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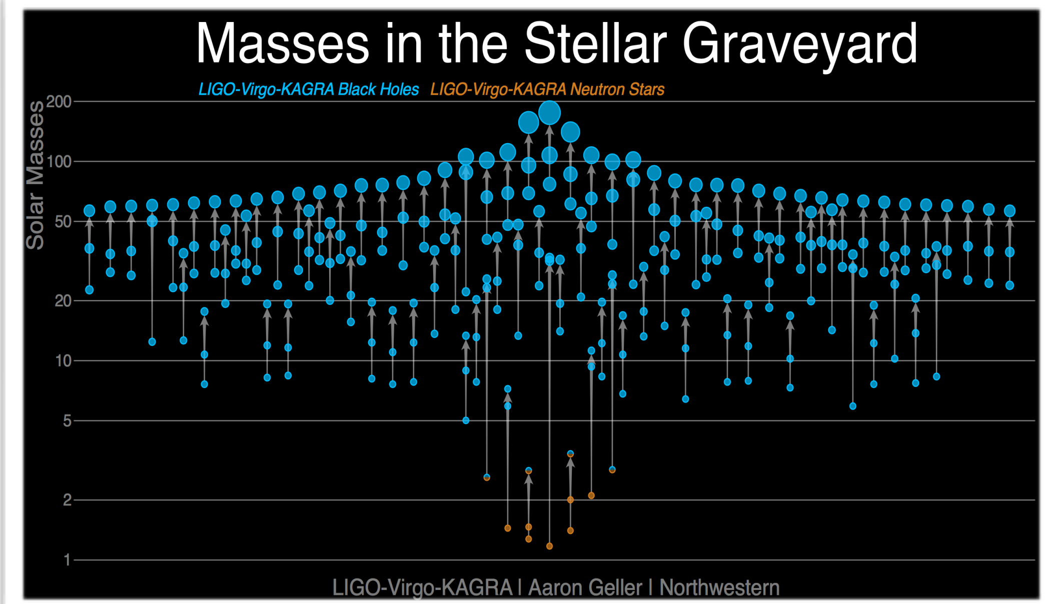
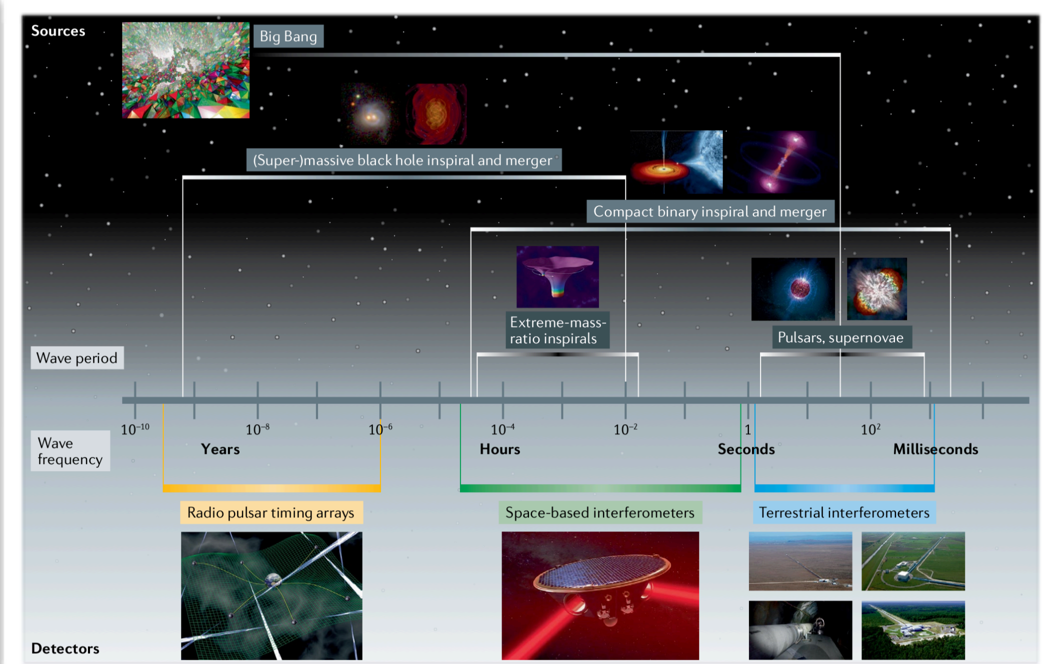
Hidden symmetries and Love numbers of black holes

based on
arXiv: 2010.00593, 2105.01069, 2203.08832, 2304.03743
and other works in progress in collaboration with
L. Hui, A. Joyce, R. Penco, M. J. Rodriguez, A. Solomon, L. F. Temoche

Gravitational-wave astronomy

- The past few years have witnessed a revolution in astronomy: direct detection of gravitational waves (GWs).
- The ever-increasing number of GW observations of merging binary systems is providing us with a unique opportunity to test General Relativity (GR) in the strong-field regime, shed light on the fundamental aspects of gravity and black holes, probe the fundamental nature of astrophysical compact objects.
- Extraordinary scientific potential of upgraded detectors and future facilities.
- Possibility of measuring several frequencies for single merging events at high SNR.





[Nature Reviews Physics, 3, 344–366 (2021)]

- We are witnessing the dawn of the era of precision physics with gravitational waves.

[Berti et al. '15], [Barack et al. '18], [Cardoso and Pani '19], [Baibhav et al. '19], [Barausse et al. '20], [Perkins, Yunes and Berti '20], [Bailes et al. '21], [Berti et al. '22]…

Symmetries of black holes

- Symmetries can help us shed light on the fundamental aspects of black holes and gravity, and constrain broad classes of theories beyond General Relativity in a model-independent way.

Symmetries of black holes

- Black hole perturbation theory has a long history starting from the work of Regge and Wheeler, Zerilli, Teukolsky, Chandrasekhar…
- Interestingly, recent investigations suggest the subject has depths yet to be plumbed.

Outline

I will focus on the static response and Love numbers (LNs) of black holes.

I. Ladder symmetries of black holes and the vanishing of the Love numbers
II. Love numbers for rotating black holes in higher dimensions

I.

Symmetries of vanishing Love numbers

Symmetries of black holes

- "The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time."
  (S. Chandrasekhar, in "*The mathematical theory of black holes*")

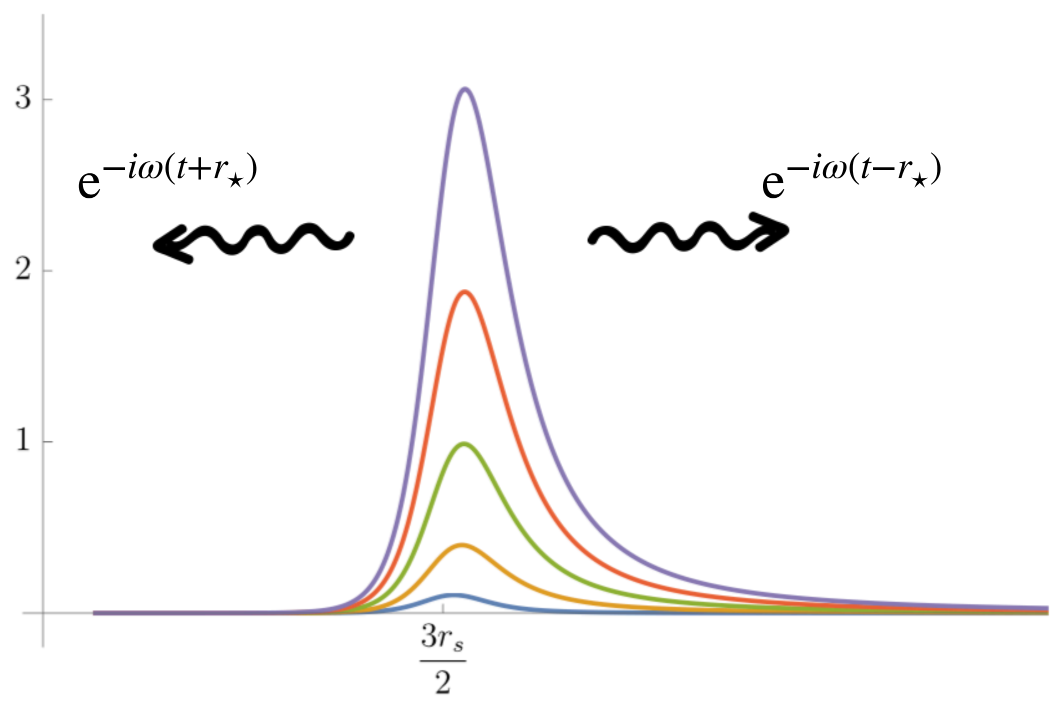
- Black holes are among the simplest and most robust objects in nature: uniquely determined by their mass and spin (and charge).

- This *simplicity* is inherited by the perturbations.

- Some aspects of this *simplicity* are well understood in terms of (hidden) symmetries of General Relativity.

Hidden symmetries of black holes

- In GR, the two d.o.f. in the gravitational wave emitted by a perturbed black hole have the same characteristic frequencies, i.e. are *isospectral*.
- Isospectrality has been known to follow from a duality of the linearized equations of motion (a.k.a. Chandrasekhar relation) since the 1970s [Chandrasekhar '75].



- Symmetry behind the vanishing of the Love numbers unclear until very recently.

Static response and tidal deformability

- The Love numbers are the coefficients encoding the (static) tidal deformability of a compact object (analogous to the electric and magnetic susceptibilities in EM).

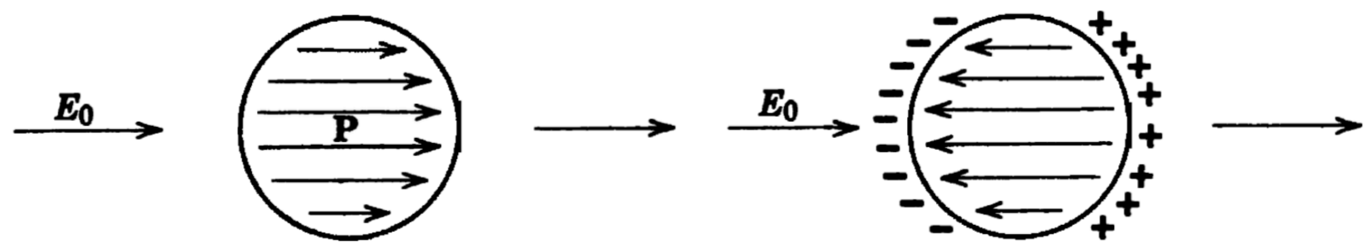


Figure 4.7 Dielectric sphere in a uniform field E_0, showing the polarization on the left and the polarization charge with its associated, opposing, electric field on the right.

- In EM we solve $\vec{\nabla}^2\Phi = 0$:

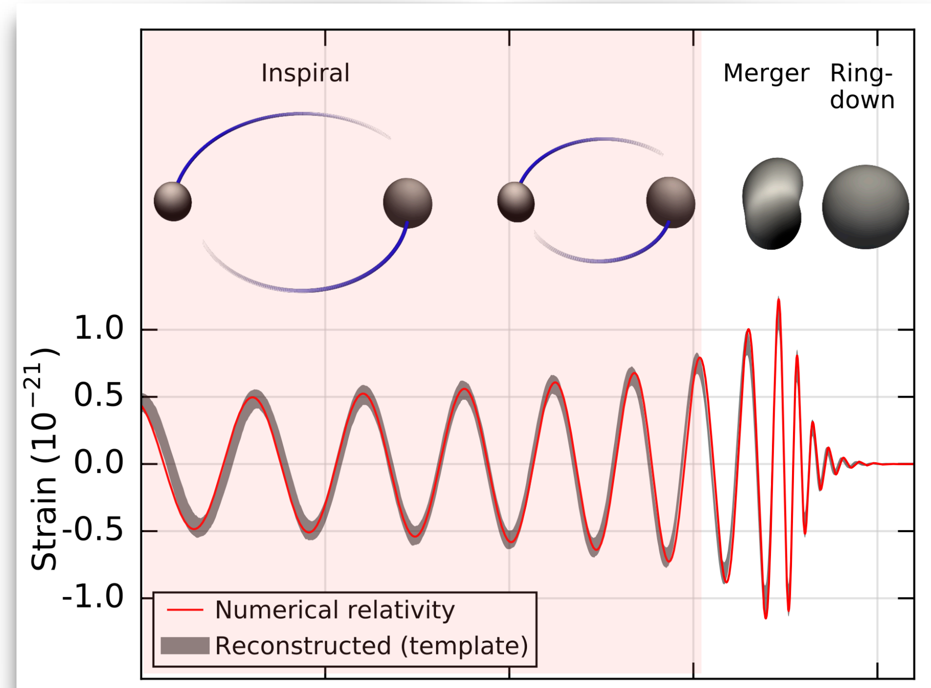
$$\Phi_{\text{ext}} = \sum_\ell A_\ell \left[r^\ell + k_\ell r^{-\ell-1}\right] P_\ell(\cos\theta)\,, \qquad \Phi_{\text{int}} = \sum_\ell B_\ell r^\ell P_\ell(\cos\theta)\,.$$

- The boundary condition at $r = +\infty$ fixes A_ℓ, while k_ℓ and B_ℓ are determined by regularity conditions across the surface (continuity of $\vec{E}_\parallel$ and $\vec{D}_\perp$).
- For instance, if $\vec{E}_0 = A_1\hat{z}$, one finds $k_{\ell=1} = -\dfrac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2} r_0^3$ (ϵ_0 and ϵ are the vacuum and dielectric permittivities).
- k_ℓ are the coefficients of the induced response.

Tidal Love numbers

- Tidal deformability affects the dynamics during the inspiral.

- An alteration in the phase of the GW signal can be used to constrain the tidal deformability of the objects.



- A novel and important channel to test GR and compact objects in the strong-field regime.

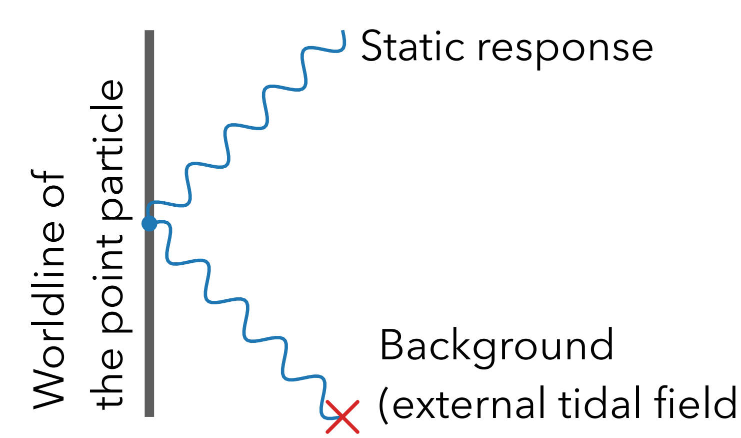
- An explicit calculation in GR (in *D*=4) shows that $k_\ell = 0$ for a black hole, as opposed to other types of compact objects.
[Fang and Lovelace '05], [Binnington and Poisson '09], [Damour and Nagar '09], [Kol and Smolkin '11].

Vanishing Love Numbers

- A conceptually clean way to define the (conservative) LNs is in terms of the worldline effective action [Goldberger and Rothstein '04, '05, …], [Kol and Smolkin '11], [Porto '16].
- At distances large compared to the characteristic size of an object, there is an effective description where the object is modeled as a point particle. Corrections due to the object's finite size and its internal structure are encoded in higher-derivative operators in the effective theory.
- Let's consider e.g. a scalar field around a black hole:

$$S = -\frac{1}{2}\int \mathrm{d}^4x\,(\partial\phi)^2 - M\int \mathrm{d}\tau + \int \mathrm{d}\tau\left[-g\phi + \sum_{\ell=0}^{\infty}\frac{\lambda_\ell}{2\ell\,!}\left(\partial_{(a_1}\cdots\partial_{a_\ell)_T}\phi\right)^2\right].$$

- λ_ℓ are the LN coefficients.
- One generically expects: $\lambda_\ell \sim \mathcal{O}(1) r_s^{2\ell-1}$ and to find (classical) RG running.
- After matching with the UV result: $\lambda_\ell = 0$ in D=4 and no running.
- Generically non-zero in D>4.
  [Kol and Smolkin '11], [Hui, Joyce, Penco, LS and Solomon '21], [Charalambous and Ivanov '23], [Rodriguez, LS, Solomon and Temoche '23]



Vanishing Love Numbers

- Following 't Hooft's naturalness principle, the vanishing of the Love numbers is a naturalness puzzle from an EFT perspective. [Rothstein '14], [Porto '16]

$$S = -\frac{1}{2}\int \mathrm{d}^4x\,(\partial\phi)^2 - M\int \mathrm{d}\tau + \int \mathrm{d}\tau\left[-g\phi + \sum_{\ell=0}^{\infty}\frac{\lambda_\ell}{2\ell\,!}\left(\partial_{(a_1}\cdots\partial_{a_\ell)_T}\phi\right)^2\right]$$

- Looks like something that can very likely follow from a symmetry in the theory.

Symmetries of vanishing Love Numbers

- In [2105.01069] we showed that the vanishing of the Love numbers is the consequence of linearly realized symmetries governing static perturbations around black holes.

- Let's start from the Teukolsky equation with $\omega = 0$ (static limit):

$$\partial_r\left(\Delta\partial_r\phi_\ell^{(s)}\right) + s(2r - r_s)\partial_r\phi_\ell^{(s)} + \left(\frac{a^2m^2 + is(2r - r_s)am}{\Delta} - (\ell - s)(\ell + s + 1)\right)\phi_\ell^{(s)} = 0$$

- We can set $s = 0$ by virtue of ladder operators in s (which generalize the Teukolsky-Starobinsky identities).
- In fact we can also set $\ell = 0$ – ladder operators allow to extend the argument to any ℓ.
- I'll set for simplicity $a = 0$ – the generalization to Kerr is straightforward.

  The equation is simply:

$$\partial_r\left(\Delta\partial_r\phi_0\right) = 0\,, \qquad\qquad \Delta = r(r - r_s)\,,$$

  which is $\Box\phi = 0$ on Schwarzschild with $\omega = 0 = \ell$.
- $P_0 \equiv \Delta\partial_r\phi_0$ is the conserved charge associated with a symmetry of the (static) scalar action.
- It is useful because it allows to connect asymptotics:

$$\phi_0 \sim r^0 \quad \text{as} \quad r \to +\infty \quad \to \quad P_0 = 0 \quad \to \quad \phi_0 \sim \text{const.} \quad \text{as} \quad r \to r_s$$

$$\phi_0 \sim r^{-1} \quad \text{as} \quad r \to +\infty \quad \to \quad P_0 \neq 0 \quad \to \quad \phi_0 \sim \log(r - r_s) \quad \text{as} \quad r \to r_s$$

Symmetries of vanishing Love Numbers

[Hui, Joyce, Penco, LS and Solomon '21]

- Generalization to all ℓs through ladder operators: $\phi_{\ell\pm1} \propto D^{\pm}_{\ell}\phi_\ell$

$$D^{+}_{\ell} \equiv -\Delta\partial_r + \frac{\ell+1}{2}(r_s - 2r), \qquad D^{-}_{\ell} \equiv \Delta\partial_r + \frac{\ell}{2}(r_s - 2r)$$

$$\phi_\ell \sim r^\ell \quad \text{as} \quad r \to +\infty \quad \to \quad P_\ell = 0 \quad \to \quad \phi_\ell \sim \text{const.} \quad \text{as} \quad r \to r_s$$

$$\phi_\ell \sim r^{-(\ell+1)} \quad \text{as} \quad r \to +\infty \quad \to \quad P_\ell \neq 0 \quad \to \quad \phi_\ell \sim \log(r - r_s) \quad \text{as} \quad r \to r_s$$

- The vanishing of the Love numbers follows from two facts: (1) the purely decaying solution ($\sim 1/r^{\ell+1}$ at large r) is divergent at the horizon, and (2) the solution that is regular at the horizon is a finite polynomial going as $\sim 1 + r + \ldots + r^\ell$.

- The growing branch respects the symmetry, while the decaying branch spontaneously breaks the symmetry.
  (See also [Achour, Livine, Mukohyama, Uzan '22])

- Fact (1) is consistent with the no-hair theorem (a black hole cannot sustain static, scalar profile that decays at infinity [Bekenstein '72]).

From Schwarzschild to AdS, with Love

[Hui, Joyce, Penco, LS and Solomon '21]

- The symmetry has a geometric origin: it arises from the (E)AdS isometries of a dimensionally reduced black hole spacetime.

Let's consider a static scalar ϕ in a Schwarzschild background,

$$S = \frac{1}{2}\int \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}r \sqrt{g}\, \phi \Box \phi \,, \qquad \mathrm{d}s^2 = \mathrm{d}r^2 + \Delta \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right).$$

After a Weyl rescaling, the metric becomes EAdS_3 with

$$\tilde{g}_{ij} = \Omega^2 g_{ij} \,, \qquad \tilde{\phi} = \Omega^{-\frac{1}{2}}\phi \,, \qquad \text{where} \qquad \Omega \equiv L^2/\Delta \,,$$

$$S = \frac{1}{2}\int \mathrm{d}^3x \sqrt{\tilde{g}}\left(\tilde{\phi}\,\tilde{\Box}\,\tilde{\phi} + \frac{r_s^2}{4L^4}\tilde{\phi}^2\right), \qquad \mathrm{d}\tilde{s}^2 = \mathrm{d}r_\star^2 + \frac{4L^4}{r_s^2}\sinh^2\!\left(\frac{r_\star r_s}{2L^2}\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right)$$

where $\mathrm{d}r_\star = (L^2/\Delta)\mathrm{d}r$. The space has 6 Killing vectors: 3 rotations and 3 translations (or "boosts"). The translation that mixes $r_\star$ and θ acts on the original ϕ as

$$\delta\phi = -2\Delta\cos\theta\partial_r\phi + (r_s - 2r)\partial_\theta(\sin\theta\,\phi)$$

or, equivalently,

$$\delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1} \,.$$

From Schwarzschild to AdS, with Love

[Hui, Joyce, Penco, LS and Solomon '21]

- At large *r*, $\delta\phi$ reduces to a SCT, $\delta\phi = c_i\left(x^i - \overrightarrow{x}^2\partial^i + 2x^i\overrightarrow{x}\cdot\overrightarrow{\partial}\right)\phi.$
- We claim that this is the sought-after infrared symmetry that forbids Love number (and hair) couplings in the point-particle effective action.

Ladder in Kerr: static limit

[Hui, Joyce, Penco, LS and Solomon '21]

- The previous algebraic ladder structure has a direct analog in a Kerr background:

$$\mathrm{d}s^2 = -\frac{\rho^2 - r_s r}{\rho^2}\mathrm{d}t^2 - \frac{2ar_s r\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\mathrm{d}\varphi^2$$

with $\rho^2 \equiv r^2 + a^2\cos^2\theta$ and $\Delta \equiv r^2 - rr_s + a^2$.

- The static Klein-Gordon equation, $\partial_r(\Delta\partial_r\phi_\ell) + \frac{a^2m^2}{\Delta}\phi_\ell - \ell(\ell+1)\phi_\ell = 0$, has both *ladder* and *horizontal* symmetries.

- The ladder symmetries $D_\ell^\pm$ descend from a CKV of the 3D-static metric:

$$\mathrm{d}s_K^2 = \frac{\rho^2 - rr_s}{\Delta}\left(\mathrm{d}r^2 + \Delta\mathrm{d}\theta^2 + \frac{\Delta^2\sin^2\theta}{\rho^2 - rr_s}\mathrm{d}\varphi^2\right).$$

- $\xi^\mu = (0, \Delta\cos\theta, \frac{1}{2}(2r - r_s)\sin\theta, 0)$ is the CKV that induces

$$\delta\phi = \xi^\mu\partial_\mu\phi + \frac{1}{2}(2r - r_s)\cos\theta\,\phi \qquad \Rightarrow \qquad \delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1}\,,$$

- The conserved charges P_ℓ associated with the horizontal symmetries, evaluated for the "growing branch", are non-zero (and imaginary), unlike in the Schwarzschild case:

$$P_\ell \propto iq\prod_{k=1}^{\ell}\left(k^2 + 4q^2\right)\,, \qquad q \equiv \frac{am}{r_+ - r_-}\,,$$

which reproduces the dissipative response [Le Tiec and Casals '20].

Ladder in Spin: From Scalar to Vector and Tensor

[Hui, Joyce, Penco, LS and Solomon '21, '22]

- Ladder operators in the spin, $E_s^\pm$, raise and lower *s* in the Teukolsky equation ($E_s^\pm \phi_\ell^{(s)} = \phi_\ell^{(s\pm1)}$),

$$\partial_r\Big(\Delta\partial_r\phi_\ell^{(s)}\Big) + s(2r - r_s)\partial_r\phi_\ell^{(s)} + \Big(\frac{a^2m^2 + is(2r - r_s)am}{\Delta} - (\ell - s)(\ell + s + 1)\Big)\phi_\ell^{(s)} = 0\,,$$

- Allow to extend the previous results from scalar to vector and tensor fields.

- $E_s^\pm$ are related to what are known as Teukolsky–Starobinsky identities.
  In Chandrasekhar's notation,

$$\phi^{(-1)} = \Delta\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\Delta\phi^{(1)}, \quad \phi^{(1)} = \mathscr{D}_0\mathscr{D}_0\phi^{(-1)}, \quad \phi^{(-2)} = \Delta^2\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\mathscr{D}_0^\dagger\Delta^2\phi^{(2)}, \quad \phi^{(2)} = \mathscr{D}_0\mathscr{D}_0\mathscr{D}_0\mathscr{D}_0\phi^{(-2)},$$

  where $\mathscr{D}_0 \equiv \partial_r + i[am - \omega(r^2 + a^2)]/\Delta$.

  The new twist we are adding is that, in the static limit, we can truncate these operations, enabling us to increment *s* by unity, $E_s^\pm \phi_\ell^{(s)} = \phi_\ell^{(s\pm1)}$.

Hidden symmetries at finite frequency

[Hui, Joyce, Penco, LS and Solomon '22]

- In binary systems, the induced deformation in a compact object, as well as the perturbing tidal field of the companion, is never exactly static.
  Is it possible to extend the ladder symmetries beyond the static limit?

- The scalar action is

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}r\mathrm{d}\Omega_{S^2}\left[\frac{r^4}{\Delta}(\partial_t\phi)^2 - \Delta(\partial_r\phi)^2 + \phi\,\nabla^2_{\Omega_{S^2}}\phi\right].$$

- Define the *near-zone* approximation by replacing $(r^4/\Delta)\partial_t^2\phi$ with $(r_s^4/\Delta)\partial_t^2\phi$.

- This has the virtue of preserving the correct singularity as $r \to r_s$, while still accurately capturing the dynamics at larger r, as long as $\omega r \ll 1$.

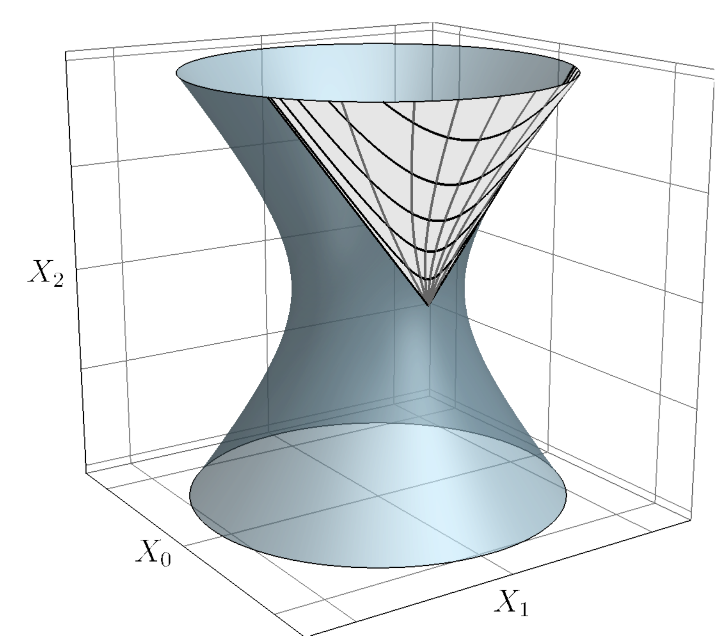
Hidden symmetries at finite frequency

[Hui, Joyce, Penco, LS and Solomon '22]

- In this limit, the scalar action is the same as that of a massless scalar minimally coupled to an *effective near-zone metric*:

$$\mathrm{d}s^2_{\text{near-zone}} = -\frac{\Delta}{r_s^2}\mathrm{d}t^2 + \frac{r_s^2}{\Delta}\mathrm{d}r^2 + r_s^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) .$$

- This metric has the following main properties:
  _ only extension of dynamics at finite ω that retains the (static) ladder generators;
  _ is a conformally-flat $\mathrm{AdS}_2 \times S^2$ spacetime ($\Rightarrow$ 6 KVs + 9 CKVs).



Hidden symmetries at finite frequency

[Hui, Joyce, Penco, LS and Solomon '22]

- The 6 KVs and 9 CKVs are:

$$T = 2r_s\,\partial_t$$
$$L_\pm = e^{\pm t/2r_s}(2r_s\,\partial_r\sqrt{\Delta}\partial_t \mp \sqrt{\Delta}\partial_r)$$
$$J_{23} = \partial_\varphi$$
$$J_{12} = \cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi$$
$$J_{13} = \sin\varphi\,\partial_\theta + \cot\theta\cos\varphi\,\partial_\varphi$$

$$J_{01} = -\frac{2\Delta}{r_s}\cos\theta\,\partial_r - \frac{\partial_r\Delta}{r_s}\sin\theta\,\partial_\theta$$
$$J_{02} = -\cos\varphi\left[\frac{2\Delta}{r_s}\sin\theta\,\partial_r + \frac{\partial_r\Delta}{r_s}\left(\frac{\tan\varphi}{\sin\theta}\partial_\varphi - \cos\theta\partial_\theta\right)\right]$$
$$J_{03} = -\sin\varphi\left[\frac{2\Delta}{r_s}\sin\theta\,\partial_r - \frac{\partial_r\Delta}{r_s}\left(\frac{\cot\varphi}{\sin\theta}\partial_\varphi + \cos\theta\partial_\theta\right)\right]$$
$$K_\pm = e^{\pm t/2r_s}\frac{\sqrt{\Delta}}{r_s}\cos\theta\left(\frac{r_s^3}{\Delta}\partial_t \mp \partial_r\Delta\partial_r \mp 2\tan\theta\partial_\theta\right)$$
$$M_\pm = e^{\pm t/2r_s}\cos\varphi\left[\frac{r_s^2}{\sqrt{\Delta}}\sin\theta\partial_t \mp \frac{\sqrt{\Delta}\partial_r\Delta\sin\theta}{r_s}\partial_r \pm \frac{2\sqrt{\Delta}}{r_s}\cos\theta\partial_\theta \mp \frac{2\sqrt{\Delta}}{r_s}\frac{\tan\varphi}{\sin\theta}\partial_\varphi\right]$$
$$N_\pm = e^{\pm t/2r_s}\sin\varphi\left[\frac{r_s^2}{\sqrt{\Delta}}\sin\theta\partial_t \mp \frac{\sqrt{\Delta}\partial_r\Delta\sin\theta}{r_s}\partial_r \pm \frac{2\sqrt{\Delta}}{r_s}\cos\theta\partial_\theta \pm \frac{2\sqrt{\Delta}}{r_s}\frac{\cot\varphi}{\sin\theta}\partial_\varphi\right]$$

- Different perspective on the vanishing of the LNs proposed by [Charalambous, Dubovsky and Ivanov '21].
- This unifies the different sets of symmetries.
- Only T, J_{ij} and J_{0i} remain good symmetries in the static limit ($\omega = 0$).
- J_{01} recovers precisely the ladders: $\delta\phi = \xi^\mu\partial_\mu\phi + \frac{1}{4}\nabla_\mu\xi^\mu\phi$, or equivalently $\delta\phi_\ell = c_{\ell+1}D^-_{\ell+1}\phi_{\ell+1} - c_\ell D^+_{\ell-1}\phi_{\ell-1}$, $D^+_\ell \equiv -\Delta\partial_r + \frac{\ell+1}{2}(r_s - 2r)$ and $D^-_\ell \equiv \Delta\partial_r + \frac{\ell}{2}(r_s - 2r)$.

II.

Love numbers for rotating black holes in higher dimensions

Love numbers for rotating black holes in higher dimensions

- To understand why the vanishing of black hole Love numbers in general relativity is special and *not generic*, consider a higher-dimensional rotating black hole.
- Qualitative features of Love numbers, such as their multipolar and dimensional dependence, do not often care about the field's spin.
- Scalar field on Myers–Perry spacetime (single plane of rotation) in $(n+4)$-dimensions:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2 + (r^2+a^2)\sin^2\theta\,\mathrm{d}\varphi^2 + \frac{\mu}{r^{n-1}\Sigma}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi)^2 + r^2\cos^2\theta\,\mathrm{d}\Omega_n^2\,,$$

$$\Delta = r^2 + a^2 - \frac{\mu}{r^{n-1}}\,, \qquad \Sigma = r^2 + a^2\cos^2\theta, \qquad D = n+4\,.$$

- Given the symmetries of the metric, we can decompose:

$$\Phi(t,r,\theta,\varphi,\theta_1,\ldots,\theta_n) = \mathrm{e}^{-i\omega t}\mathrm{e}^{im\varphi}\phi(r)S_{\ell m}(\theta)Y_L(\theta_1,\ldots,\theta_n)\,,$$

  where $\ell \geq L + |m|$.

- Radial equation in the static limit:

$$\frac{1}{r^n}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^n\Delta\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + \left[\frac{m^2a^2}{\Delta} - \frac{L(L+n-1)a^2}{r^2} - \ell(\ell+n+1)\right]\phi = 0\,.$$

Love numbers of Myers–Perry black holes in 5D

- Complex singularity structure. Focus on $D = 5$.
- The $D = 5$ equation has three regular singular points at $r^2 = 0$, $\mu - a^2$, ∞, i.e. it can be recast in hypergeometric form:

$$\phi \to r^{-\frac{3}{2}} x^{\frac{2\ell+1}{4}} (1-x)^{\frac{iam}{2\sqrt{\mu - a^2}}} \phi \,, \qquad x \equiv \frac{\mu - a^2}{r^2} \,,$$

$$x(1-x)\phi''(x) + [\mathfrak{c} - (\mathfrak{a} + \mathfrak{b} + 1)x]\phi'(x) - \mathfrak{a}\,\mathfrak{b}\,\phi(x) = 0 \,,$$

$$\mathfrak{a} = 1 + \frac{\ell}{2} + \frac{ia(m-L)}{2\sqrt{\mu - a^2}} \,, \qquad \mathfrak{b} = 1 + \frac{\ell}{2} + \frac{ia(m+L)}{2\sqrt{\mu - a^2}} \,, \qquad \mathfrak{c} = \ell + 2 \,.$$

- Expanding the solution that is regular at the horizon $r = \sqrt{\mu - a^2}$ at large radii, $\phi \sim r^\ell + \lambda_\ell r^{-\ell-2}$:

$$\lambda_\ell = (-1)^\ell \frac{2\Gamma(\mathfrak{a})\Gamma(\mathfrak{b})}{\ell\,!\,\Gamma(\ell+2)\Gamma(\mathfrak{a}-\ell-1)\Gamma(\mathfrak{b}-\ell-1)} \ln\left(\frac{r_0}{r}\right) \in \mathrm{Re} \,.$$

- The Love numbers are non-vanishing and have log running.
  [Rodriguez, LS, Solomon and Temoche '23], [Charalambous and Ivanov '23]

Love numbers of rotating black ring in 5D

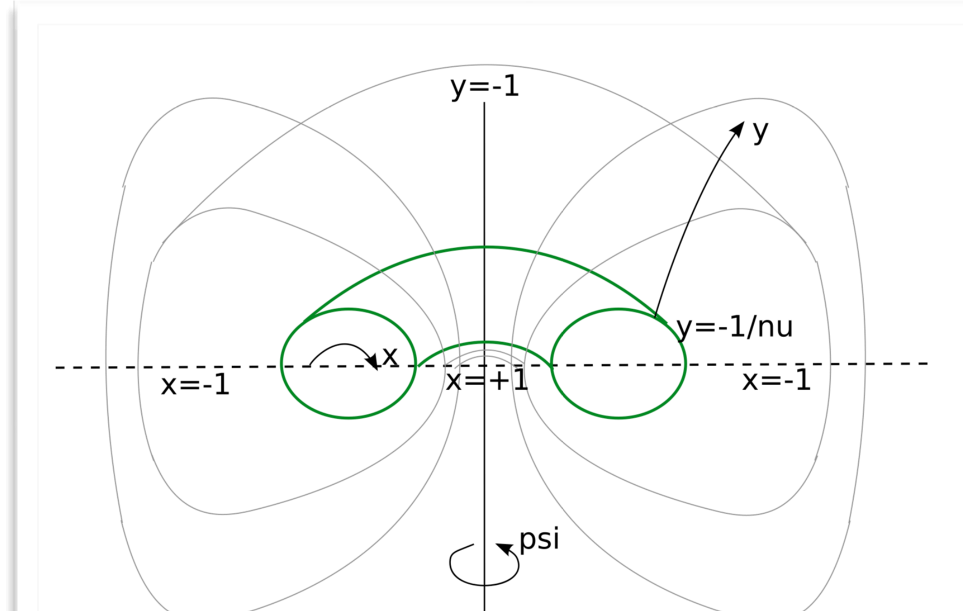
[Rodriguez, LS, Solomon and Temoche '23]

- The landscape of vacuum solutions in higher-dimensional GR is richer than in $4D$. In $D > 4$ there exist black objects with extended horizons (black strings and black p-branes) as well as solutions presenting horizons with non-trivial topology (black rings).
- Black rings are smooth rotating solutions (no conical singularity) with horizon topology $S^1 \times S^2$, where the tension and gravitational self-attraction are balanced by the centrifugal repulsion. [Emparan and Reall '02]
- In suitable coordinates the metric is:

$$\mathrm{d}s^2 = -\frac{\hat{f}}{\hat{g}}\left(\mathrm{d}t - r_0 \sinh\sigma\cosh\sigma\sqrt{\frac{R + r_0\cosh^2\sigma}{R - r_0\cosh^2\sigma}}\,\frac{\frac{r}{R}-1}{r\hat{f}}\,R\,\mathrm{d}\psi\right)^2 +$$

$$+\frac{\hat{g}}{\left(1+\frac{r\cos\theta}{R}\right)^2}\left[\frac{f}{\hat{f}}\left(1-\frac{r^2}{R^2}\right)R^2\mathrm{d}\psi^2 + \frac{\mathrm{d}r^2}{(1-\frac{r^2}{R^2})f} + \frac{r^2}{g}\mathrm{d}\theta^2 + \frac{g}{\hat{g}}r^2\sin^2\theta\,\mathrm{d}\phi^2\right]$$

$$f = 1 - \frac{r_0}{r}, \quad \hat{f} = 1 - \frac{r_0\cosh^2\sigma}{r}, \quad g = 1 + \frac{r_0}{R}\cos\theta, \quad \hat{g} = 1 + \frac{r_0\cosh^2\sigma}{R}\cos\theta.$$



[Schnülle '12]

- The horizon is at $r = r_0$. For thin large rings, R is roughly the radius of the S^1 circle. To avoid conical singularities the spin cannot be arbitrary: $\cosh^2\sigma = 2/(1+(r_0/R)^2)$.

Love numbers of rotating black ring in 5D

[Rodriguez, LS, Solomon and Temoche '23]

- We can decompose:

$$\Phi(t,r,\theta,\phi,z) = \mathrm{e}^{-i\omega t + im\phi + i\nu z}\left(1+\frac{r}{R}\cos\theta\right)\Psi(r,\theta)\,.$$

- The equation becomes separable in the static limit $\omega = 0$.
  The radial equation is:

$$\partial_r\left[r\,(r-r_0)\left(1-\frac{r^2}{R^2}\right)\partial_r\Psi\right] + \nu^2\frac{r^2(r-r_0\cosh^2\sigma)}{(r_0-r)\left(1-\frac{r^2}{R^2}\right)}\Psi - (2r-r_0)\frac{r}{R^2}\Psi = K\Psi\,.$$

- The exact solution involves generalized Heun functions.
  Instead of solving numerically, use method of *matched asymptotic expansion*: first solve in the near and far zones, and then match along a surface of an intermediate overlap region.

- For the ring, analytic solutions in asymptotic regions exist for $\dfrac{r_0}{R} \ll 1$ (large thin ring).

- In this limit, the black ring resembles a black string that has been bent into a circular shape.

Love numbers of rotating black ring in 5D

[Rodriguez, LS, Solomon and Temoche '23]

- Focus on $\frac{r_0}{R} \ll 1$. For the computation of the Love numbers, the complete matching procedure will actually be unnecessary. In the near region, we shall approximate:

$$\Delta \partial_r \left(\Delta_0 \partial_r \Psi \right) + \left(V(r_0) - K \Delta_0 \right) \Psi = 0 \,,$$

$$\Delta \equiv r(r - r_0), \qquad \Delta_0 \equiv r(r - r_0)(1 - r_0^2/R^2) \,,$$

such that the eq. becomes:

$$\partial_r \left[r(r - r_0) \partial_r \Psi \right] + \left[\frac{r_0^2 \mathscr{W}^2}{r(r - r_0)} - \ell(\ell + 1) \right] \Psi = 0, \qquad \mathscr{W} = \frac{\nu \, r_0 \sinh \sigma}{1 - \frac{r_0^2}{R^2}} \,.$$

- The equation can be solved analytically in terms of hypergeometric functions. In the overlap region between near and far zones, $\Psi \sim r^\ell + \lambda_\ell^{\rm BR} r^{-\ell-1}$ and (after analytic continuation in real space for ℓ):

$$\lambda_{\ell \in \mathbb{N}}^{\rm BR} = (-1)^{\ell+1} \frac{\Gamma(\ell + 1)^2 \Gamma(\ell - 2i\mathscr{W} + 1)}{2\,\Gamma(2\ell + 1)\Gamma(2\ell + 2)\Gamma(-\ell - 2i\mathscr{W})} \in \mathrm{Im}$$

- The Love numbers (conservative response) vanish, like for a Kerr black hole in 4D.
- In the limit of zero spin $\sigma = 0$, $\lambda_\ell^{\rm BR}$ recover $\lambda_\ell^{\rm Sch}$ of Schwarzschild 4D black holes.

Love numbers of boosted black string in 5D

[Rodriguez, LS, Solomon and Temoche ’23]

- Add a flat direction z to a Myers–Perry black hole and then boost it with parameter σ:

$$
\begin{aligned}
\mathrm{d}s^2 = &-\left(1-\frac{\mu\, r^{1-n}\cosh^2\sigma}{\Sigma}\right)\mathrm{d}t^2+\frac{\mu\, r^{1-n}\sinh(2\sigma)}{\Sigma}\mathrm{d}t\mathrm{d}z+\left(1+\frac{\mu\, r^{1-n}\sinh^2\sigma}{\Sigma}\right)\mathrm{d}z^2\\
&+\frac{r^2\Sigma}{\Delta}\mathrm{d}r^2+\Sigma\,\mathrm{d}\theta^2+\frac{r^2(r^2+a^2)^2-\Delta a^2\sin^2\theta}{r^2\Sigma}\sin^2\theta\,\mathrm{d}\phi^2\\
&-\frac{2\mu\, r^{1-n}\cosh\sigma}{\Sigma}a\sin^2\theta\,\mathrm{d}t\mathrm{d}\phi-\frac{2\mu\, r^{1-n}\sinh\sigma}{\Sigma-\mu\, r^{1-n}}a\sin^2\theta\,\mathrm{d}z\mathrm{d}\phi+r^2\cos^2\theta\,\mathrm{d}\Omega^2_{S^n}\,,
\end{aligned}
$$

$$
\Delta = r^2(r^2+a^2-\mu\, r^{1-n})\,,\qquad \Sigma = r^2+a^2\cos^2\theta\,,\qquad D=n+5\,.
$$

- The Klein–Gordon equation on this metric admits separation of variables:

$$
\Phi = \mathrm{e}^{im\phi+i\nu z}\Psi(r)S^m_\ell(\theta)Y_L(\Omega)\,.
$$

- The radial equation is: $\dfrac{\Delta}{r^{n+2}}\partial_n\left(r^{n-2}\Delta\partial_r\Psi\right)+[\Delta V_1(r)+V_2(r)]\Psi=0\,.$

Love numbers of boosted black string in 5D

[Rodriguez, LS, Solomon and Temoche '23]

- Let us compute the static response using the method of matched asymptotic expansion.
- Let us first define a near zone valid in the range $r_+ \leq r \ll |1/\nu|$: replace $r \to r_+$ while preserving the structure of the singularity at $r = r_+$,

$$\Delta V_1(r) + V_2(r) \to \Delta V_1(r_+) + V_2(r_+) \, .$$

- The (radial) scalar equation in the near-zone approximation in 5D ($n = 0$, $L = 0$) is

$$\frac{\Delta}{r^3}\partial_r \left[r^{-1}\,\Delta\partial_r\Psi\right] + \Bigg[-\frac{\Delta}{r^2}\left(\kappa^2 r_+^2 + A_{lm}\right) + a^2m^2\cosh^2\sigma + \mu r_+\left(r_+^2 + a^2\right)\nu^2\sinh^2\sigma$$

$$-m^2a^2\sinh^2\sigma - 2\nu m a\mu r_+\sinh\sigma\Bigg]\Psi = 0\,.$$

- This can be solved exactly. The static response can be defined as the coefficients of the decaying falloff in the region that overlaps with the far zone. To leading order in $1/\nu$:

$$\Psi \sim r^\ell + \lambda_\ell r^{-\ell-1}\,, \qquad \lambda_{\ell m} = -\frac{\Gamma(-2\ell)\Gamma(\ell+1)\Gamma(1+\ell+\frac{2iam}{r_+-r_-})}{\Gamma(2\ell+2)\Gamma(-\ell)\Gamma(-\ell+\frac{2iam}{r_+-r_-})}\,,$$

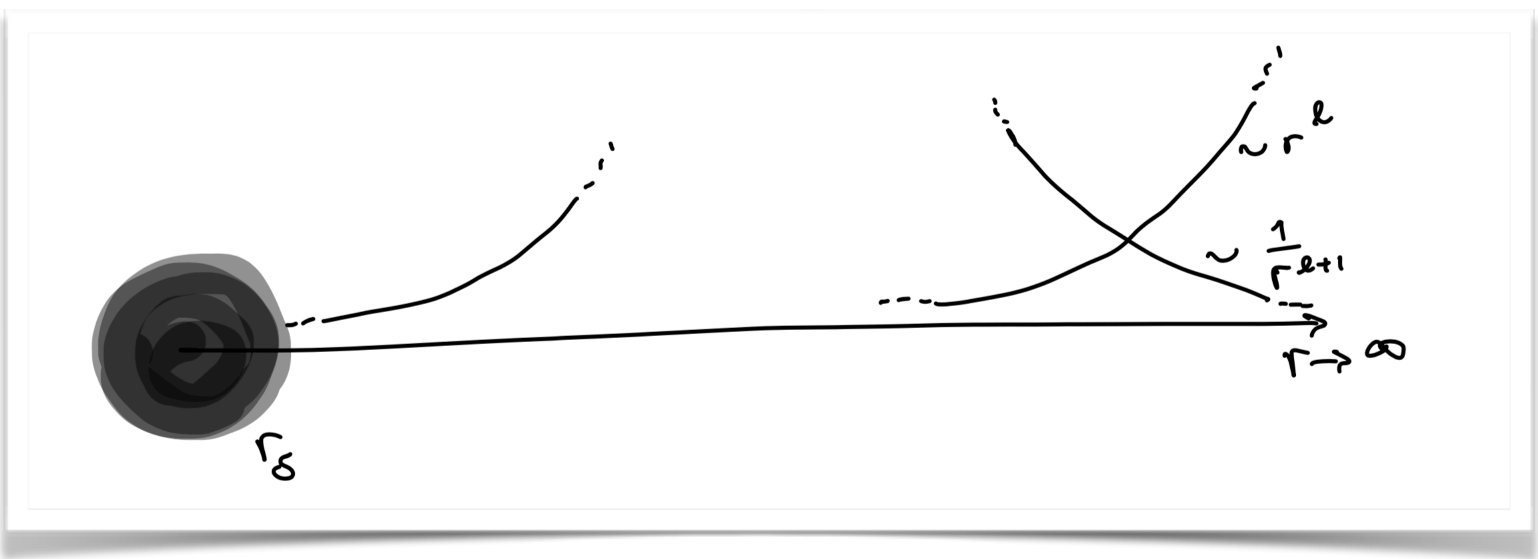
which reproduce the (scalar) static response of a Kerr black hole in 4D.

Love numbers of Kerr black holes

[Rodriguez, LS, Solomon and Temoche '23]

$$\lambda_{\ell m}^{\rm Kerr} = -\frac{\Gamma(-2\ell)\Gamma(\ell+1)\Gamma(1+\ell+\frac{2iam}{r_+-r_-})}{\Gamma(2\ell+2)\Gamma(-\ell)\Gamma(-\ell+\frac{2iam}{r_+-r_-})}$$

- The calculation of Love numbers of Kerr black holes in general relativity ($D=4$) is affected by an intrinsic ambiguity: subleading corrections in the falloff of the source happen to have the same power exponent as the leading tidal response contribution.



- This source/response ambiguity is usually resolved through an analytic continuation in ℓ.
[Le Tiec and Casals '20], [Charalambous, Dubovsky and Ivanov '21]
- Performing the calculation in higher dimensions for a black string with tunable parameter ν has no ambiguity and provides an alternative robust way of deriving $\lambda_{\ell m}^{\rm Kerr}$ in general relativity.

Love numbers of boosted black string in $6D$

[Rodriguez, LS, Solomon and Temoche '23]

- It is possible to extend the previous result to $6D$.
- The (radial) scalar equation in the near-zone approximation ($r_+ \leq r \ll |1/\nu|$) is:

$$\frac{\Delta}{r^3}\partial_r\left(r^{-1}\,\Delta\partial_r\Psi\right)+\left[-\frac{\Delta}{r^2}\left(\nu^2 r_+^2+A_{\ell m}+\frac{a^2L^2}{r^2}\right)+m^2a^2\cosh^2\sigma\right.$$
$$\left.+\mu\left(r_+^2+a^2\right)\nu^2\sinh^2\sigma-m^2a^2\sinh^2\sigma-2\nu m a\mu\sinh\sigma\right]\Psi=0.$$

- This can be solved exactly. The static response coefficients defined in the region that overlaps with the far zone are, to leading order in $1/\nu$:

$$\Psi\sim r^{\ell}+\lambda_\ell r^{-\ell-2}\,,\qquad \lambda_\ell=\left[(-1)^\ell\frac{2\Gamma(\mathfrak{a})\Gamma(\mathfrak{b})}{\ell\,!\,\Gamma(\ell+2)\Gamma(\mathfrak{a}-\ell-1)\Gamma(\mathfrak{b}-\ell-1)}\ln\left(\frac{r_0}{r}\right)\right]_{\nu=0},$$

which reproduce the response of Myers–Perry black holes in $5D$.

Conclusions

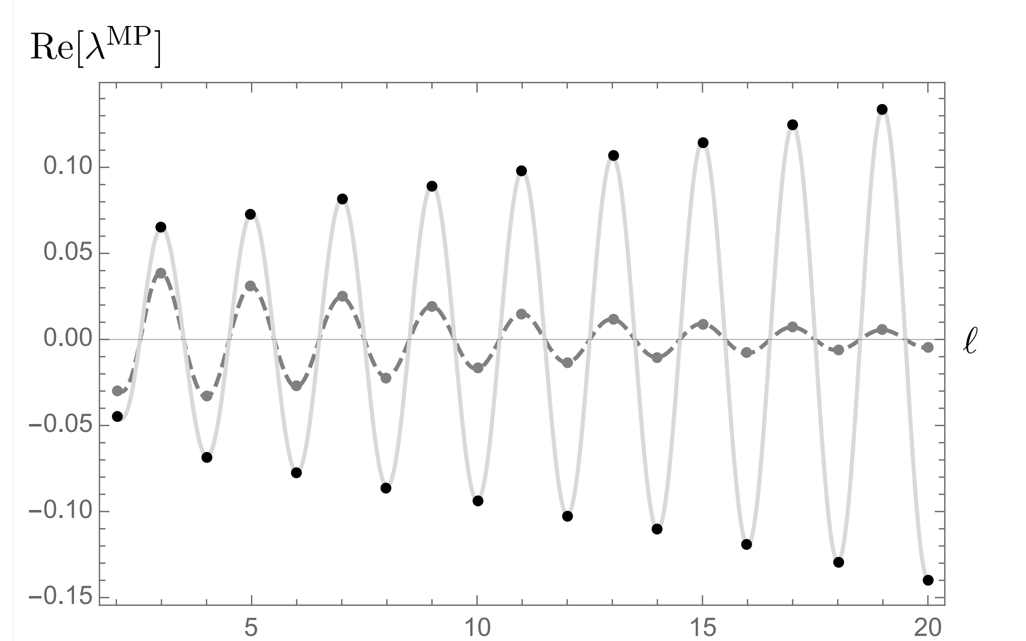
Conclusions and open directions

- Symmetries are key tools to shed light on the fundamental aspects of gravity and compact objects, and constrain broad classes of theories beyond GR.
- Isospectrality and the vanishing of the Love numbers in GR are examples of properties that follow from hidden symmetries in the theory.
- What can we learn from (hidden) symmetries of gravity about the regime beyond linear perturbation theory?

Conclusions and open directions

- The direct detection of GWs is a unique opportunity to test GR in the strong gravity regime.
- Love numbers are a useful observable to characterize and constrain the properties of black holes and neutron stars.
- The study of the static response of higher dimensional rotating black holes is relevant because:
  _ it provides a robust and unambiguous way of computing Love numbers of 4*D* objects;
  _ the rich phenomenology of Love numbers in higher dimensions can teach us important lessons about gravity.
- There seems to be a critical region around $(J/M^2)_{\rm crit} \simeq 0.286$ where the behavior in ℓ of the dissipative coefficients changes.
- This suggests that tidal deformations for the faster spinning Myers–Perry black holes may play an important role in elucidating the stability of these objects.

Love numbers as functions of ℓ for $J/M^2 = 0.26, 0.29$.



Conclusions and open directions

- How about the static response of rotating black holes in $D > 6$? And for higher spins?

- Matching with EFT? (See also [Charalambous and Ivanov '23])
  Black holes solutions are no longer unique in vacuum, hence a complete "point-particle" EFT interpretation should reflect this fact.