Neutrino Oscillation: an Avenue to Probe the Universe

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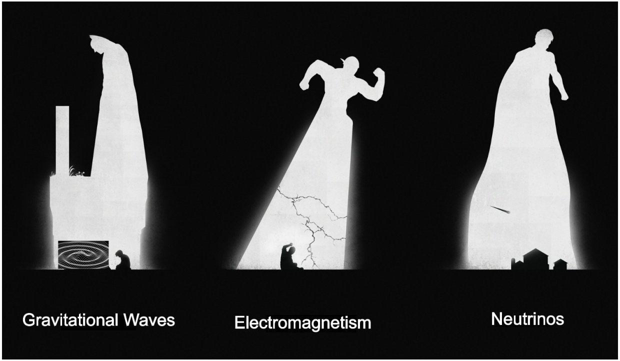
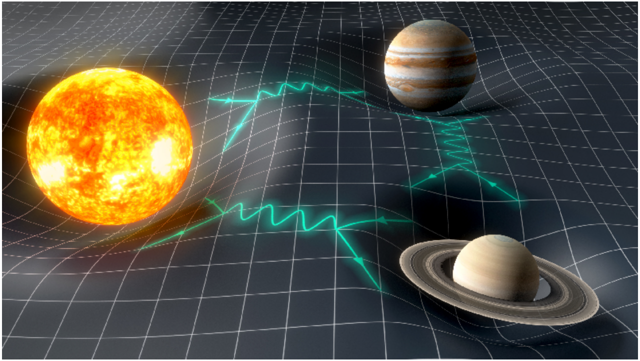
In collaboration with Raul Jimenez at ICCUB
Refs:2010.08181, 2105.07973 and 2111.15249

Multimessenger Alliance



Source:downloadwallpapers.net

Gravity at the Quantum Realm



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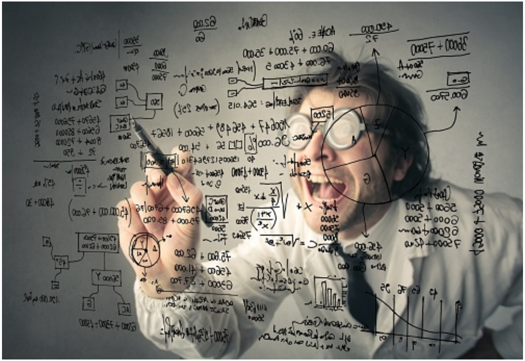

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Mel Schwartz, Jack Steinberger and Léon Lederman

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- More details: *"Fundamentals of Neutrino Physics and Astrophysics"*; Giunti, Carlo; Kim, Chung W.

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The most impactful interaction is the one with e, having an interaction potential $V_I \sim G_F E N_e$.

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$$V_I \sim G_F E N_e$$

$$\mathcal{H} = \mathcal{H}_V + \mathcal{H}_I; \quad \mathcal{H}_V|\nu_j\rangle = E_j|\nu_j\rangle \text{ and } \mathcal{H}_I|\nu_\alpha\rangle = V_I \delta_{\alpha e}|\nu_\alpha\rangle$$

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$$\ldots \Rightarrow \; i\frac{d}{dt}\begin{pmatrix}\psi_{ee}\\\psi_{e\mu}\end{pmatrix} = \left(U\mathcal{M}^2U^\dagger + V_I\right)\begin{pmatrix}\psi_{ee}\\\psi_{e\mu}\end{pmatrix} \equiv \mathcal{H}_F\begin{pmatrix}\psi_{ee}\\\psi_{e\mu}\end{pmatrix}$$
$$\psi_{\alpha\beta} = \langle\nu_\beta|\nu_\alpha(t)\rangle; \; \mathcal{M}^2 = \begin{pmatrix}m_1^2 & 0\\0 & m_2^2\end{pmatrix}; \; V_I = \begin{pmatrix}2\sqrt{2}EG_FN_e & 0\\0 & 0\end{pmatrix}$$

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Diagonalizing $\mathcal{H}_F$ gives $\tilde{\theta}(\theta, \Delta m^2, V_I)$ and $\Delta\tilde{m}^2(\theta, \Delta m^2, V_I)$

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If $\tilde{\theta}$ is time-independent(*adiabatic regime*):

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This is known as the *Mikheev-Smirnov-Wolfenstein*(MSW) effect

Stanislav Mikheev, Alexei Smirnov and Lincoln Wolfenstein

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References:chapter 3.8 of Quantum Fields in Curved Space by N.D.Birell and P.C.W.Davies, or chapter 5 of Quantum Information in Gravitational Fields by M.Lanzagorta.

Part I:General Formalism

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Oscillation probability as a function of redshift. Gravity and φ alter neutrino oscillations.

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- Current-Velocity coupling:$\Theta = i\hbar\bar{\psi}\gamma^\mu\psi\partial_\mu\varphi$;
  Kinetic-Potential coupling:$\Theta = i\hbar\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi\varphi^2$;
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- At order $\hbar^0$: $\Bigg[-\Big(\gamma^{\mu}\partial_{\mu}S + m\Big) + \frac{\lambda}{2}\gamma^{\mu}\partial_{\mu}\varphi\Bigg]\psi_0 = 0$

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- Non-trivial solution $\Leftrightarrow$ $\det\Big[\gamma^{\mu}\partial_{\mu}\Big(S - \tfrac{\lambda}{2}\varphi\Big) + m\Big] = 0$

$$\Rightarrow \quad \partial_{\mu}\Big(S - \tfrac{\lambda}{2}\varphi\Big)\partial^{\mu}\Big(S - \tfrac{\lambda}{2}\varphi\Big) = -m^2$$

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  $\Rightarrow \quad \partial_\mu \left(S - \frac{\lambda}{2}\varphi \right) \partial^\mu \left(S - \frac{\lambda}{2}\varphi \right) = -m^2$
- Canonical 4-momentum satisfies usual geodesic equation: $\frac{dp^\alpha}{d\tau} + \frac{1}{m}\Gamma^\alpha{}_{\beta\gamma} p^\beta p^\gamma = 0$

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- At order $\hbar^0$: $\left[-\left(\gamma^\mu \partial_\mu S + m\right) + \frac{\lambda}{2}\gamma^\mu \partial_\mu \varphi \right] \psi_0 = 0$
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  $\Rightarrow \quad \partial_\mu \left(S - \frac{\lambda}{2}\varphi\right) \partial^\mu \left(S - \frac{\lambda}{2}\varphi\right) = -m^2$
- Canonical 4-momentum satisfies usual geodesic equation: $\frac{dp^\alpha}{d\tau} + \frac{1}{m}\Gamma^\alpha{}_{\beta\gamma} p^\beta p^\gamma = 0$
- At order $\hbar$: $\frac{dp^\alpha}{d\tau} + \frac{1}{m}\Gamma^\alpha{}_{\beta\gamma} p^\beta p^\gamma = m f^\alpha \propto \hbar p^\alpha R_{\mu\beta\gamma\delta}$

Part I: WKB approximation- Kinetic-Potential Coupling

- $\Theta = i\hbar\bar{\psi}\gamma^{\mu}\mathcal{D}_{\mu}\psi\varphi^2 \Rightarrow \left(i\hbar\not{\mathcal{D}} - m\right)\psi = \frac{i\hbar\lambda}{2}\not{\mathcal{D}}\psi\varphi^2; \not{\mathcal{D}} = \gamma^{\mu}\mathcal{D}_{\mu}$

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- Change in matter-radiation equality:

$$1 + z_{eq} = \frac{\Omega_{m0}}{\Omega_{\gamma 0}}\left[1 + N_\nu\left(\frac{8}{11}\right)^{1/3}\left(1 - \frac{\lambda\varphi^2}{2}\right)^{-1}\right]^{-1}.$$

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Part I:General Formalism

- $S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) + i\hbar(\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi - \mathcal{D}_\mu\bar{\psi}\gamma^\mu\psi) - 2m\bar{\psi}\psi + \lambda\Theta\right]$

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\bar{\psi}}\Downarrow$$

- $\left(i\hbar\gamma^\mu(\partial_\mu - \Gamma_\mu) - mc\right)\psi = -\frac{\lambda}{2}\left(\frac{\partial\Theta}{\partial\bar{\psi}} - (\partial^\mu - \Gamma^\mu)\frac{\partial\Theta}{\partial X^\mu_{\bar{\psi}}}\right) \equiv -\frac{\lambda}{2}\frac{\delta\Theta}{\delta\bar{\psi}}$
- Current-Velocity coupling:$\Theta = \bar{\psi}\gamma^\mu\psi\partial_\mu\varphi$;
  Kinetic-Potential coupling:$\Theta = i\hbar\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi\varphi^2$;
  Kinetic-Kinetic coupling: $\Theta = i\hbar\bar{\psi}\gamma^\nu\mathcal{D}_\nu\psi\partial_\mu\varphi\partial^\mu\varphi$.

$\swarrow \quad \searrow$

Solve Dirac equation using WKB approximation at 0^{th} and 1^{st} order in $\hbar$.

Oscillation probability as a function of redshift. Gravity and φ alter neutrino oscillations(NO).

Part II: NO & DE-Interaction term

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$\nearrow$ Flavor-invariant coupling $\nwarrow$ Flavor-dependent Coupling

Focus later on Λ and Scalar field DE, with Current-Velocity coupling$(\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\varphi)$

Part II: NO & DE-Dirac Equation

For 2-flavor system (ν_e, ν_μ), define $\boldsymbol{\psi} = \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$

$$\left(i\gamma^\mu \mathcal{D}_\mu - \mathcal{M}_f\right)\boldsymbol{\psi} = \left(\xi F(\varphi, X^\mu_\varphi) + \xi_f \gamma^\mu G_\mu(\varphi, X^\mu_\varphi)\right)\boldsymbol{\psi}$$

where $\mathcal{M}_f \equiv$ vaccum mass matrix in flavor space;

$$\mathcal{M}_f^2 = U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger$$

and $U \equiv$ mixing matrix $= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$; $\theta \equiv$ mixing angle

Part II: NO & DE-Flavor state

Recall: in flat S.T, $|\nu_\alpha\rangle = \sum_{j=1,2} U_{\alpha j} e^{-i\frac{m_j^2}{2E}L} |\nu_j\rangle$.

Part II: NO & DE-Flavor state

Recall: in flat S.T, $|\nu_\alpha\rangle = \sum_{j=1,2} U_{\alpha j} e^{-i\frac{m_j^2}{2E}L}|\nu_j\rangle$.

In curved S.T, $|\nu_\alpha\rangle = \sum_{j=1,2} U_{\alpha j} e^{i\Phi(\lambda)}|\nu_j\rangle$

where $\Phi(\lambda) = \int_{\lambda_0}^{\lambda} \boldsymbol{P}.\boldsymbol{p}_{\text{null}} d\lambda'$; $\boldsymbol{P} \equiv$ 4-momentum operator;

$\boldsymbol{p}_{\text{null}} \equiv$null vector tangent to worldline.

Part II: NO & DE-Flavor state

$$|\nu_\alpha\rangle = \sum_{j=1,2} U_{\alpha j} e^{i\Phi(\lambda)}|\nu_j\rangle \Leftrightarrow i\frac{d}{d\lambda}|\nu_\alpha(\lambda)\rangle = \Phi(\lambda)|\nu_\alpha(\lambda)\rangle$$

Part II: NO & DE-Transition Amplitude

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$$\Psi_{\alpha\beta} \equiv \langle\nu_\beta|\nu_\alpha(\lambda)\rangle \Longrightarrow i\frac{d}{d\lambda}\Psi_{\alpha\beta} = \left[\frac{1}{2}\tilde{\mathcal{M}}_f^2 + V_I\right]\Psi_{\alpha\beta}$$

$$\tilde{\mathcal{M}}_f^2 = U \begin{pmatrix} \left(m_1 - \xi F\right)^2 & 0 \\ 0 & (m_2 - \xi F)^2 \end{pmatrix} U^\dagger;\ V_I \propto \xi_f G_\mu p^\mu_{\text{null}}$$

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Gravitational MSW effect

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Define $\phi_{\alpha j} = \tilde{U}_{j\beta}\Psi_{\alpha\beta} \Rightarrow i\frac{d}{d\lambda}\begin{pmatrix}\phi_{e-} \\ \phi_{e+}\end{pmatrix} = \begin{pmatrix} v_- & -i\frac{d\tilde{\theta}}{d\lambda} \\ i\frac{d\tilde{\theta}}{d\lambda} & v_+ \end{pmatrix}\begin{pmatrix}\phi_{e-} \\ \phi_{e+}\end{pmatrix}.$

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For Λ and scalar DE with linear derivative coupling, we have adiabatic regime, i.e. $\frac{d\tilde{\theta}}{d\lambda} = 0$.

Part II: NO & DE-Probability

Solving for $\phi_{e+,-} \Rightarrow \Psi_{e\mu}$ by inverse transformation.

Final Result

$$P_{\nu_e \to \nu_\mu} \equiv |\Psi_{e\mu}|^2 = \mathcal{F}(\xi F, \xi_f G) \sin^2 2\theta \sin^2\left(\frac{\omega_- - \omega_+}{2}\right),$$

$$\omega_- - \omega_+ \approx$$

$$\frac{m_2^2 - m_1^2}{2}(\lambda_0 - \lambda) + V_I \cos 2\theta(\xi_e - \xi_\mu)(\lambda - \lambda_0) + \xi \Delta m \int_{\lambda_0}^{\lambda} F d\lambda'.$$

Compare to flat S.T: $P^{\text{std}}_{\nu_e \to \nu_\mu} = \sin^2 2\theta \sin^2\left(\frac{(m_2^2 - m_1^2)L}{4E}\right)$

Part II: NO & ΛCDM

In Particle Physics (Minkowski spacetime):

$$P_{\nu_e \to \nu_\mu} = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

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$$\text{FRW} \Big\Downarrow \quad L \to d_L;\, E \to E_0/a$$

$$P_{\nu_e \to \nu_\mu} = \sin^2 2\theta \sin^2 \left(\frac{\Delta m^2 d_L a}{4E_0}\right)$$

$$d_L = (1+z_e) H_0^{-1} \int_0^{z_e} \left(\Omega_{m0}(1+z)^3 + \Omega_{\Lambda_0}\right)^{-1/2}$$

Part II: NO & ΛCDM

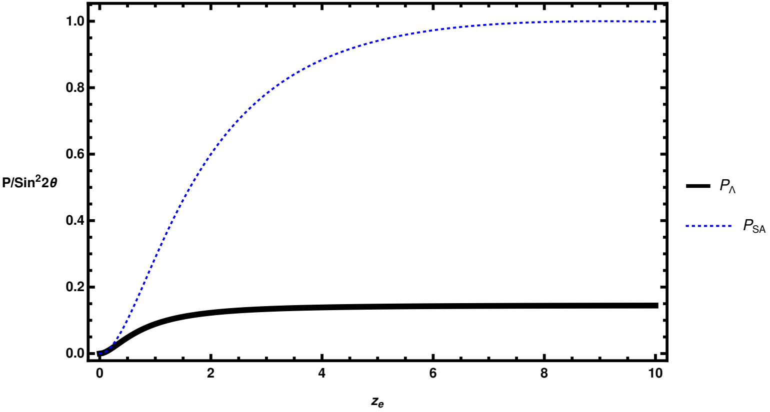
Quantum spinors in flat FLRW universe with cosmological constant DE:

$P_\Lambda = \sin^2 2\theta \sin^2 \omega_\Lambda;$

$$\omega_\Lambda = \frac{\Delta m^2}{2} \int \frac{d\lambda}{E} = \frac{\Delta m^2}{2H_0 E_0} \int_0^{z_e} \left(\Omega_{m_0}(1+z)^7 + \Omega_{\Lambda_0}(1+z)^4 \right)^{-1/2} dz$$

- H_0 ≡Hubble constant today;
- z_e ≡emission redshift;
- $\Omega_{m_0}(\Omega_{\Lambda_0})$ ≡matter(DE) density parameter.

Part II: NO & ΛCDM



Part II: NO & Quintessence

- In flat FLRW with scalar field DE(e.g. quintessence, modified gravity) coupled to neutrinos via $\mathcal{L}_{\text{int}} = \xi\bar{\psi}\gamma^{\mu}\psi\partial_{\mu}\varphi$

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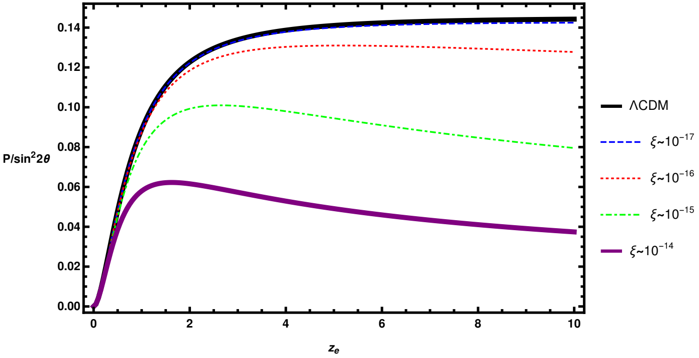
- Does not alter the dynamics $\Rightarrow$ Klein-Gordon equation remains the same.
- Slowly rolling scalar field: $\dot{\varphi}^2 \ll V(\varphi)$
- $P_Q = \frac{\sin^2 2\theta}{D_Q}\sin^2(\omega_Q/2)$

  $D_Q = 1 + 4E_0\sqrt{\epsilon(1+z_e)}\cos 2\theta(\xi_e - \xi_\mu)(\Delta m^2)^{-1}$

  $\omega_Q \approx \frac{\Delta m^2}{2E_0 H_0}\int_0^{z_e} D_Q^{-1/2}\left(\Omega_{m_0}(1+z)^7 + \Omega_{\varphi_0}(1+z)^4\right)^{-1/2} dz$

Part II: NO & Quintessence



Part III: NO & the Hubble Tension(HT)-Plan

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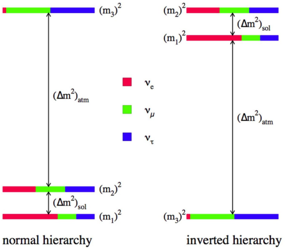

Schematic difference between the two neutrino hierarchies. $m_{1,2,3}$ are eigenvalues for neutrino mass states, and $\Delta m_{ij}^2 = m_i^2 - m_j^2$

Part III: NO & HT-Plan

- Generalize to three neutrino flavors in ΛCDM
- Effect of different H_0 values on the oscillation probability.
- Distinguish between Normal Hierarchy(NH) and Inverted Hierarchy(IH)
- Show results in terms of Ternary diagrams and neutrino flux vs. redshift plots.

Part III: NO & HT-Equations Needed

- ΛCDM:
  $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)\big(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\big),$$
  $$H^2(z) = \frac{8\pi G}{3}\big(\rho_m + \rho_\Lambda\big) = H_0^2\left(\Omega_m(1+z)^3 + \Omega_\Lambda\right)$$

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  $H^2(z) = \frac{8\pi G}{3}\big(\rho_m + \rho_\Lambda\big) = H_0^2\left(\Omega_m(1+z)^3 + \Omega_\Lambda\right)$
- Transition amplitudes' evolution:

$$i\frac{d}{d\lambda}\begin{pmatrix}\Psi_{\alpha e}\\ \Psi_{\alpha\mu}\\ \Psi_{\alpha\tau}\end{pmatrix} = \frac{1}{2}U\begin{pmatrix}m_1^2 & 0 & 0\\ 0 & m_2^2 & 0\\ 0 & 0 & m_3^2\end{pmatrix}U^\dagger\begin{pmatrix}\Psi_{\alpha e}\\ \Psi_{\alpha\mu}\\ \Psi_{\alpha\tau}\end{pmatrix}$$

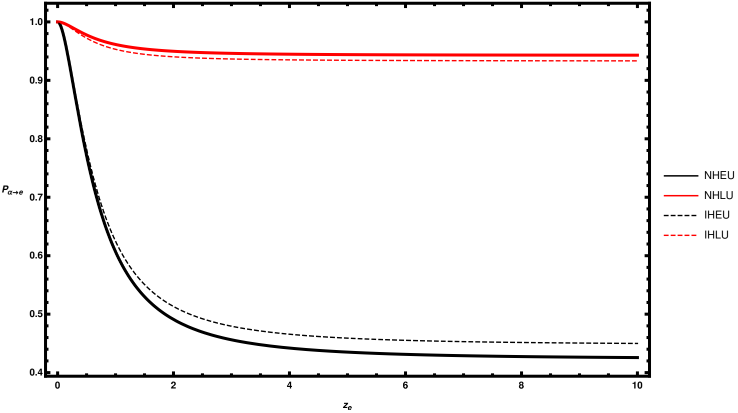
Part III: NO & HT-Probability

- Transition Probability:

$$P_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{i<j} \left[a_{\alpha\beta;ij} \sin^2\left(\frac{\Delta m_{ij}^2 \Delta\lambda}{4}\right) + b_{\alpha\beta;ij} \sin\left(\frac{\Delta m_{ij}^2 \Delta\lambda}{2}\right)\right]$$

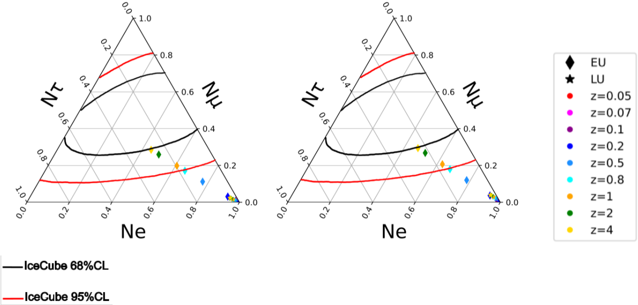
$$\Delta\lambda \equiv \frac{1}{E_0}\int_0^{z_e} \frac{dz}{H(z)(1+z)^2},$$

Part III: NO & HT-Probability



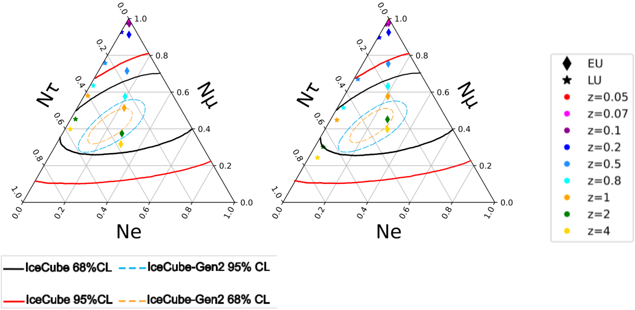
Part III: NO & HT-Pure Electron Neutrino I.C.

$$(\nu_e, \nu_\mu, \nu_\tau) = (1, 0, 0)$$



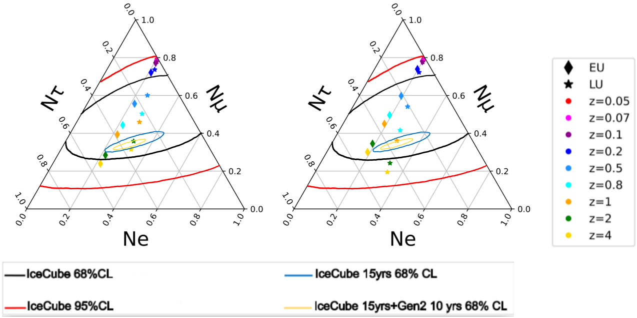
Part III: NO & HT-Pure Muon Neutrino I.C.

$$(\nu_e, \nu_\mu, \nu_\tau) = (0, 1, 0)$$

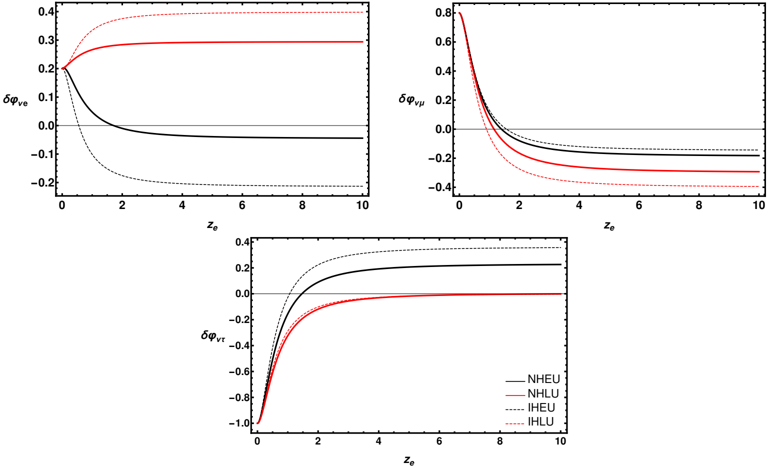


Part III: NO & HT-Pion Decay I.C.

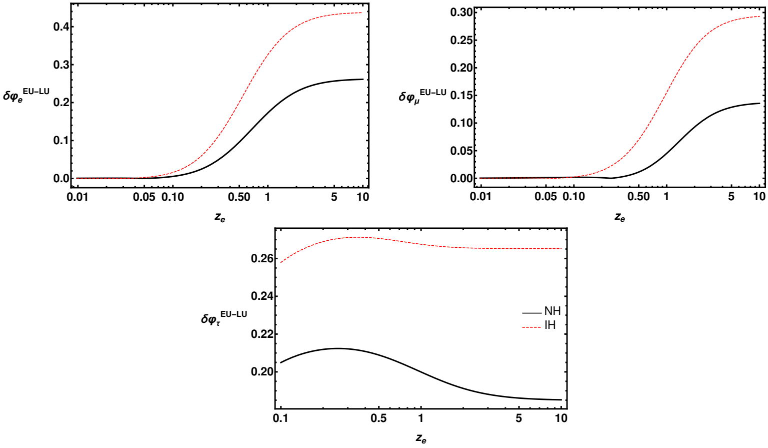
$$(\nu_e, \nu_\mu, \nu_\tau) = (1/3, 2/3, 0)$$



Part III: NO & HT-Flux emitted vs. observed



Part III: NO & HT-Flux EU vs. LU



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- Analysis is made assuming flat spacetime. But gravity now must be included.

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- One must not make a simple substitution of cosmological quantities into transition probability formula.
- Different H_0 values can make a few % difference in transition probability.

What's Next?

- More work on the observational front: MCMCs.
- Look at wave-packets of neutrinos.
- Look at 1^{st} order perturbations and effect of power spectra.
- Neutrinos traveling near Dark Matter halos.
- Apply to other fermionic entities: electrons or DM(?).
- ...
- Quantum field theory in curved spacetime has many applications still to be explored.
- It is a further step in generalizing our analysis of the Universe.

The End!

Questions or comments?

Refs:2010.08181, 2105.07973 and 2111.15249

Spinors in Curved Spacetime

- The solution is to introduce tetrad fields, ${e_a}^{\mu}(x)$, that covers the entire spacetime. These fields link local flat coordinates to the global curved ones. Latin indices⇔local coordinates; Greek indices⇔global coordinates.
- $\left\{\gamma^a, \gamma^b\right\} = -2\eta^{ab} \Leftrightarrow \left\{\gamma^\mu, \gamma^\nu\right\} = -2g^{\mu\nu}; \gamma^\mu(x) = {e_a}^{\mu}(x)\gamma^a$
- $\mathcal{D}_\mu\psi \equiv \left(\partial_\mu - \Gamma_\mu\right)\psi; \Gamma_\mu = -\frac{1}{4}\gamma_a\gamma_b e^{a\alpha}(x)\nabla_\mu {e^b}_\alpha(x)$
- $\gamma^a e_a^\mu \Gamma_\mu = \frac{i}{\hbar}\gamma^a e_a^\mu A_\mu; A^\mu = \frac{1}{4}\sqrt{-g}\, e_a^\mu \epsilon^{abcd}(\partial_\sigma e_{b\nu} - \partial_\nu e_{b\sigma}) e_c^\nu e_d^\sigma$
- $\left(i\hbar\gamma^\mu\mathcal{D}_\mu - mc\right)\psi = 0 \Leftrightarrow \left[i\hbar\gamma^\mu\left(\partial_\mu - \frac{i}{\hbar}A_\mu\right) - m\right]\psi = 0$