Institut d'Astrophysique de Paris
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RELATIVISTIC KHRONON THEORY for MOND & LARGE-SCALE COSMOLOGY

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Based on a recent collaboration with

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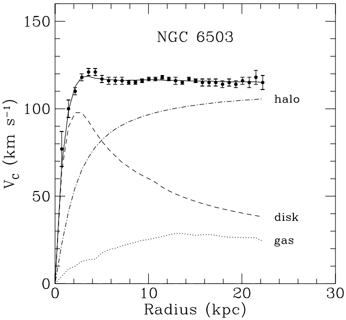
Plan of the talk

1. The MOND empirical formula
2. Relativistic theory based on the Khronon
3. Non relativistic limit and MOND
4. Agreement with large scale cosmology

THE MOND EMPIRICAL FORMULA

Rotation curves of galaxies are flat [Bosma 1981; Rubin 1982]



- For a circular orbit we expect

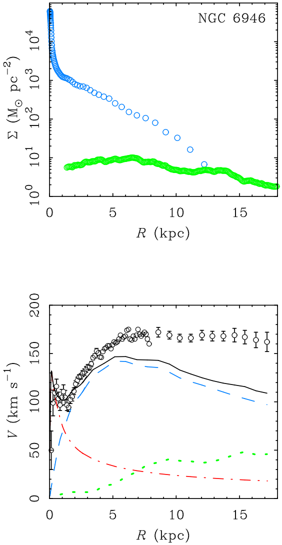
$$v(r) = \sqrt{\frac{G\,M(r)}{r}}$$

- The fact that $v(r)$ is constant implies that beyond the optical disk

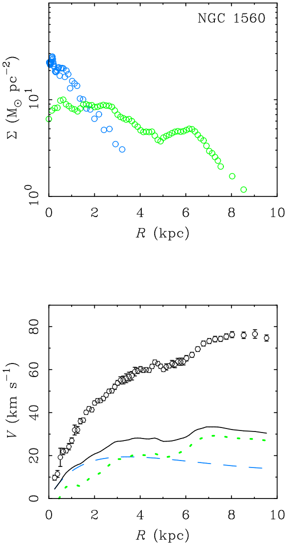
$$M_{\text{halo}}(r) \simeq r \qquad \rho_{\text{halo}}(r) \simeq \frac{1}{r^2}$$

HSB versus LSB galaxies

High Surface Brightness



Low Surface Brightness

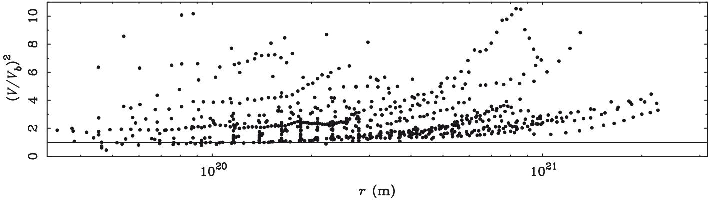


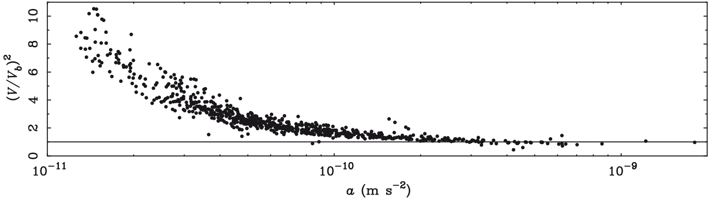
Challenges with cold dark matter at galactic scales

[McGaugh & Sanders 2004; Famaey & McGaugh 2012]

1. Unobserved predictions
   - Numerous but unseen satellites of large galaxies
   - Phase-space correlation of galaxy satellites
   - Generic formation of dark matter cusps in galaxies
   - Tidal dwarf galaxies dominated by dark matter
2. Unpredicted observations
   - Correlation between mass discrepancy and acceleration
   - Surface brightness of galaxies and the Freeman limit
   - Flat rotation curves of galaxies
   - Baryonic Tully-Fisher relation for spirals
   - Faber-Jackson relation for ellipticals

Mass discrepancy versus acceleration [Milgrom 1983]





A critical acceleration scale $a_0 \simeq 1.2 \times 10^{-10} \mathrm{m/s^2}$ is present in the data

The critical acceleration scale a_0

1. The mass discrepancy is given by

$$\frac{M_{\text{dyn}}}{M_{\text{b}}} \simeq \left(\frac{V}{V_{\text{b}}}\right)^2 \sim \sqrt{\frac{a_0}{g}}$$

2. The scale a_0 gives the transition between galaxies with high and low central surface brightness

$$\Sigma \gtrsim \frac{a_0}{G} \quad \text{for HSB galaxies (baryon dominated)}$$
$$\Sigma \lesssim \frac{a_0}{G} \quad \text{for LSB galaxies (DM dominated)}$$

3. The measured value of a_0 is very close to typical cosmological values

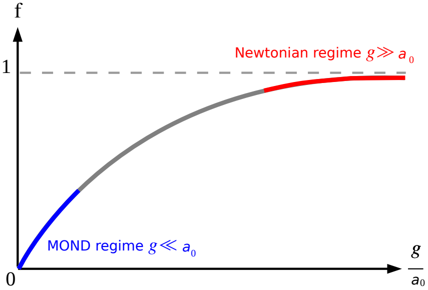
$$a_0 \simeq 1.3\, a_\Lambda \quad \text{with} \quad \overbrace{a_\Lambda = \frac{c^2}{2\pi}\sqrt{\frac{\Lambda}{3}}}^{\text{[Gibbons \& Hawking 1977]}}$$

The MOND formula [Milgrom 1983abc; Bekenstein & Milgrom 1984]

The previous challenges are mysteriously solved by the MOND empirical formula

$$\boxed{\nabla \cdot \Big[\underbrace{f\left(\frac{g}{a_0}\right)}_{\text{MOND function}} \boldsymbol{g} \Big] = -4\pi\, G\, \rho_{\text{baryon}} \quad \text{with} \quad \boldsymbol{g} = \nabla U}$$

- The Newtonian regime is recoved when $g \gg a_0$
- In the MOND regime $g \ll a_0$ we have $\boxed{f \simeq g/a_0}$



Flat rotation curves and baryonic Tully-Fisher relation

- For a spherical mass $g^2/a_0 = g_\mathrm{N}$ with $g_\mathrm{N} = \frac{G\,M}{r^2}$ hence

$$\boxed{g \approx \frac{\sqrt{G\,M\,a_0}}{r} \quad \text{or} \quad U \approx -\sqrt{G\,M\,a_0}\ln r}$$

- For circular motion $\frac{v^2}{r} = g$ thus v is constant and we get

$$\boxed{v_\mathrm{flat} \approx \left(G\,M\,a_0\right)^{1/4}}$$

which naturally explains the baryonic Tully-Fisher relation

- The MOND transition radius (at which $g = a_0$) is

$$r_M \approx \sqrt{\frac{GM}{a_0}}$$

We have $r_M \sim 7100\,\text{AU}$ for the Sun and $r_M \sim 10\,\mathrm{kpc}$ for the Galaxy

The external field effect in MOND [Milgrom 1983; Bekenstein & Milgrom 1984]

- An isolated galaxy in MOND has a logarithmic potential and this would mean that there is no escape velocity from that galaxy
- However the galaxy is embedded into the external field g_e of other galaxies and the MOND potential becomes asymptotically Newtonian

$$U \approx \frac{GM/f_\mathrm{e}}{r} \qquad \text{when} \qquad g \lesssim g_\mathrm{e}$$

- This is a particular case of the external field effect (EFE) due to the non-linearity of the MOND equation

$$f\left(\frac{|\boldsymbol{g}+\boldsymbol{g}_\mathrm{e}|}{a_0}\right)\boldsymbol{g} \approx \boldsymbol{g}_\mathrm{N}$$

- In addition the potential is deformed in the direction of the external field

$$\boxed{U = \boldsymbol{g}_\mathrm{e}\cdot\mathbf{x} + \frac{GM/f_\mathrm{e}}{r\sqrt{1+f'_\mathrm{e}\sin^2\theta}} + \mathcal{O}\left(\frac{1}{r^2}\right)}$$

Some failures of the MOND formula

1. Large scales
   - Does not account for all the dark matter measured in galaxy clusters [Gerbal, Durret *et al* 1992; Sanders 1999]
2. Stellar system scales
   - EFE predicts an orbital precession of outer planets in the Solar system constrained by planetary ephemerides [Milgrom 2009; Blanchet & Novak 2011; Hees *et al.* 2014]
   - Fails to reproduce the orbital distribution of long period comets and trans-Neptunian objects [Vokrouhlický, Nesvorný & Tremaine 2024]
   - Debate on the EFE and dynamics of wide binaries seen by GAIA [Hernandez 2023; Chae 2023; 2024] (pros) [Banik, Famaey, Ibata *et al.* 2023] (cons)

Different approaches to the DM problem

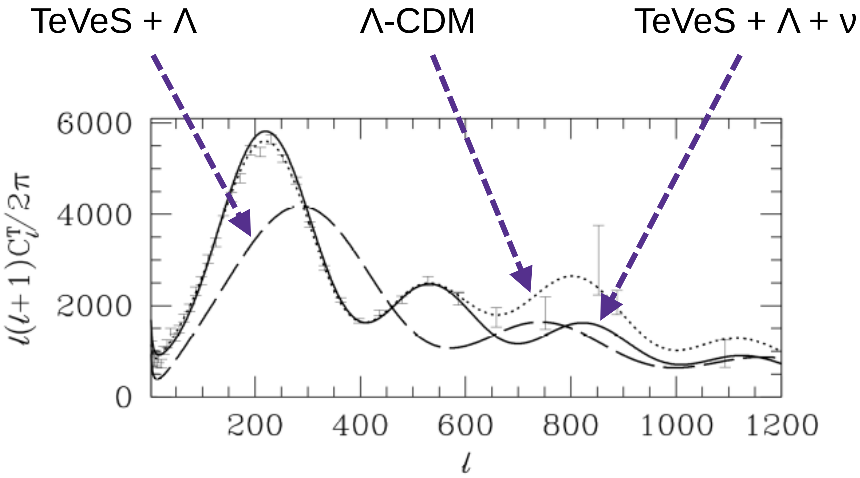
1. Standard: MOND could be explained within the Λ-CDM model

2. Modification of gravity: There is a fundamental modification of the law of gravity in a regime of weak gravity
   - Tensor-vector-scalar theory (TeVeS) [Bekenstein 2004; Sanders 2005]
   - Aether-scalar-tensor theory (AeST) [Skordis & Złośnik 2021]
   - Relativistic Khronon theory [this talk]

3. Non-standard dark matter: DM is endowed with special non-standard properties which make it able to explain MOND [Blanchet & Le Tiec 2008, 2009]

Importance of matching the standard cosmology

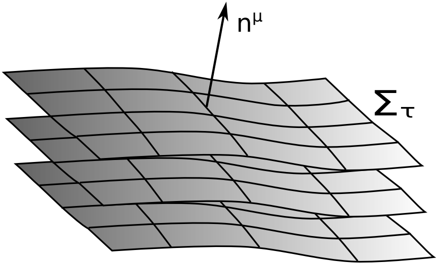
[Skordis, Mota, Ferreira & Boehm 2006]



RELATIVISTIC THEORY BASED ON THE KHRONON[1]

[1] Chronos or Khronos (Χρονος) is the personification of time in ancient Greece, not to be confused with the paronym Cronos or Kronos (king of the Titans, father of Zeus)

Space-time foliation by spatial hypersurfaces



- Assume a foliation of space-time by a family of spatial hypersurfaces with hypersurface orthogonal vector field

$$n_\mu = -\frac{c}{\mathcal{Q}}\partial_\mu \tau$$

where the scalar field τ is the Khronon

$$\mathcal{Q} = c\sqrt{-g^{\rho\sigma}\partial_\rho\tau\partial_\sigma\tau}$$

- The space like acceleration of the congruence of worldlines n_μ is

$$A_\mu = n^\nu \nabla_\nu n_\mu = -q^\nu_\mu \nabla_\nu \ln \mathcal{Q}$$

where the projector onto the foliation is $q^\nu_\mu = \delta^\nu_\mu + n_\mu n^\nu$

Coordinates adapted to the foliation

- Adopting a coordinate system for which $t \equiv x^0/c = \tau$ is equal to the Khronon field we have

$$\mathcal{Q} = \frac{1}{N} \quad \text{where} \quad N = (-g^{00})^{-1/2} \quad \text{is the lapse}$$

- The unit vector is $n_\mu = (-N, \mathbf{0})$ and introducing the shift $N_i = g_{0i}$ and spatial metric $q_{ij} = g_{ij}$ we have the usual 3+1 form for the metric

$$\boxed{\mathrm{d}s^2 = -c^2 N^2 \mathrm{d}t^2 + q_{ij}\left(\mathrm{d}x^i + cN^i\mathrm{d}t\right)\left(\mathrm{d}x^j + cN^j\mathrm{d}t\right)}$$

- To fix notations GR takes the form

$$S_{\text{GR}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{q}N\Big[\mathcal{R} + K_{ij}K^{ij} - K^2\Big] + S_m$$

where $\mathcal{R}$ is the $3d$ scalar curvature and K_{ij} the extrinsic curvature

$$K_{ij} = \frac{1}{2N}\left(\frac{1}{c}\partial_t q_{ij} - D_i N_j - D_j N_i\right)$$

Hořava-Lifshitz gravity [Hořava 2009; Blas, Pujolas & Sibiryakov 2010, 2011]

$$S_{\text{HL}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{q}N\Big[\mathcal{R} + (1-\beta)K_{ij}K^{ij} - (1+\lambda)K^2 - \mathcal{V}\big(q_{ij}, A^i\big)\Big] + S_m$$

- Motivated by quantum gravity in the high energy (UV) limit
- Arbitrary constants λ, β labeling local Lorentz invariance (LLI) violation
- "Potential" $\mathcal{V}(q_{ij}, A^i)$ contains terms of high order (fourth and sixth) in spatial derivatives of q_{ij} but with no time derivatives [e.g. the Cotton tensor squared $C_{ij}C^{ij}$] to ensure the power-counting renormalizability
- Dependence on acceleration $A^i = D^i \ln N$ added to provide stability of the scalar degree of freedom

$$\boxed{\mathcal{V}\big(q_{ij}, A^i\big) = -2\alpha\mathcal{Y} + \cdots} \quad \text{where} \quad \mathcal{Y} \equiv \frac{A_i A^i}{c^4} \quad \text{and} \quad 0 < \alpha < 1$$

Hořava-Lifshitz gravity [Hořava 2009; Blas, Pujolas & Sibiryakov 2010, 2011]

- The effective Newton's constant in the low energy limit is

$$G_{\mathrm{N}} = \frac{G}{1-\alpha}$$

- The PPN parameters of the theory are the same as for GR except for the preferred-frame parameters α_1 and α_2 given by [Blas & Sanctuary 2011]

$$\alpha_1 = -\frac{8(\alpha-\beta)}{1-\beta}\,,$$
$$\alpha_2 = \frac{(\alpha-\beta)\big[2\alpha(1+\beta+2\lambda)-\beta(3+\beta+3\lambda)-\lambda\big]}{(1-\alpha)(1-\beta)(\beta+\lambda)}\,.$$

- The cosmological equations (on a FLRW background) are modified

$$3\Big(1+\frac{\beta}{2}+\frac{3\lambda}{2}\Big)H^2+\frac{3\kappa}{a^2}=8\pi G\sum_I\overline{\rho}_I\,,$$
$$-\Big(1+\frac{\beta}{2}+\frac{3\lambda}{2}\Big)\left(2\dot{H}+3H^2\right)-\frac{\kappa}{a^2}=8\pi G\sum_I\overline{P}_I\,.$$

Simple relativistic MOND theory

[Blanchet & Marsat 2011; Sanders 2011; Bonetti & Barausse 2015; Flanagan 2023]

- Since MOND is a modification of gravity in the weak-acceleration regime it is natural to build a theory using the acceleration vector A^i and we pose

$$\mathcal{Y} \equiv \frac{A_i A^i}{c^4} \qquad A^i = D^i \ln N$$

- Theory inspired by Hořava gravity but with a completely different motivation: DM and the weak, low acceleration regime rather than the high energy limit

$$\boxed{S = \frac{c^3}{16\pi G} \int \mathrm{d}^4 x \sqrt{q} N \Big[\mathcal{R} + K_{ij} K^{ij} - K^2 - 2\mathcal{J}(\mathcal{Y}) \Big] + S_m[\Psi, N, N_i, q_{ij}]}$$

where N, N_i, γ_{ij} are the only dynamical variables

- Covariant formulation with the metric $g_{\mu\nu}$ and the Khronon τ

$$\boxed{S = \frac{c^3}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} \Big[R - 2\mathcal{J}(\mathcal{Y}) \Big] + S_m \big[\Psi, g\big]}$$

Simple relativistic MOND theory

[Blanchet & Marsat 2011; Sanders 2011; Bonetti & Barausse 2015; Flanagan 2023]

- The theory reproduces MOND in the low acceleration regime for stationary systems and standard GR for high accelerations with the function

$$\mathcal{J}(\mathcal{Y}) = \begin{cases} 0 & \text{when} \quad \mathcal{Y} \gg a_0^2/c^4 \\ \Lambda - \mathcal{Y} + \dfrac{2c^2}{3a_0}\mathcal{Y}^{3/2} + \mathcal{O}\left(\mathcal{Y}^2\right) & \text{when} \quad \mathcal{Y} \ll a_0^2/c^4 \end{cases}$$

  where Λ is the cosmological constant
- It has $\alpha = \beta = \lambda = 0$ in the PN regime $\Longrightarrow$ same PPN parameters as GR
- It predicts the same light deflection and gravitational lensing as GR, *i.e.* for any matter distribution whose dynamics shows the presence of DM, the gravitational lensing will also conclude at the same DM
- However the theory does not explain the effects attributed to DM which occur at large scales in cosmological perturbations

Elaborated relativistic MOND theory [Blanchet & Skordis 2024]

- In the shift-symmetric k-essence model an action for a scalar field τ which takes the form $(\mathcal{Q}-1)^2$ leads to approximate dust solutions in cosmology [Scherrer 2004; Arkani-Hamed, Cheng & Mukohyama 2004]
- With respect to the previous model we add a kinetic term for the Khronon

$$\mathcal{K}(\mathcal{Q}) = \overbrace{\mu^2\left(\mathcal{Q}-1\right)^2 + \cdots}^{\text{to be further specified later}}$$

$$\text{where} \quad \mathcal{Q} = c\sqrt{-g^{\mu\nu}\partial_\mu\tau\partial_\nu\tau}$$

- Such a term in the action plays a crucial role in AeST theory to fit cosmological observations [Skordis & Złośnik 2021]
- The dynamical variables are $g_{\mu\nu}$ and τ and the covariant formulation is

$$\boxed{S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,\Big[R - 2\mathcal{J}(\mathcal{Y}) + 2\mathcal{K}(\mathcal{Q})\Big] + S_m\left[\Psi, g\right]}$$

Elaborated relativistic MOND theory [Blanchet & Skordis 2024]

- Einstein field equations ($\mathcal{J}_\mathcal{Y} = \frac{\mathrm{d}\mathcal{J}}{\mathrm{d}\mathcal{Y}}$, $\mathcal{K}_\mathcal{Q} = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}\mathcal{Q}}$)

$$G^{\mu\nu} = \frac{8\pi G}{c^4}\Big(T^{\mu\nu} + \mathcal{T}^{\mu\nu}\Big)$$

$$\boxed{\mathcal{T}^{\mu\nu} = \frac{c^4}{8\pi G}\left\{-\left(\mathcal{J} - \mathcal{K}\right) g^{\mu\nu} + \frac{2}{c^4}\mathcal{J}_\mathcal{Y} A^\mu A^\nu + \left[-\frac{2}{c^2}\nabla_\rho\left(\mathcal{J}_\mathcal{Y} A^\rho\right) + \mathcal{Q}\mathcal{K}_\mathcal{Q}\right] n^\mu n^\nu\right\}}$$

- Khronon equation takes the form of a conserved current

$$\nabla_\mu \mathcal{S}^\mu = 0$$

$$\boxed{\mathcal{S}^\mu = \frac{2}{\mathcal{Q}}\mathcal{J}_\mathcal{Y}\, A^\mu\, n^\nu \nabla_\nu \ln \mathcal{Q} + \left[\frac{2}{\mathcal{Q}}\nabla_\nu\left(\mathcal{J}_\mathcal{Y}\, A^\nu\right) - c^2\,\mathcal{K}_\mathcal{Q}\right] n^\mu}$$

which can be derived as a consequence of $\nabla_\nu \mathcal{T}^{\mu\nu} = 0$ (*via* the Bianchi identity and the conservation of matter)

NON RELATIVISTIC LIMIT AND MOND

Non-relativistic slow motion limit

- Usual ansatz for the PN expansion of the metric of an isolated system

$$N = 1 + \frac{\phi}{c^2} + \mathcal{O}(c^{-4})$$
$$N_i = \frac{4}{c^3}\,\zeta_i + \mathcal{O}(c^{-5})$$
$$q_{ij} = \gamma_{ij}\left(1 - \frac{2\psi}{c^2}\right) + \mathcal{O}(c^{-4})$$

  motivated in GR by the PN order of the matter tensor $T^{\mu\nu} = \mathcal{O}(c^2, c, c^0)$

- We must ensure that the Khronon tensor admits the same leading PN order $\mathcal{T}^{\mu\nu} = \mathcal{O}(c^2, c, c^0)$ which is satisfied with the Khronon perturbation

$$\tau = t + \frac{\sigma}{c^2} + \mathcal{O}(c^{-4})$$

- The PN ansatz is incompatible with adapted coordinates (unitary gauge) as the shift acquires an overdominant PN term $\partial_i\sigma/c$ [Flanagan 2023]

Non-relativistic slow motion limit

- The *ij* components of the Einstein field equations give

$$\boxed{\phi = \psi + \mathcal{O}(c^{-2})}$$

The equatility of the two potentials shows that the theory is a viable alternative to DM concerning light deflection and gravitational lensing

- The 00 component gives (with $\Xi \equiv \phi - \dot{\sigma} + \frac{1}{2}|\vec{\nabla}\sigma|^2$)

$$\Delta\phi + \vec{\nabla}\cdot\left(\mathcal{J}_\mathcal{Y}\,\vec{\nabla}\Xi\right) + \mu^2\,\Xi = 4\pi G\rho_m + \mathcal{O}(c^{-2})$$

- The 0*i* components are

$$\Delta\zeta^i - \vec{\nabla}^i\vec{\nabla}_j\zeta^j - \vec{\nabla}^i\dot{\phi} - \left[\vec{\nabla}\cdot\left(\mathcal{J}_\mathcal{Y}\,\vec{\nabla}\Xi\right) + \mu^2\Xi\right]\vec{\nabla}^i\sigma = 4\pi G\rho_m v_m^i + \mathcal{O}(c^{-2})$$

Non-relativistic slow motion limit

- We introduce the Khronon “mass density” and “velocity” field [Flanagan 2023]

$$\rho_\tau \equiv -\frac{1}{4\pi G}\left[\vec{\nabla}\cdot\left(\mathcal{J}_\mathcal{Y}\vec{\nabla}\Xi\right)+\mu^2\Xi\right] \qquad\qquad v^i_\tau \equiv -\vec{\nabla}^i\sigma\,.$$

- We rewrite the Einstein field equations in the form

$$\Delta\phi = 4\pi G\left(\rho_m+\rho_\tau\right)+\mathcal{O}(c^{-2})$$

$$\Delta\zeta^i - \vec{\nabla}^i\vec{\nabla}_j\zeta^j - \vec{\nabla}^i\dot{\phi} = 4\pi G\left(\rho_m v^i_m + \rho_\tau v^i_\tau\right)+\mathcal{O}(c^{-2})$$

- These imply together with the continuity equation for matter, the continuity equation for the Khronon “fluid”

$$\dot{\rho}_\tau + \vec{\nabla}_i\left(\rho_\tau v^i_\tau\right) = \mathcal{O}(c^{-2})$$

- This equation is nothing but the slow motion approximation of the Khronon field equation $\nabla_\mu\mathcal{S}^\mu = 0$

Recovering the MOND equation for stationary systems

- For stationary systems the continuity equation for matter reduces to the constraint $\vec{\nabla}_i\left(\rho_m v_m^i\right)=0$ and the Khronon equation reduces to $\vec{\nabla}\cdot(\rho_\tau\vec{\nabla}\sigma)=0$ which we can solve by choosing $\sigma=0$
- With this choice the acceleration reduces to the Newtonian acceleration

$$A_0=\mathcal{O}(c^{-3})$$
$$A_i=\vec{\nabla}_i\phi+\mathcal{O}(c^{-2})$$

- The equation for the Newtonian potential ϕ becomes the modified Poisson (or modified Helmholtz) equation

$$\boxed{\vec{\nabla}\cdot\Big[\left(1+\mathcal{J}_{\mathcal{Y}}\right)\vec{\nabla}\phi\Big]+\underbrace{\mu^2\phi}_{\text{mass term}}=4\pi G\rho_m+\mathcal{O}(c^{-2})}$$

- From the first term we identify the MOND interpolating function as

$$\boxed{f\left(\sqrt{\mathcal{Y}}\right)=1+\mathcal{J}_{\mathcal{Y}}(\mathcal{Y})}$$

where $\sqrt{\mathcal{Y}}\equiv|\nabla\phi|/c^2$ with this approximation

Phenomenology of spherically symmetric solutions

- We obtain the same conditions on the function $\mathcal{J}$ as in the previous model

$$\mathcal{J}(\mathcal{Y}) = \begin{cases} 0 & \text{when} \quad |\nabla\phi| \gg a_0 \\ \Lambda - \frac{|\nabla\phi|^2}{c^4} + \frac{2}{3a_0}\frac{|\nabla\phi|^3}{c^4} + \cdots & \text{when} \quad |\nabla\phi| \ll a_0 \end{cases}$$

- The MOND equation [Bekenstein & Milgrom 1984] emerges when the distance to the center of a galaxy becomes larger than the MOND transition radius

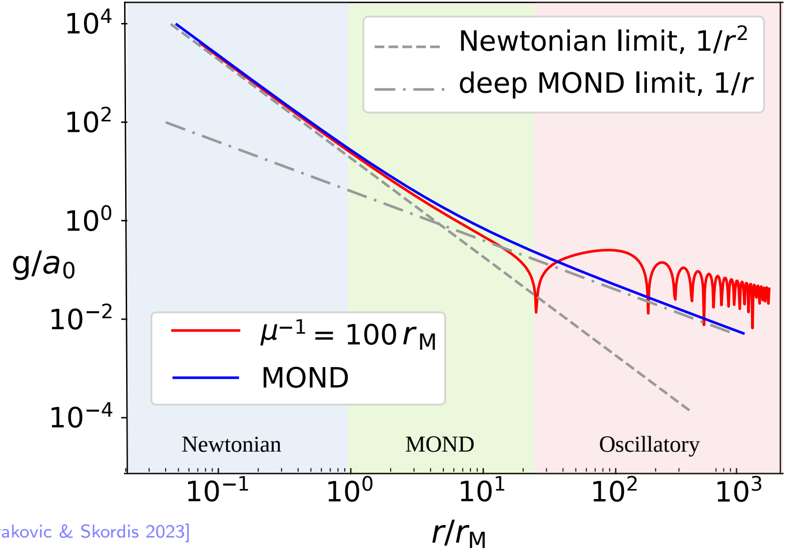
$$r_M \sim \sqrt{\frac{GM}{a_0}} \qquad r_M \sim 10\,\text{kpc}$$

- Furthermore the effect of the "mass term" is to induce an oscillatory behaviour for the potential ϕ on scales larger than the critical scale

$$r_C \sim \left(\frac{r_M}{\mu^2}\right)^{1/3} \qquad r_C \sim 200\,\text{kpc} \quad \text{for} \quad \mu^{-1} \sim 1\,\text{Mpc}$$

- This is similar to the case of the AeST theory [Verwayen, Skordis & Boehm 2023]

Phenomenology of spherically symmetric solutions



[Durakovic & Skordis 2023]

AGREEMENT WITH LARGE-SCALE COSMOLOGY

Friedmann-Lemaître-Robertson-Walker cosmology

- The Friedmann equations on the FLRW background read

$$3H^2 + \frac{3\kappa}{a^2} - \Lambda = 8\pi G \sum_{I\neq\mathcal{K}} \overline{\rho}_I + \overbrace{\overline{\mathcal{Q}}\,\overline{\mathcal{K}}_{\mathcal{Q}} - \overline{\mathcal{K}}}^{\text{Khronon contributions}}$$

$$-\left(2\dot{H} + 3H^2 + \frac{\kappa}{a^2}\right) + \Lambda = 8\pi G \sum_{I\neq\mathcal{K}} \overline{P}_I + \overline{\mathcal{K}}$$

- The Khronon contributions make it possible to defining the Khronon energy density and pressure on the FLRW background

$$\left.\begin{aligned}\overline{\rho}_{\mathcal{K}} &= \frac{1}{8\pi G}\left(\overline{\mathcal{Q}}\,\overline{\mathcal{K}}_{\mathcal{Q}} - \overline{\mathcal{K}}\right)\\ \overline{P}_{\mathcal{K}} &= \frac{1}{8\pi G}\,\overline{\mathcal{K}}\end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} w &= \frac{\overline{\mathcal{K}}}{\overline{\mathcal{Q}}\,\overline{\mathcal{K}}_{\mathcal{Q}} - \overline{\mathcal{K}}}\\ c^2_{\text{ad}} &= \frac{\overline{\mathcal{K}}_{\mathcal{Q}}}{\overline{\mathcal{Q}}\,\overline{\mathcal{K}}_{\mathcal{Q}\mathcal{Q}}}\end{aligned}}$$

Approximate dust solutions [Scherrer 2004; Arkani-Hamed, Cheng & Mukohyama 2004]

- The Khronon equation $\nabla_\mu \mathcal{S}^\mu = 0$ reduces to $\partial_t(a^3\,\overline{\mathcal{K}}_\mathcal{Q}) = 0$ hence

$$\boxed{\overline{\mathcal{K}}_\mathcal{Q} = \frac{I_0}{a^3}}$$

where I_0 is a constant set by initial conditions

- Posing $w_0 = \frac{I_0}{4\mu^2}$ and focusing on the quadratic potential $\mathcal{K}(\mathcal{Q}) = \mu^2(\mathcal{Q}-1)^2$ we get $\overline{\mathcal{Q}} = 1 + \frac{2w_0}{a^3}$ and the energy density as

$$\boxed{\overline{\rho}_\mathcal{K} = \frac{\mu^2 w_0}{2\pi G a^3}\left(1 + \frac{w_0}{a^3}\right)}$$

- The Khronon field behaves approximately as dust in the FLRW background, with the equation of state evolving as

$$w = \frac{w_0}{w_0 + a^3}$$

- In particular $w \to 1$ in the early Universe when $a \to 0$

Linear perturbations on the FLRW background

- Keeping only scalar modes ($\hat{D}_{ij} \equiv \vec{\nabla}_i \vec{\nabla}_j - \frac{1}{3}\gamma^{\kappa}_{ij}\vec{\nabla}^2$)

$$\boxed{\mathrm{d}s^2 = -(1+2\phi)\,\mathrm{d}t^2 - 2\vec{\nabla}_i\zeta\,\mathrm{d}t\,\mathrm{d}x^i + a^2\Big[(1-2\psi)\,\gamma^{\kappa}_{ij} + \hat{D}_{ij}\nu\Big]\mathrm{d}x^i\mathrm{d}x^j}$$

- Linear perturbation of the Khronon $\boxed{\tau = \overline{\tau} + \sigma}$
- Defining $\Upsilon \equiv \phi - \partial_t(\frac{\sigma}{\mathcal{Q}})$ the space-time congruence is

$$n_0 = -1 - \phi \qquad n_i = -\frac{1}{\overline{\mathcal{Q}}}\vec{\nabla}_i\sigma$$

$$A_0 = 0 \qquad A_i = \vec{\nabla}_i\Upsilon$$

- Since the acceleration A_μ vanishes in the background Υ is gauge invariant, while the gauge invariant variables associated to the metric are

$$\Psi \equiv \psi + \frac{1}{6}\vec{\nabla}^2\nu + H\left(\zeta + \frac{a^2}{2}\dot{\nu}\right)$$

$$\Phi \equiv \phi - \partial_t\left(\zeta + \frac{a^2}{2}\dot{\nu}\right)$$

The generalized dark matter model [Hu 1998; Kopp, Skordis & Thomas 2016]

- This model is defined only on the FLRW background and linearized perturbation level and determined by three parametric functions: the equation of state $w(t)$, the sound speed $c_s^2(t,k)$ and viscosity $c_{\text{vis}}^2(t,k)$
- To fit the model it suffices to prove that all matter species and all non-standard fields "I" in the RHS of the EFE can be recast as

$$T_{I\,0}^{\,0} = -\overline{\rho}_I\big(1+\delta_I\big)$$

$$T_{I\,i}^{\,0} = -\big(\overline{\rho}_I+\overline{P}_I\big)\vec{\nabla}_i\theta_I$$

$$T_{I\,j}^{\,i} = \overline{\rho}_I\big(w_I+\Pi_I\big)\delta^i{}_j + \big(\overline{\rho}_I+\overline{P}_I\big)\hat{D}^i{}_j\Sigma_I$$

with density contrast $\delta_I \equiv \delta\rho_I/\overline{\rho}_I$, velocity divergence θ_I, pressure contrast Π_I and anisotropic stress Σ_I

The Khronon field fits the GDM model

- Applying these definitions to the Khronon

$$\delta_\mathcal{K} = \frac{1+w}{c^2_\text{ad}}\left(\frac{\dot\sigma}{\overline{\mathcal{Q}}}-\phi\right)+\frac{\vec\nabla^2\Upsilon}{4\pi G a^2\overline{\rho}_\mathcal{K}}$$

$$\theta_\mathcal{K} = \frac{\sigma}{\overline{\mathcal{Q}}}$$

$$\Pi_\mathcal{K} = (1+w)\left(\frac{\dot\sigma}{\overline{\mathcal{Q}}}-\phi\right)$$

$$\Sigma_\mathcal{K} = 0$$

- The perturbation variable Υ follows from the Khronon equation as

$$\vec\nabla^2\Upsilon = 4\pi G\overline{\rho}_\mathcal{K} a^2\left\{\frac{1+w}{c^2_\text{ad}}\Upsilon+\Delta_\mathcal{K}\right\}$$

with gauge-invariant co-moving density contrast

$$\Delta_\mathcal{K} \equiv \delta_\mathcal{K} + 3H(1+w)\theta_\mathcal{K}$$

The Khronon field fits the GDM model

- The Khronon fluid is no longer adiabatic in first order perturbation, and we have $\Pi_\mathcal{K} = c^2_\mathrm{ad}\,\delta_\mathcal{K} + \Pi_\mathrm{nad}$ with non-adiabatic pressure perturbation

$$\Pi_\mathrm{nad} = -\frac{c^2_\mathrm{ad}}{4\pi G a^2 \overline{\rho}_\mathcal{K}} \vec{\nabla}^2 \Upsilon = \left(c_s^2 - c^2_\mathrm{ad}\right) \Delta_\mathcal{K}$$

  where $c_s^2(t,\vec{x})$ is the speed of sound
- In the Fourier domain the speed of sound is found to be

$$\boxed{c_s^2(t,k) = c^2_\mathrm{ad}\left[1 + \frac{c^2_\mathrm{ad} k^2}{4\pi G a^2 \overline{\rho}_\mathcal{K}(1+w)}\right]^{-1}}$$

  When $k \to 0$, we have $c_s^2 \to c^2_\mathrm{ad}$ and the Khronon is adiabatic in this limit
- Thus the Khronon behaves as a GDM fluid with zero viscosity $c^2_\mathrm{vis} = 0$, non-adiabatic sound speed c_s^2 and time-dependent equation of state $w(a)$

Allowed constraints on cosmology and the CMB

1. For any choice of $\mathcal{K}(\mathcal{Q})$ the Khronon equations can be recast into the framework of the GDM model
2. Furthermore, the functional choice $\mathcal{K} = \mu^2(\mathcal{Q}-1)^2$ leads to approximate dust solutions in the late universe ($a \to \infty$)
3. However, the equation of state for the quadratic potential $w = \frac{w_0}{w_0+a^3}$ is not exactly zero but tends to $w \to 1$ in the early universe ($a \to 0$)
4. Using the GDM framework we can use the allowed constraints on cosmology and the CMB (namely the cosmological DM should be close enough to a "pure dust" model) which give essentially [Kopp, Skordis, Thomas & Ilić 2018, 2021]

$$\boxed{w \lesssim 0.0164 \quad \text{around} \quad a \sim 10^{-4.5}} \qquad \text{(99\% confidence level)}$$

Tension between the cosmology and MOND

1. The latter constraint will be satisfied if we choose

$$w_0 \lesssim 5.3 \times 10^{-16}$$

   which represents the equation of state of the Khronon today ($a = 1$)

2. The expression of the energy density of the Khronon DM today gives

$$w_0 = \frac{2\pi G\, \overline{\rho}_{\mathcal{K},0}}{\mu^2 c^2} = \frac{3H_0^2\, \Omega_{\mathcal{K},0}}{4\mu^2 c^2}$$

   With $H_0 \sim 70\,\mathrm{km}/s/\mathrm{Mpc}$ and $\Omega_{\mathcal{K},0} \sim 0.26$ (the DM fraction today) we find that cosmology places the bound

$$\mu^{-1} \lesssim 0.22\,\mathrm{kpc}$$

3. This is clearly incompatible with having a MOND limit in galaxies, since for our own galaxy we should have MOND behaviour out to tens of kpc, and we had chosen previously $\mu^{-1} \sim 100\, r_M \sim 1\,\mathrm{Mpc}$

Dirac-Born-Infeld type of functions [Born & Infeld 1933,1934; Dirac 1962]

- We look for a function which has a well-defined Taylor expansion at $\mathcal{Q} = 1$ in order to reproduce the dust solutions and keep the correct non-relativistic PN limit, but tends to a different behaviour when $\mathcal{Q} \gg 1$
- The Dirac-Born-Infeld (DBI) inspired function

$$\mathcal{K}(\mathcal{Q}) = \frac{2\mu^2}{\lambda_D}\left[1 - \sqrt{1 - \lambda_D\left(\mathcal{Q} - 1\right)^2}\right]$$

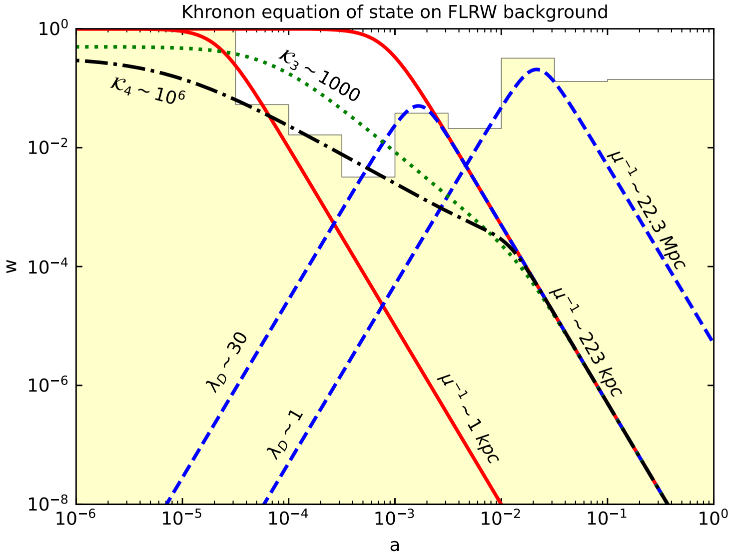
will play the role, where λ_D is a new dimensionless parameter

- Redoing the previous analysis this model behaves approximatively as dust not only today but also in the early universe

$$w \approx \begin{cases} \dfrac{w_0}{a^3} & \text{when } a \to \infty \\ \dfrac{a^3}{2w_0\sqrt{\lambda_D}(1+\sqrt{\lambda_D})} & \text{when } a \to 0 \end{cases}$$

hence it will easily pass the constraints from the CMB

The Khronon equation of state



Conclusions

Although this theory cannot be considered as "fundamental" (it contains an arbitrary function in the action) it owns a number of attractive features

- It is based on only two dynamical fields, the metric and the scalar Khronon field (plus ordinary matter)
- It recovers MOND at the scale of galaxies (in the weak acceleration regime), with however the restriction to systems being stationary
- In the strong acceleration regime it recovers GR and in particular has the same PPN parameters as GR for tests in the Solar System
- The theory has no propagating GW with helicity 0 or helicity 1, so gravitational waves are the same as in GR
- It can be arbitrarily close to the Λ-CDM cosmological model at the level of linear cosmological perturbations, where it retrieves the full observed spectrum of CMB anisotropies (for a wide range of parameters)