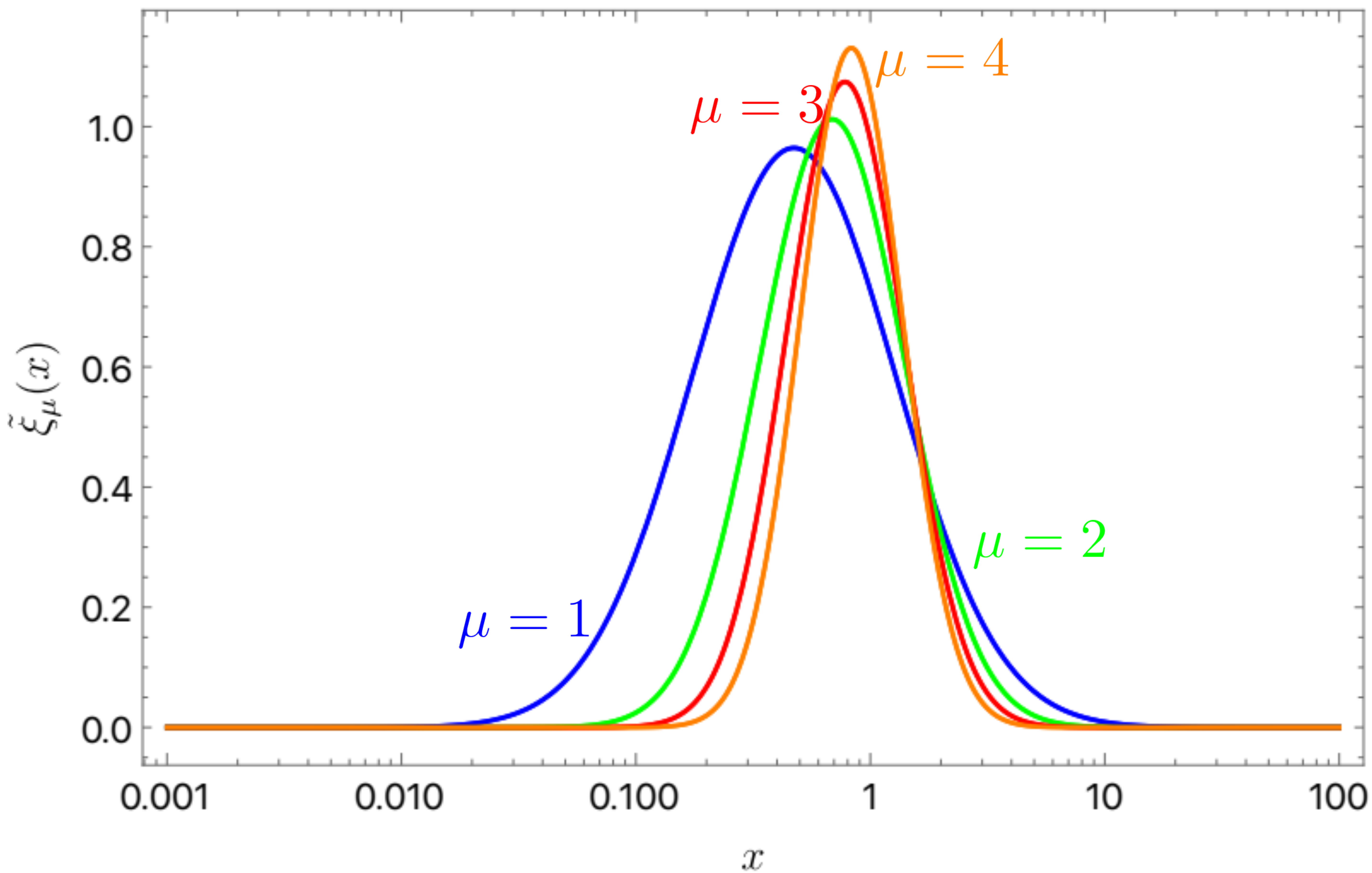
$\longrightarrow \langle q,p|\widehat{Q}^\alpha\widehat{P}^2|q,p\rangle = \tilde{\rho}(-\alpha-1)q^\alpha p^2 + i\alpha\tilde{\rho}(-\alpha)q^{\alpha-1}p + \left[\tilde{\sigma}(-\alpha-1) + \frac{\alpha(1-\alpha)}{2}\tilde{\rho}(-\alpha+1)\right]q^{\alpha-2}$

Necessary choice: $\tilde{\rho}(-2) = 1$ to ensure $\langle\tilde{\xi}|\widehat{Q}|\tilde{\xi}\rangle = \int |\tilde{\xi}(x)|^2 x\mathrm{d}x = 1$

Example of fiducial vector family $\tilde{\xi}_\mu(x) = \left(\frac{\mu}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{x}} \exp\left[-\frac{\mu}{2}\left(\ln x + \frac{1}{4\mu}\right)^2\right]$

$$\sigma_\nu(\alpha) = \left[\frac{\nu}{2} + \left(\frac{\alpha+2}{2}\right)^2\right] \exp\left[\frac{\alpha(\alpha+3)}{4\nu}\right]$$

$$\tilde{\rho}_\mu(\alpha) = \exp\left[\frac{(\alpha+1)(\alpha+2)}{4\mu}\right]$$




$$\tilde{\rho}(1) = e^{3/(2\mu)}$$

$$\neq e^{1/(2\mu)} = \tilde{\rho}(0)$$

not for quantization

$\tilde{\rho}_\mu(-2) = 1$ ✓

back to the quantum Hamiltonian $\widehat{H}^{(0)} = 2\kappa_0 \left(\widehat{P}^2 + \hbar^2 \mathfrak{c}_0 \widehat{Q}^{-2}\right)$

$$\longrightarrow \quad H_{\text{sem}} = 2\kappa_0 \left(p^2 + \frac{\hbar^2 \mathfrak{K}}{q^2}\right) \quad \text{with} \quad \mathfrak{K} = \mathfrak{c}_0 \tilde{\rho}(1) + \tilde{\sigma}(-1)$$

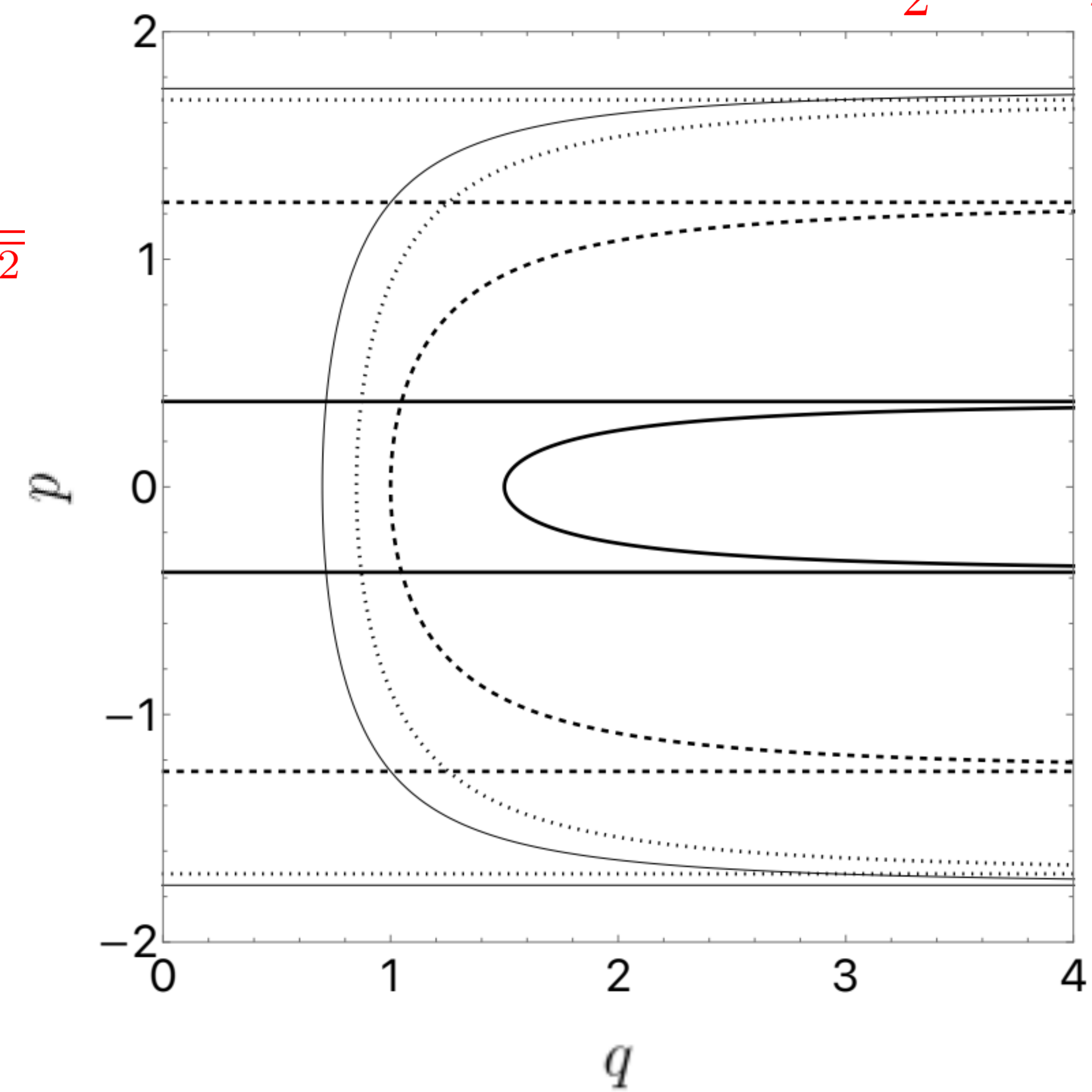
$$\mathfrak{K} = \frac{\nu}{2} + \frac{2\mu+1}{4} \exp\left(\frac{3}{2\mu}\right)$$

semiclassical trajectory in phase space

$$\begin{cases} q = q_{\text{B}} \sqrt{1 + (\omega\tau)^2} \\ p = \dfrac{q_{\text{B}}\omega^2}{4\kappa_0} \dfrac{\tau}{\sqrt{1+(\omega\tau)^2}} \end{cases}$$

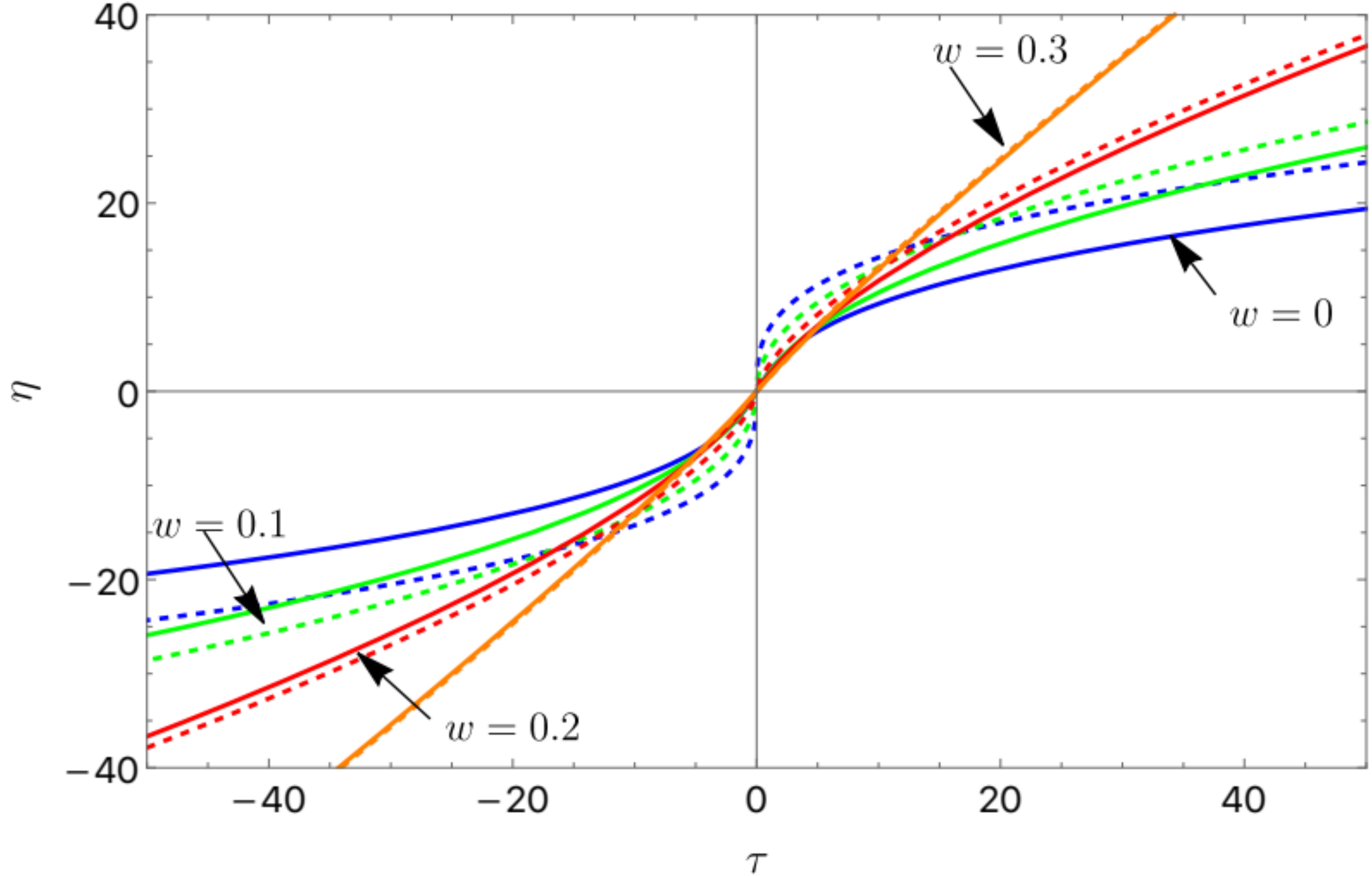
$$q_{\text{B}}^2 = \frac{2\kappa_0 \hbar^2 \mathfrak{K}}{H_{\text{sem}}}$$

$$\omega = \frac{2H_{\text{sem}}}{\hbar\sqrt{\mathfrak{K}}}$$



**conformal time** $\eta(\tau)$

$$\eta = (1+w)\tau\left(\frac{q_{\mathrm{B}}}{\gamma}\right)^{2r_1} F\left[\frac{1}{2}, -r_1; \frac{3}{2}; -(\omega\tau)^2\right]$$



Full quantum dynamics: $|\psi(\tau)\rangle = |\psi_\mathrm{B}(\tau)\rangle \otimes |\psi_\mathrm{P}(\tau)\rangle$

$$|q(\tau), p(\tau)\rangle \qquad\qquad |\psi_\mathrm{P}\rangle = \prod_{\boldsymbol{k}} |\psi_{\boldsymbol{k}}\rangle$$

2nd order quantum action

$$\mathcal{S}^{(0)+(2)} = \mathcal{S}_\mathrm{B} + \mathcal{S}_\mathrm{P} = \int \langle\psi(\tau)| \left( i\hbar\frac{\partial}{\partial\tau} - \widehat{H}^{(0)} - \sum_{\boldsymbol{k}} \widehat{H}_{\boldsymbol{k}}^{(2)} \right) |\psi(\tau)\rangle \mathrm{d}\tau$$

$$\int \langle\psi_\mathrm{P}| \left( i\hbar\frac{\partial}{\partial\tau} - \sum_{\boldsymbol{k}} \underbrace{\langle q(\tau), p(\tau)|\widehat{H}_{\boldsymbol{k}}^{(2)}|q(\tau), p(\tau)\rangle}_{\widehat{H}_{\boldsymbol{k}}} \right) |\psi_\mathrm{P}\rangle \mathrm{d}\tau$$

Schrödinger equations for the perturbations $\quad i\hbar\frac{\partial}{\partial\tau}|\psi_{\boldsymbol{k}}\rangle = \widehat{H}_{\boldsymbol{k}}|\psi_{\boldsymbol{k}}\rangle$

From that point on, business as usual...

## F-*parameterization quantization*

$$\mathfrak{L}_{\rm S} = \mathfrak{L}_{\rm Q}\tilde{\rho}\left(-4r_1 - 1\right)$$

$$\widehat{H}_{\boldsymbol{k}} = \frac{1}{2}|\widehat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\rm S}}{2}w(1+w)^2\left(\frac{q}{\gamma}\right)^{4r_1}k^2|\widehat{\phi}_{\boldsymbol{k}}|^2$$

Heisenberg e.o.m.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\phi}_{\boldsymbol{k}} = -\widehat{\pi}_{\phi,\boldsymbol{k}}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\pi}_{\phi,\boldsymbol{k}} = \mathfrak{L}_{\rm S}w(1+w)^2\left(\frac{q}{\gamma}\right)^{4r_1}k^2\widehat{\phi}_{\boldsymbol{k}}$$

mode expansion

$$\widehat{\phi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}\phi_k^*(\tau) + a_{-\boldsymbol{k}}^{\dagger}\phi_k(\tau)\right]$$

$$\widehat{\pi}_{\phi,\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}\left[a_{\boldsymbol{k}}\dot{\phi}_k^*(\tau) + a_{-\boldsymbol{k}}^{\dagger}\dot{\phi}_k(\tau)\right]$$

canonical commutation relations → $\dot{\phi}_k\phi_k^* - \phi_k\dot{\phi}_k^* = 2i$

→ 2nd order o.d.e $\ddot{\phi}_k \cdots$

Mukhanov-Sasaki variable in the F-parameterization $v_k^{\rm F} = Z\phi_k$

$$\frac{\mathrm{d}^2 v_k^{\rm F}}{\mathrm{d}\eta^2} + \left[k_{\rm F}^2 - \mathcal{V}_{\rm F}(\eta)\right]v_k^{\rm F} = 0$$

F-*parameterization quantization*

$$\widehat{H}_{\boldsymbol{k}} = \frac{1}{2}|\widehat{\pi}_{\phi,\boldsymbol{k}}|^2 + \frac{\mathfrak{L}_{\mathrm{S}}}{2} w(1+w)^2 \left(\frac{q}{\gamma}\right)^{4r_1} k^2 |\widehat{\phi}_{\boldsymbol{k}}|^2$$

$$\frac{\mathrm{d}^2 v_k^{\mathrm{F}}}{\mathrm{d}\eta^2} + \left[k_{\mathrm{F}}^2 - \mathcal{V}_{\mathrm{F}}(\eta)\right] v_k^{\mathrm{F}} = 0$$

$$k_{\mathrm{F}}^2 = \mathfrak{L}_{\mathrm{S}} w k^2$$

$$\mathcal{V}_{\mathrm{F}}(\eta) = \frac{8}{9q^2 Z^4} \frac{(2\kappa_0)^2(1-3w)}{(1-w)^2} \left[p^2 - \frac{3(1-w)\mathfrak{K}}{2q^2}\right]$$

$$= \frac{(q^{r_1})''}{q^{r_1}}$$ upon using the background e.o.m.

$$\lim_{\eta\to\infty} \mathcal{V}_{\mathrm{F}}(\eta) = \mathcal{V}_{\mathrm{cl}}(\eta)$$

## C-*parameterization quantization*

$$\widehat{H}_{\boldsymbol{k}} = \frac{1}{2} Z^2 \mathfrak{M}_{\mathrm{S}} \left\{ |\widehat{\pi}_{v,k}|^2 + \left[ w k^2 - \frac{8 \mathfrak{M}_{\mathrm{S}}^{-1}}{9 q^2 Z^4} \frac{(2\kappa_0)^2 (1-3w)}{(1-w)^2} \left( \mathfrak{N}_{\mathrm{S}} p^2 + \frac{\hbar^2 \mathfrak{T}_{\mathrm{S}}}{q^2} \right) \right] |\widehat{v}_k|^2 \right\}$$

$$\mathfrak{M}_{\mathrm{Q}} \tilde{\rho} \left(-2r_1 - 1\right)$$

$$\mathfrak{N}_{\mathrm{Q}} \tilde{\rho} \left(2r_2 - 1\right)$$

$$\mathfrak{N}_{\mathrm{Q}} \tilde{\sigma} (2r_2 - 1) + \mathfrak{T}_{\mathrm{Q}} \tilde{\rho} \left(2r_2 + 1\right)$$

Wronskian $\dot{v}_k v_k^* - v_k \dot{v}_k^* = 2 i Z^2 \mathfrak{M}_{\mathrm{S}}$

$$v_k' v_k^* - v_k v_k^{*\prime} = 2 i \mathfrak{M}_{\mathrm{S}}$$

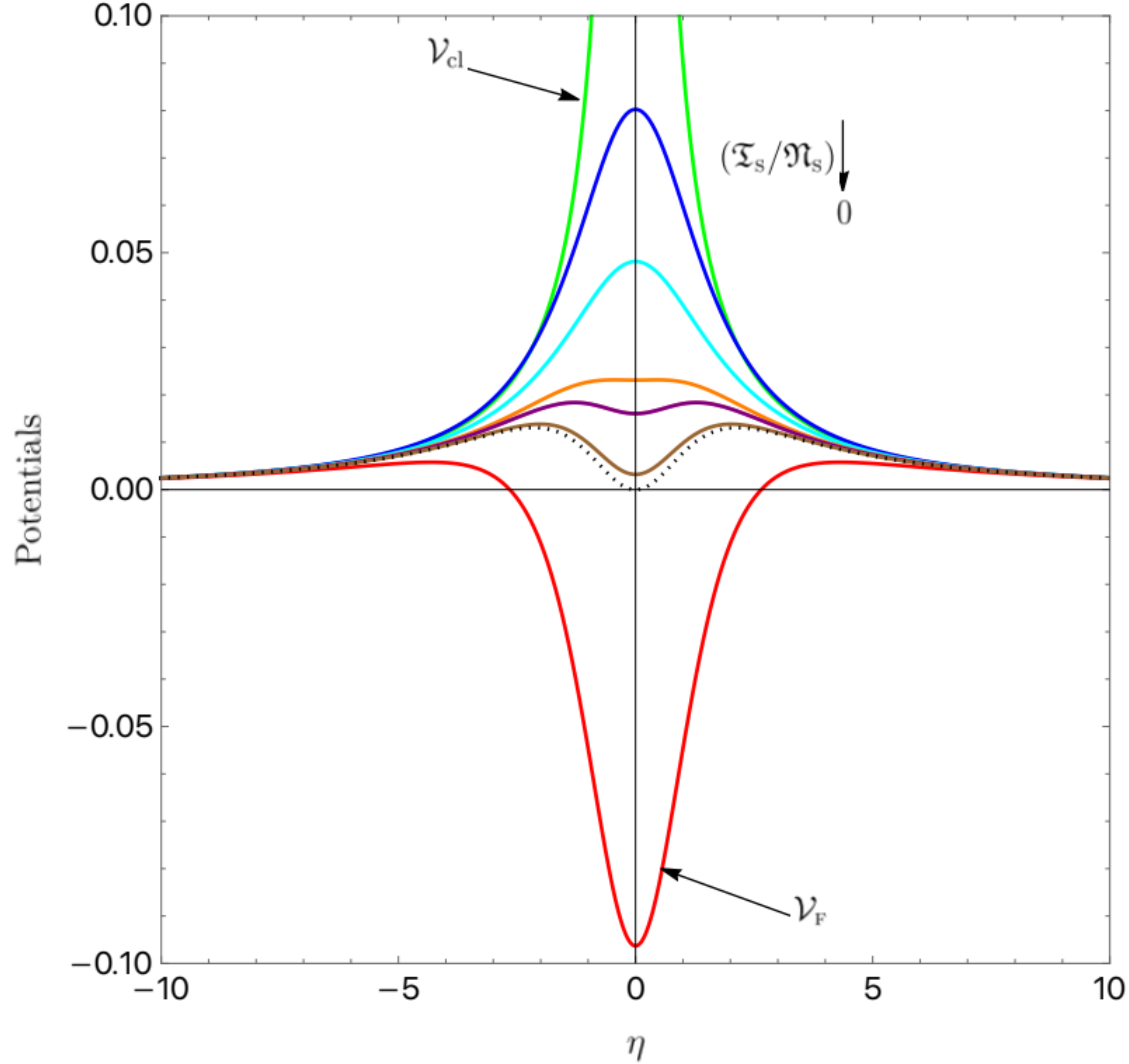
$$\frac{\mathrm{d}^2 v_k^{\mathrm{C}}}{\mathrm{d}\varsigma^2} + \left[ k_{\mathrm{C}}^2 - \mathcal{V}_{\mathrm{C}}(\varsigma) \right] v_k^{\mathrm{C}} = 0$$

$$\varsigma = \sqrt{\mathfrak{M}_{\mathrm{S}} \mathfrak{N}_{\mathrm{S}}} \eta$$

rescaled

$$w \frac{\mathfrak{M}_{\mathrm{S}}}{\mathfrak{N}_{\mathrm{S}}} k^2$$

$$\frac{8}{9 q^2 Z^4} \frac{(2\kappa_0)^2 (1-3w)}{(1-w)^2} \left( p^2 + \frac{\hbar^2 \mathfrak{T}_{\mathrm{S}} / \mathfrak{N}_{\mathrm{S}}}{q^2} \right)$$

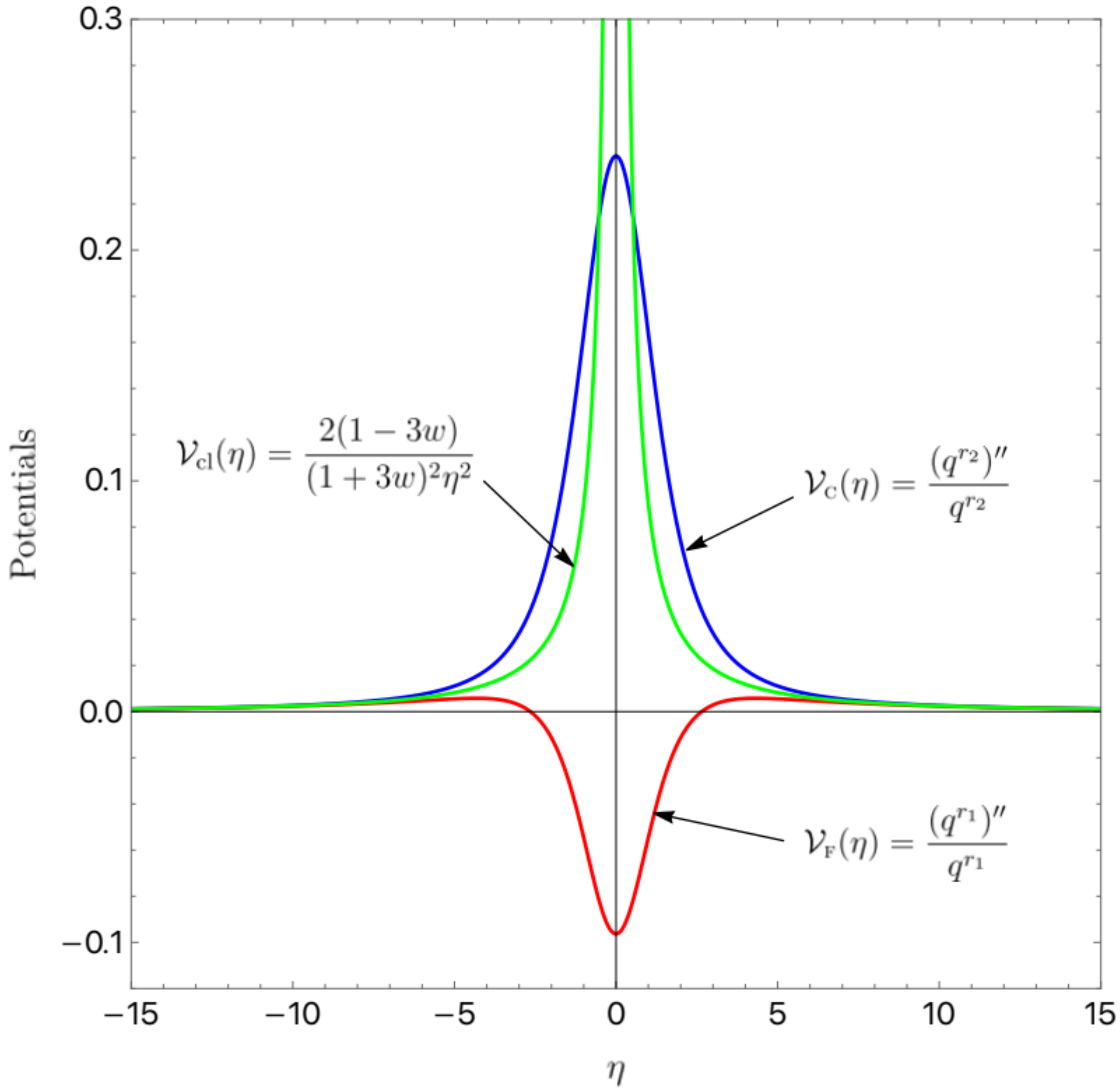

0.10
0.05
0.00
−0.05
−0.10
Potentials
$\mathcal{V}_{\rm cl}$
$(\mathfrak{T}_{\rm s}/\mathfrak{N}_{\rm s})$
0
$\mathcal{V}_{\rm F}$
−10
−5
0
5
10
$\eta$

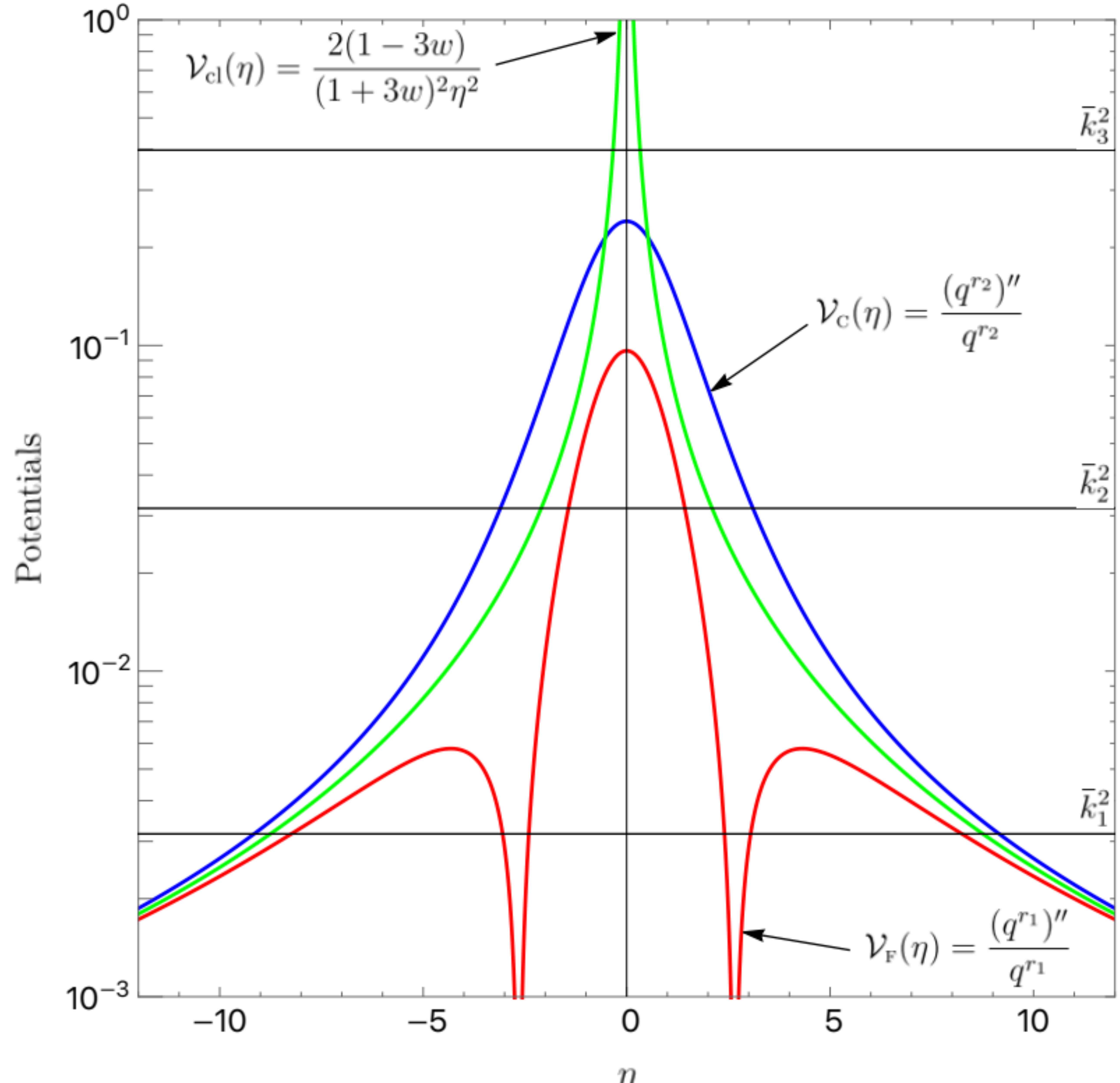
special case

$$\frac{\mathfrak{T}_{\rm S}}{\mathfrak{N}_{\rm S}} = \frac{3\mathfrak{K}(1-w)}{1-3w}$$

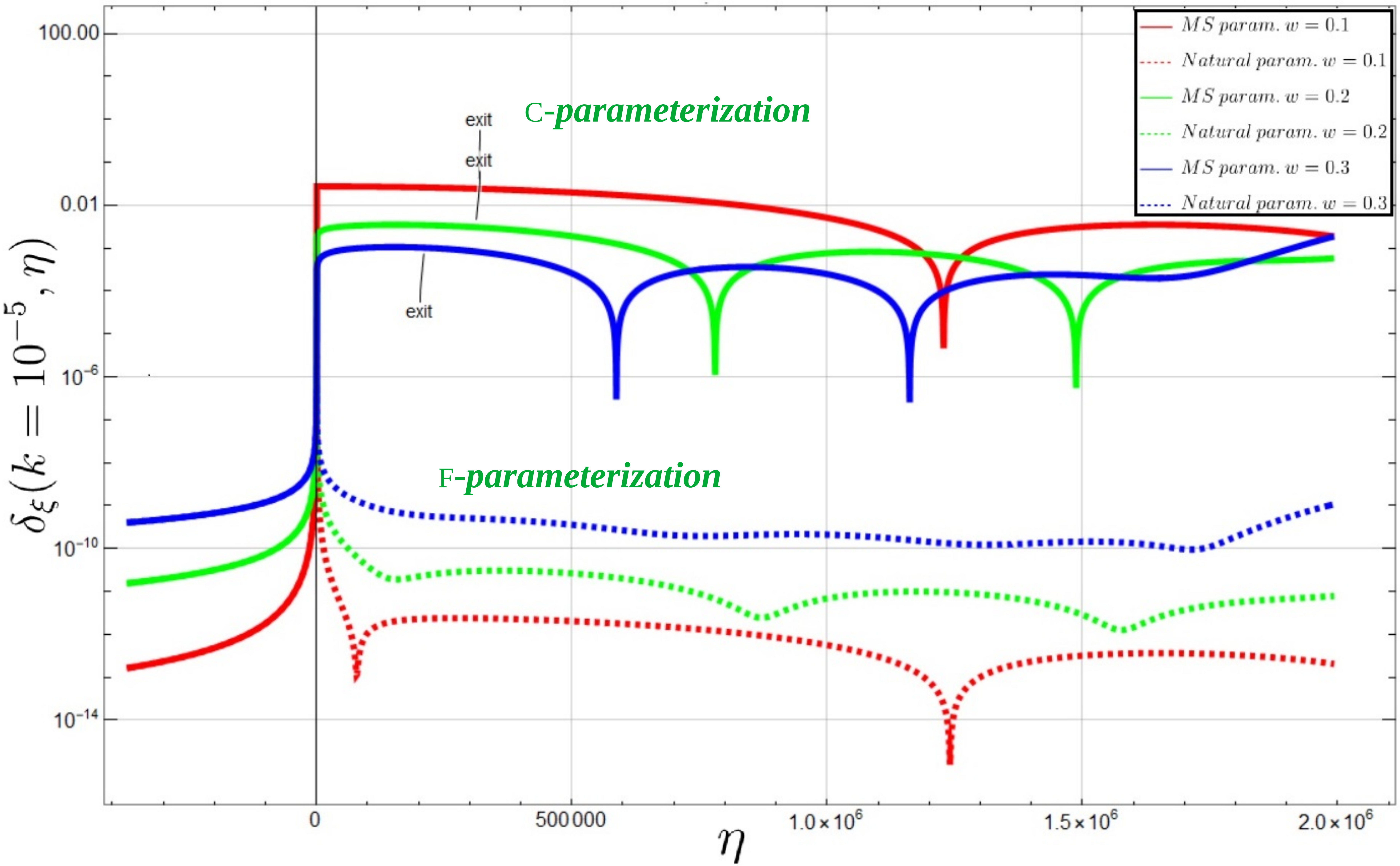
↓

$$\mathcal{V}_{\rm C} = \frac{(q^{r_2})''}{q^{r_2}}$$

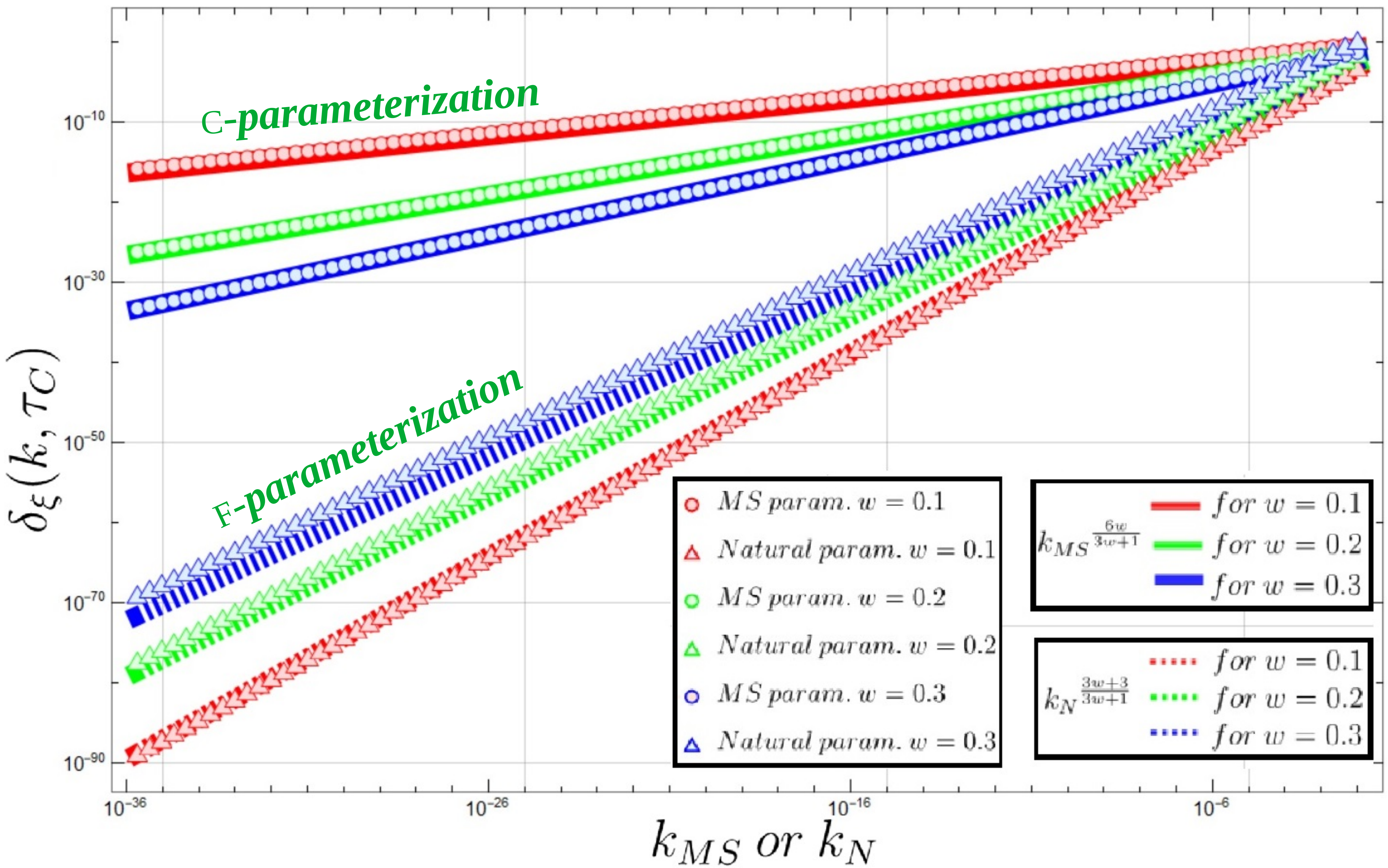




$\mathcal{V}_{\rm cl}(\eta) = \frac{2(1-3w)}{(1+3w)^2\eta^2}$
$\bar{k}_3^2$
$\mathcal{V}_{\rm C}(\eta) = \frac{(q^{r_2})''}{q^{r_2}}$
$\bar{k}_2^2$
$\bar{k}_1^2$
$\mathcal{V}_{\rm F}(\eta) = \frac{(q^{r_1})''}{q^{r_1}}$
Potentials
$10^0$
$10^{-1}$
$10^{-2}$
$10^{-3}$
−10
−5
0
5
10
$\eta$



MS param. w = 0.1
Natural param. w = 0.1
MS param. w = 0.2
Natural param. w = 0.2
MS param. w = 0.3
Natural param. w = 0.3
C-parameterization
F-parameterization
exit
exit
exit
100.00
0.01
$10^{-6}$
$10^{-10}$
$10^{-14}$
0
500000
$1.0\times10^{6}$
$1.5\times10^{6}$
$2.0\times10^{6}$
$\eta$
$\delta_\xi(k=10^{-5},\eta)$



C-parameterization
F-parameterization
$\delta_\xi(k, \tau_C)$
$10^{-10}$
$10^{-30}$
$10^{-50}$
$10^{-70}$
$10^{-90}$
$10^{-36}$
$10^{-26}$
$10^{-16}$
$10^{-6}$
$k_{MS}$ or $k_N$
$MS\ param.\ w = 0.1$
$Natural\ param.\ w = 0.1$
$MS\ param.\ w = 0.2$
$Natural\ param.\ w = 0.2$
$MS\ param.\ w = 0.3$
$Natural\ param.\ w = 0.3$
$k_{MS}^{\frac{6w}{3w+1}}$
$for\ w = 0.1$
$for\ w = 0.2$
$for\ w = 0.3$
$k_N^{\frac{3w+3}{3w+1}}$
$for\ w = 0.1$
$for\ w = 0.2$
$for\ w = 0.3$

# Conclusions

- **GR + perfect fluid = simple, classically singular model**
- **Background quantization yields naturally bouncing solutions**
- **Unitarily inequivalent parameterizations**
- **Two classes of predictions: simple but not predictive model!**