Non-linear treatment of cosmological perturbations

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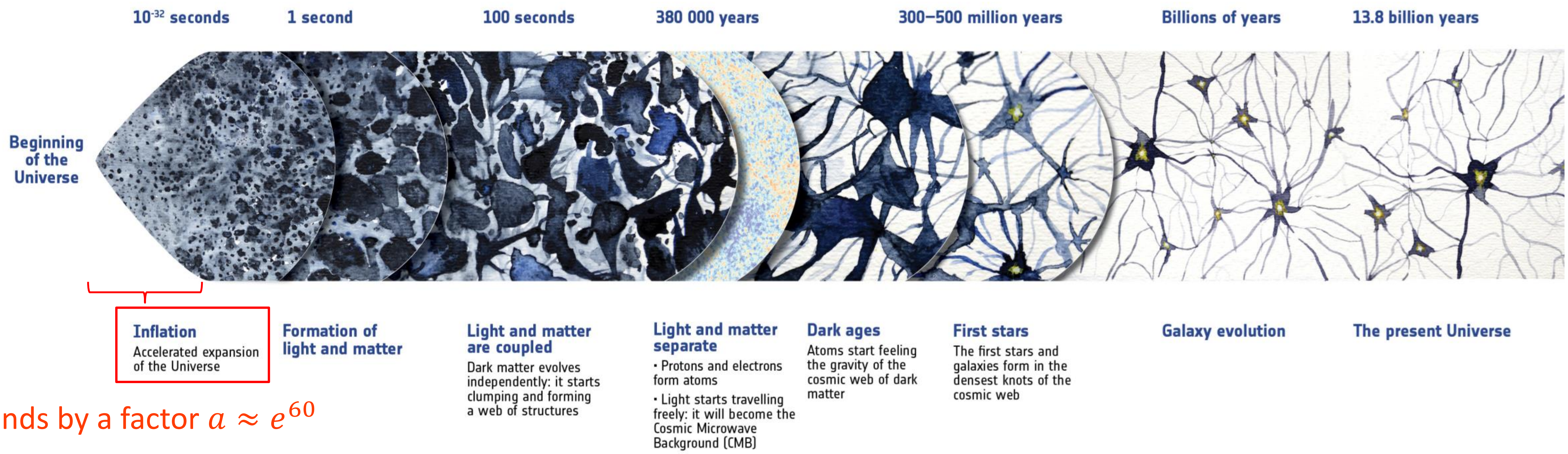
Inflation

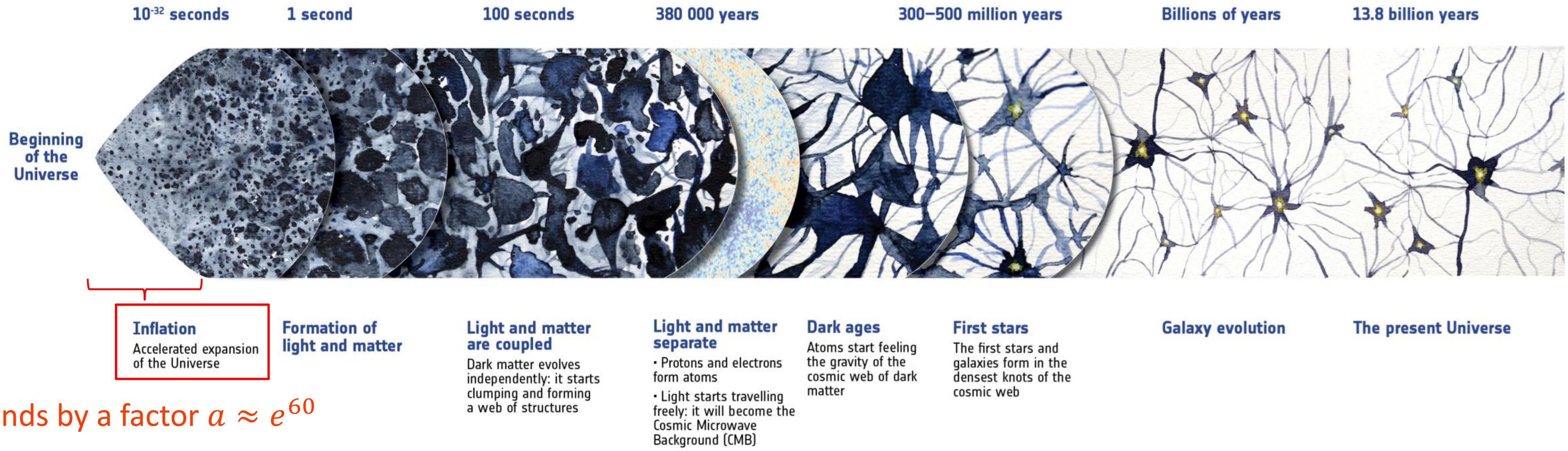


Expands by a factor $a \approx e^{60}$

$a(t) \coloneqq \exp(\lvert\alpha\rvert t)$	$H(t) \coloneqq \dfrac{\dot{a}(t)}{a(t)} = \lvert\alpha\rvert$	$R_H(t) \approx 1/H(t)$

Number of e-folds

Inflation



Expands by a factor $a \approx e^{60}$

Tons of models

Tools to make observational predictions

Observations

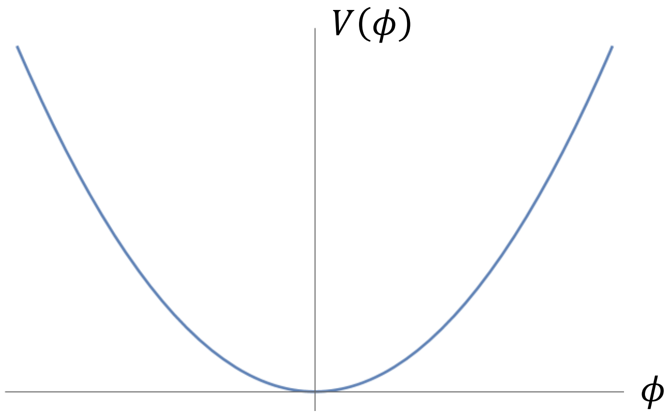
Inflation

- Scalar field ϕ obeying to the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0$$

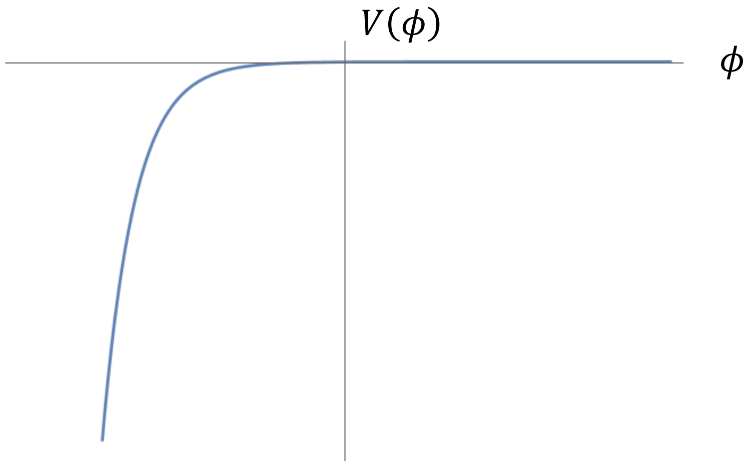
Slow roll

$$3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} \approx 0$$



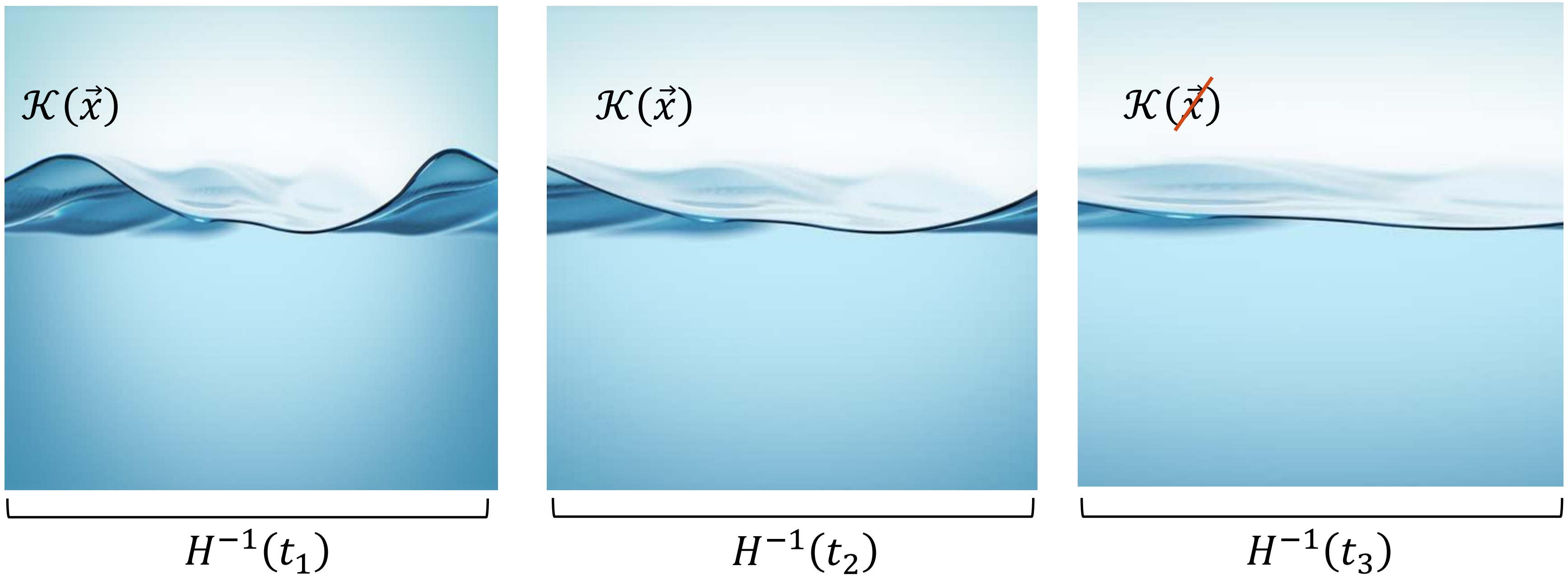
Ultra-slow roll

$$\ddot{\phi} + 3H\dot{\phi} \approx 0$$



Inflation

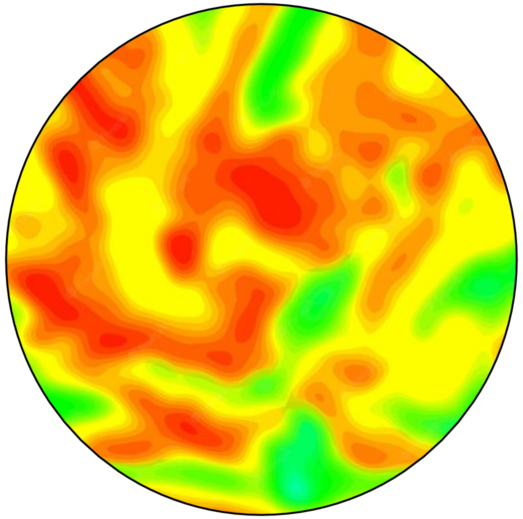
As the universe inflates, it becomes flat.



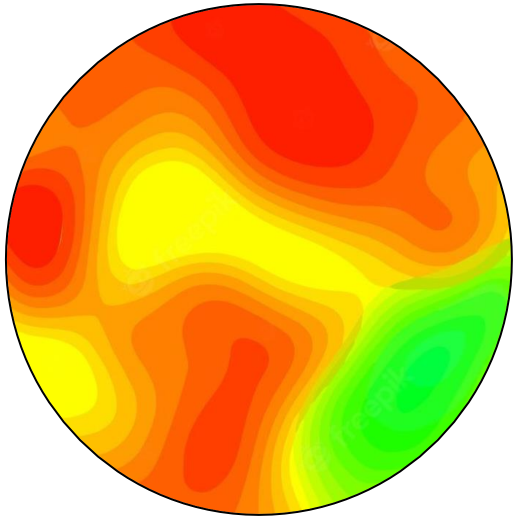
$a(t) \coloneqq \exp(\lvert\alpha\rvert t)$	$H(t) \coloneqq \frac{\dot{a}(t)}{a(t)} = \lvert\alpha\rvert$	$R_H(t) \approx 1/H(t)$

Inflation

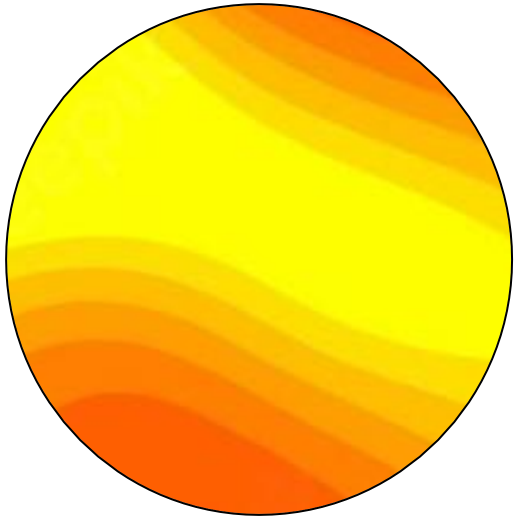
As the universe inflates, it becomes homogeneous.



$H^{-1}(t_1)$



$H^{-1}(t_2)$



$H^{-1}(t_3)$

$a(t) \coloneqq \exp(\|\alpha\|t)$	$H(t) \coloneqq \dfrac{\dot{a}(t)}{a(t)} = \|\alpha\|$	$R_H(t) \approx 1/H(t)$

$\epsilon_1(t) := -\dot{H}(t)/H^2(t)$

Inflation

- The curvature perturbation is $\zeta_k = v_k/z$

where $z = \sqrt{2\epsilon_1(t)}\, a(t)$ and

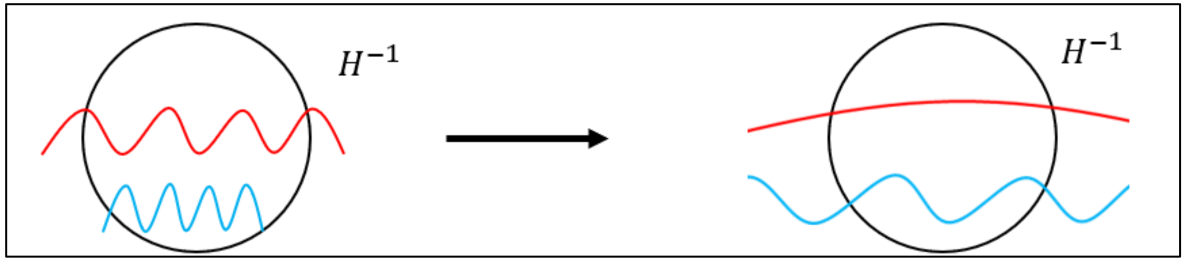
$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

High-frequency modes

$$\zeta_k = \frac{B_1(k)}{a(t)} e^{-ikt} + \frac{B_2(k)}{a(t)} e^{ikt}$$

Low-frequency modes

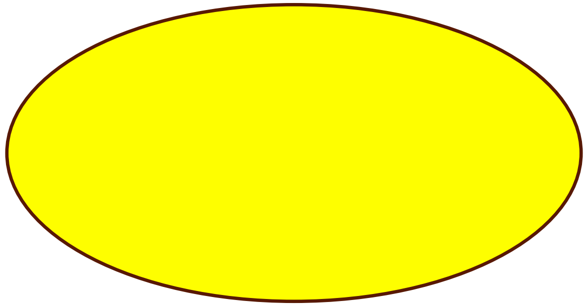
$$\zeta_k \approx C_1(k)$$



Inflation

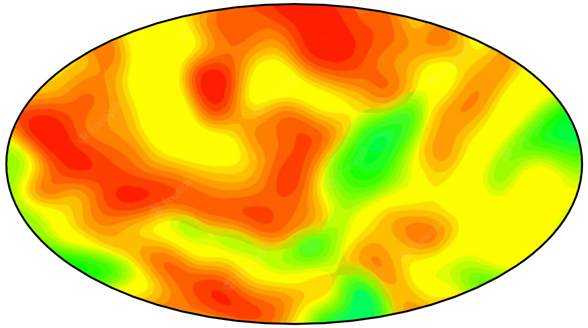
- Linear perturbations

$\phi(t)$



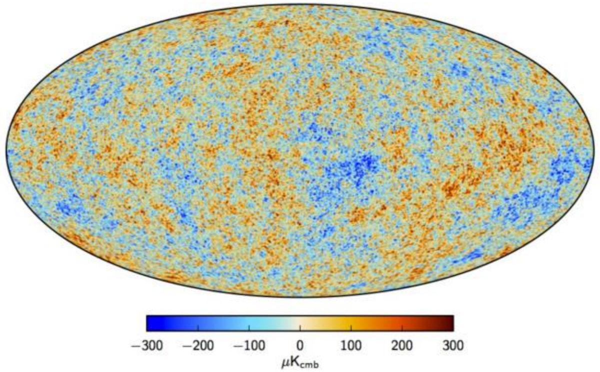
$+$

$\delta\phi(t,\vec{x})$



$\longrightarrow$

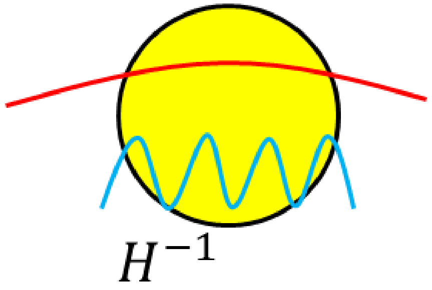
[Planck Collaboration (2016)]



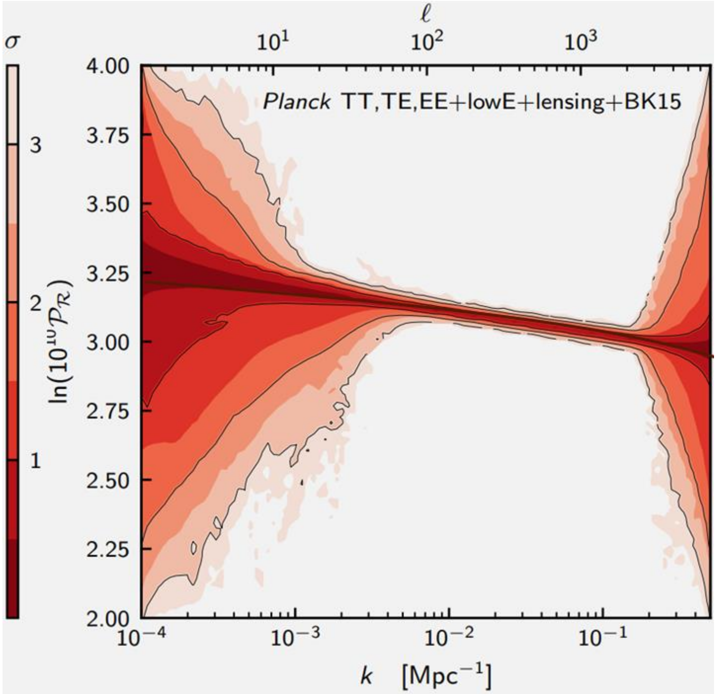
$$\langle \zeta_k^2 \rangle \approx \left\langle \frac{\delta T^2}{T^2} \right\rangle \approx 10^{-10}$$

Inflation

- For Bunch-Davies vacuum the power spectrum is scale invariant: $P_\zeta(k) \approx 10^{-9}$
- The PDF is Gaussian



The longest wavelengths correspond
to the earliest modes to exit the horizon

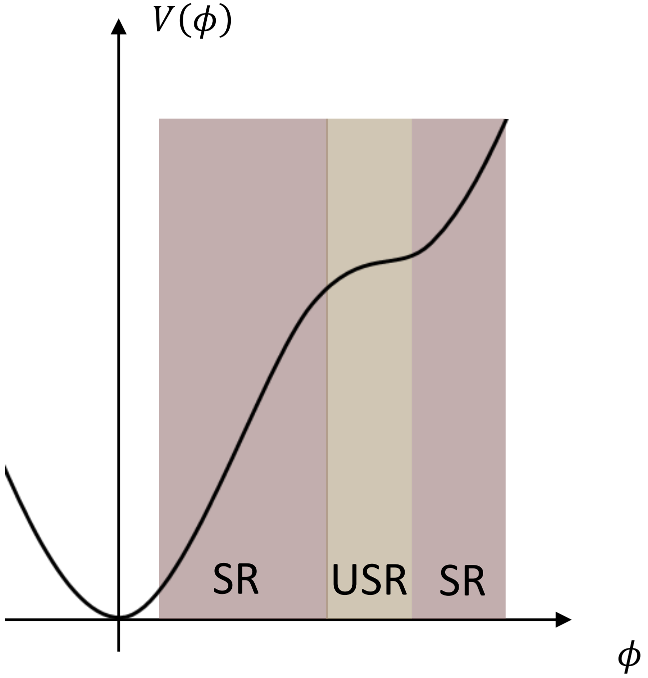


First e-folds of inflation

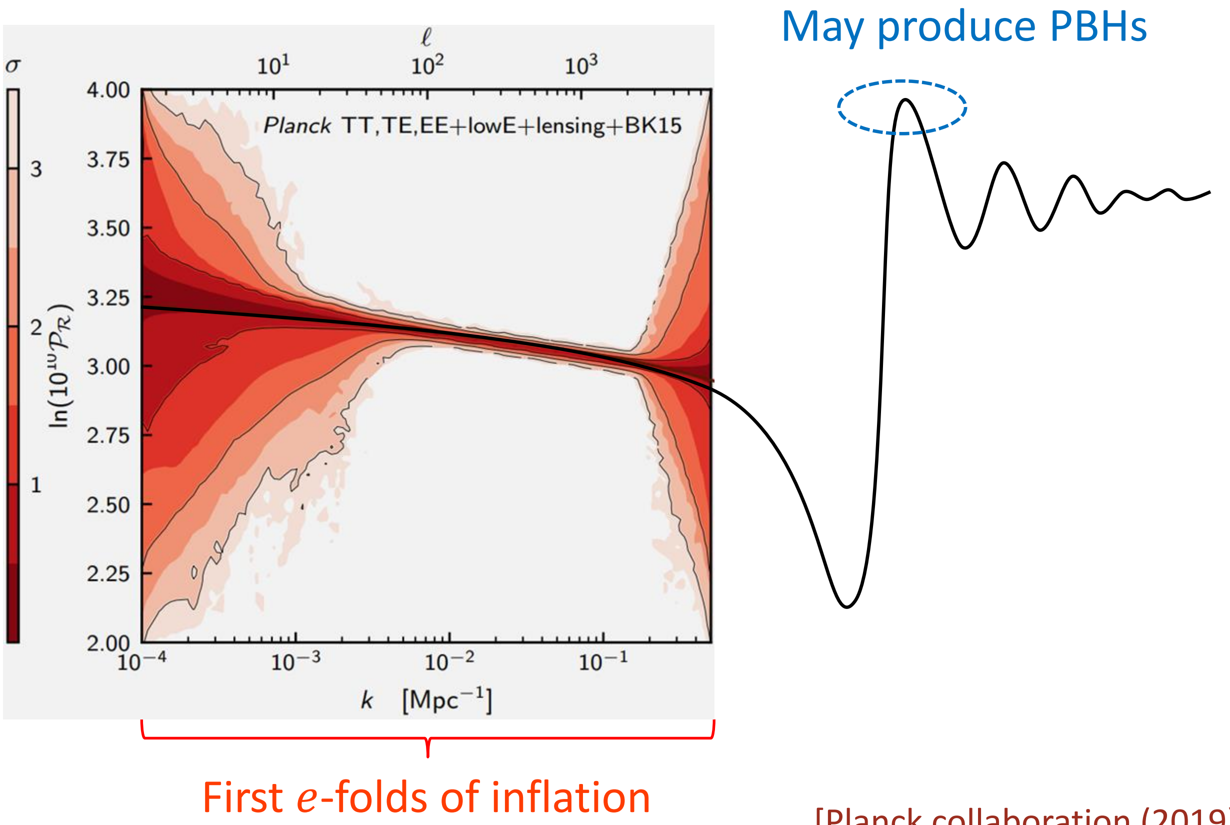
[Planck collaboration (2019)]

Inflation

- For Bunch-Davies vacuum the power spectrum is scale invariant: $P_\zeta(k) \approx 10^{-9}$
- The PDF is Gaussian

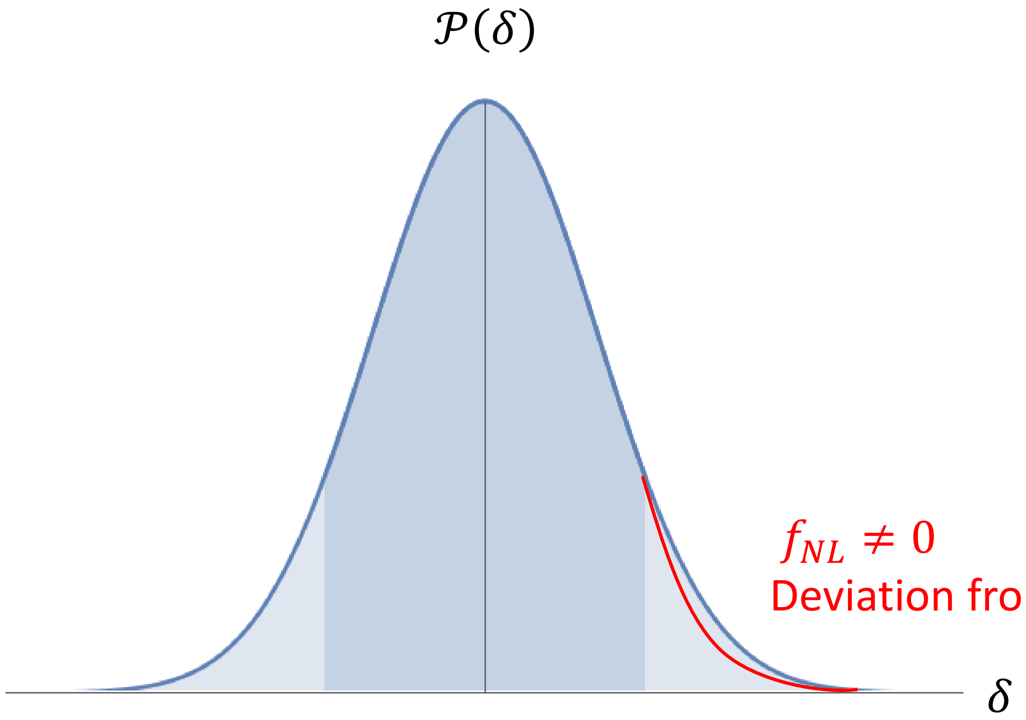


[Starobinsky (1992)]
[Byrnes, Cole, Patil (2018)]



[Planck collaboration (2019)]

Inflation



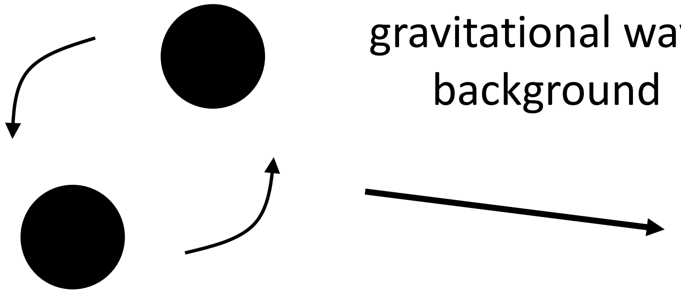
$$\zeta = \zeta_G + \frac{3}{5} f_{NL}\, \zeta_G^2$$

$f_{NL} \neq 0$

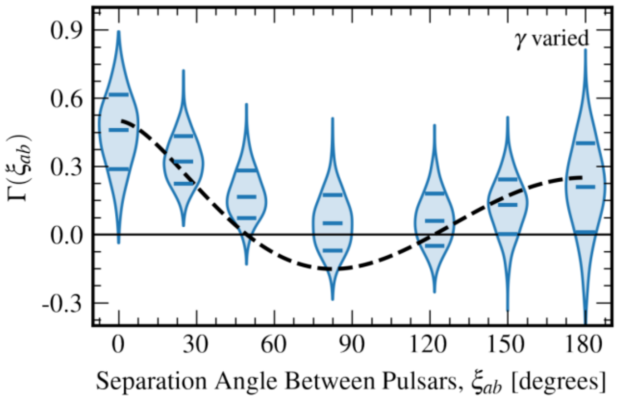
Deviation from Gaussianity $\longrightarrow$ PBHs

[Yokoyama, Suyama, Tanaka (2008)]
[Fujita et al. (2013)]]
[Byrnes, Cole, Patil (2018)]
[Ezquiaga, Garcia-Bellido, Vennin (2019)]
[Kitajima et al. (2021)]

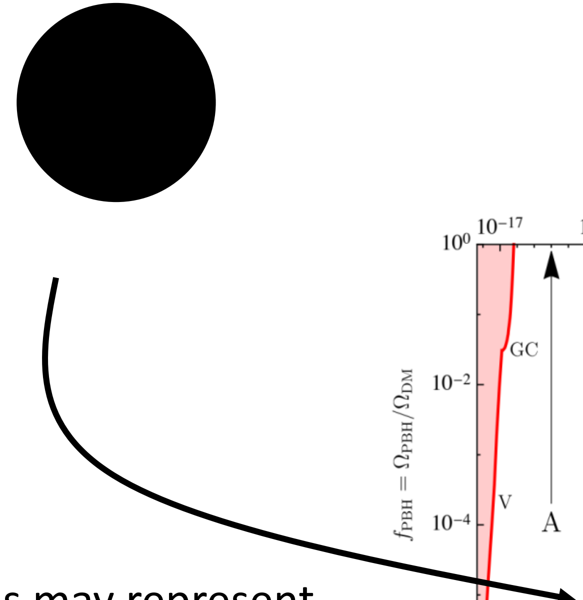
Inflation

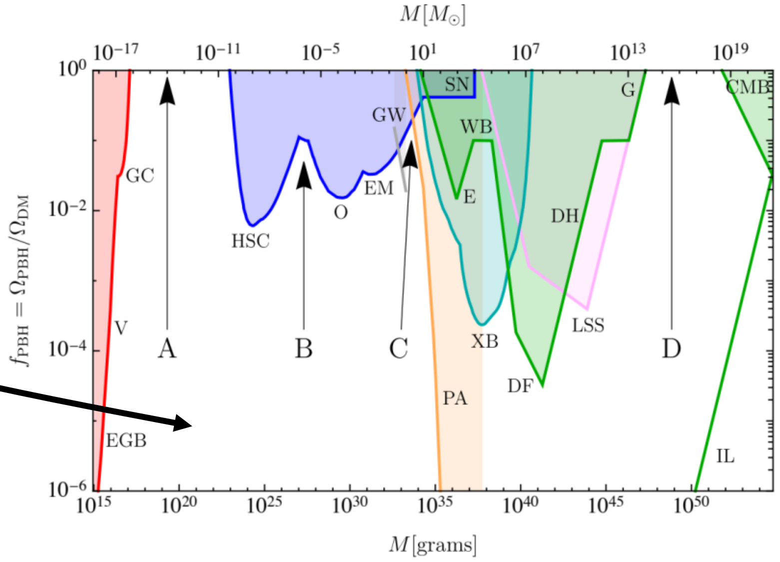


PBH-binaries contribute to the stochastic gravitational wave background



[NANOGrav collaboration (2023)]





PBHs may represent dark matter

[Escrivà, Kühnel, Tada (2022)]

[Yokoyama, Suyama, Tanaka (2008)]
[Fujita et al. (2013)]]
[Byrnes, Cole, Patil (2018)]
[Ezquiaga, Garcia-Bellido, Vennin (2019)]
[Kitajima et al. (2021)]

Contents

Curvature perturbation

- The curvature perturbation obeys

$$\zeta_k'' + \frac{2\,z'}{z}\zeta_k' + k^2\zeta_k = 0$$

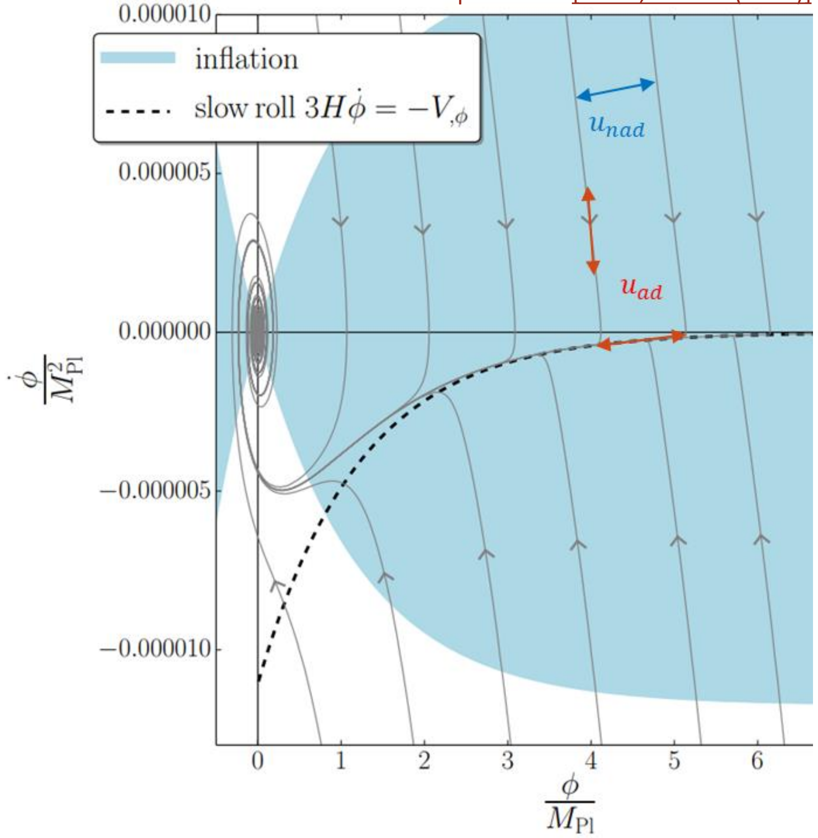
- Decompose the solution into adiabatic and non-adiabatic modes

$$\zeta(\eta) = \zeta_*\, u_{ad}(\eta) + \zeta_*'\, u_{nad}(\eta)$$

$$u_{ad}(\eta) = 1 - k^2 \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \int_{\eta_*}^{\tilde{\eta}} d\tilde{\tilde{\eta}}\, z^2(\tilde{\tilde{\eta}})$$

$$u_{nad}(\eta) = z^2(\eta_*)\left[\int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} - \mathcal{O}(k^2)\right]$$

Adapted from [Grain, Vennin (2017)]



Gradient expansion

Separate universe

- Perform a 1+3 splitting of the metric

$$g_{00} = -N^2 + \underbrace{\mathrm{N^i N_i}}_{=0} \quad , \quad g_{0i} = \underbrace{N_i}_{=0} \quad , \quad g_{ij} = \gamma_{ij}$$

- Define the integrated expansion rate

$$\mathcal{N} = \frac{1}{6}\int \gamma^{ij}\dot{\gamma}_{ij}\, d\tau$$

- At large scales $k \to 0$, the anisotropic part of the extrinsic curvature decays with the expansion

$$\dot{A}^i_j = -\frac{1}{2}\left(\gamma^{mn}\dot{\gamma}_{mn}\right)A^i_j \quad \longrightarrow \quad A^i_j \propto \gamma^{-1/2}$$

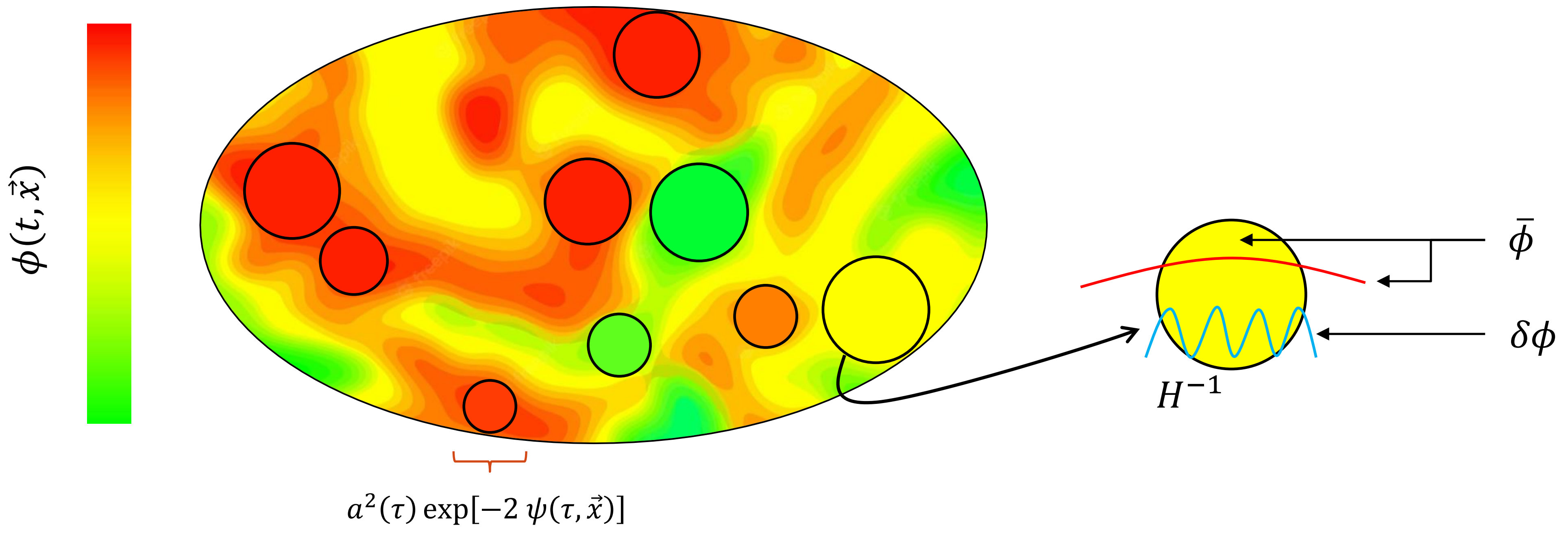
- The most general metric with vanishing anisotropy and N^i is

$$\gamma_{ij}(\tau,\vec{x}) = a^2(\tau)\exp[-2\,\psi(\tau,\vec{x})]\, \underbrace{h_{ij}(\vec{x})}_{=\delta_{ij}\text{ locally}}$$

[Salopek, Bond (1990)]

Gradient expansion

Separate universe



[Starobinsky (1983)]
[Salopek, Bond (1990)]
[Sasaki, Stewart (1996)]
[Sasaki, Tanaka (1998)]
[Wands, Malik, Lyth, Liddle (2000)]

Gradient expansion

Separate universe

- Nonlinearly:

$$\mathcal{N}(\tau_{in}, \tau_f, \vec{x}) = \ln\left(\frac{a_f\, e^{-\psi_f}}{a_{in}\, e^{-\psi_{in}}}\right) = \bar{\mathcal{N}}(\tau_{in}, \tau_f) + \ln\left(\frac{e^{-\psi_f}}{e^{-\psi_{in}}}\right)$$

$-\psi_f = \zeta_f$

$-\psi_{in} = 0$

$$\longrightarrow \ \delta\mathcal{N}(\tau_{in}, \tau_f, \vec{x}) = \zeta(\tau_f, \vec{x})$$

- Take a set of FLRW universes

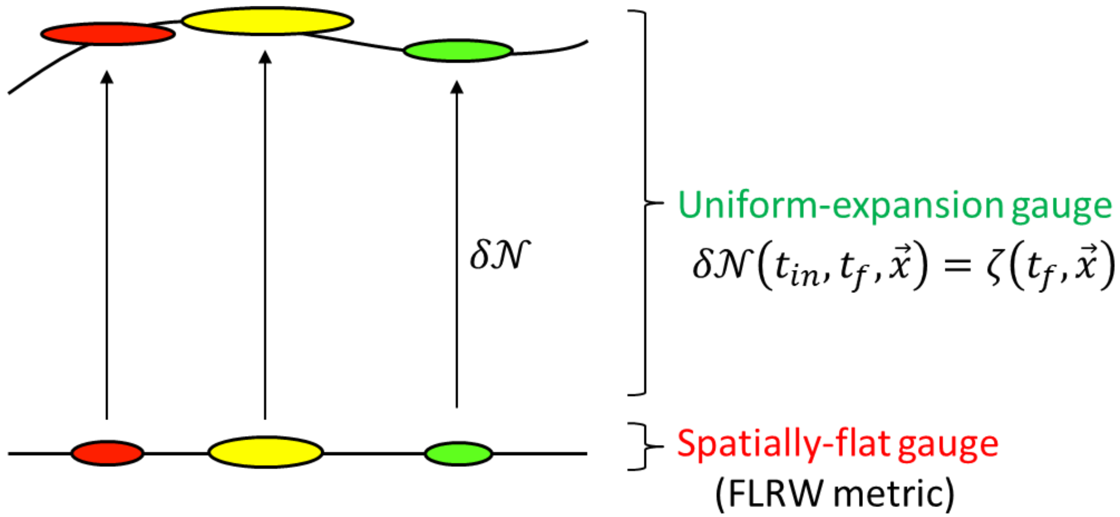
$$\gamma_{ij}(\tau, \vec{x}) = a^2(\tau)\, \delta_{ij}$$

- Perturb the FLRW equations

$$a + \delta a \ , \qquad H + \delta H \ ,$$
$$\phi + \delta\phi \ , \qquad \pi_\phi + \delta\pi_\phi$$

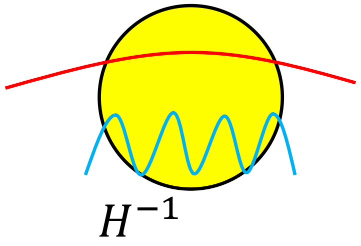
- The perturbed integrated expansion rate is

$$\delta\mathcal{N} = \delta a / a$$



Non-adiabatic counterpart

- The standard gradient expansion only captures the adiabatic mode.



This mode evolves at large scales depending on u_{nad}

$$\zeta(\eta) = \zeta_* \, u_{ad}(\eta) + \zeta'_* \, u_{nad}(\eta)$$

$$u_{ad}(\eta) = 1$$

$$u_{nad}(\eta) = z^2(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})}$$

[Gordon, Wands, Bassett, Maartens (2000)]

- Gauges used in the gradient expansion (spatially flat) are inconsistent with non-slow roll phases.

[**DA**, Grain, Vennin (2022)]
[**DA**, Grain, Vennin (2023)]
[**DA**, Frion, Miranda, Vennin, Wands (in prep.)]

- But this term is relevant in non-slow-roll inflation (e.g. ultra-slow roll).

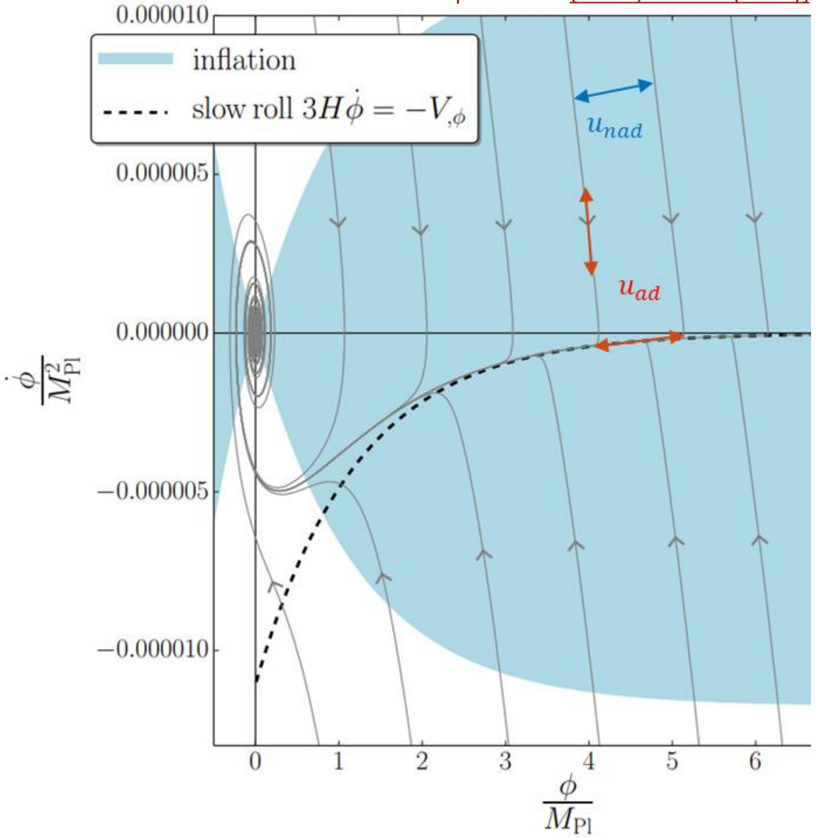
Generalised gradient expansion

$$\zeta(\eta) = \zeta_* \, u_{ad}(\eta) + \zeta'_* \, u_{nad}(\eta)$$

$$u_{ad}(\eta) = 1 - k^2 \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})} \int_{\eta_*}^{\tilde{\eta}} d\tilde{\tilde{\eta}} \, z^2(\tilde{\tilde{\eta}})$$

$$u_{nad}(\eta) = z^2(\eta_*) \int_{\eta_*}^{\eta} \frac{d\tilde{\eta}}{z^2(\tilde{\eta})}$$

Adapted from [Grain, Vennin (2017)]



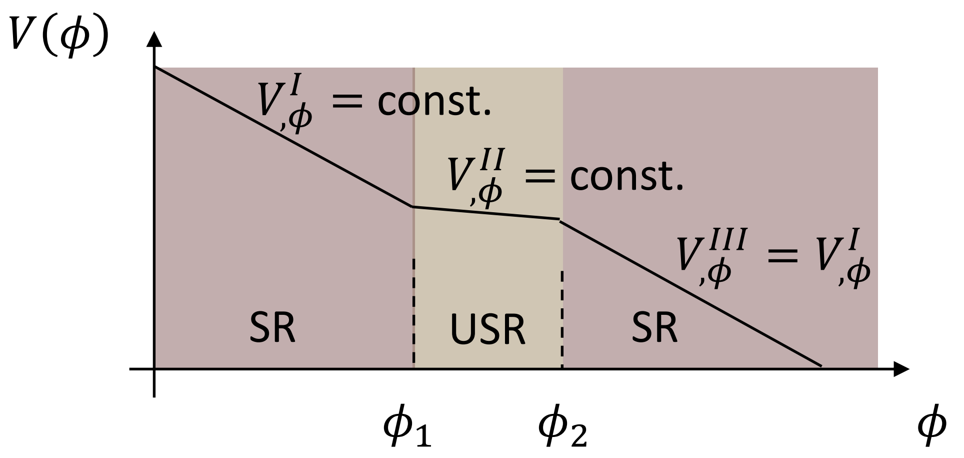
- The leading order of the non-adiabatic mode can be described as a k^2 correction to the adiabatic mode.

- Allow the gradient expansion to describe the $\mathcal{O}(k^2)$.

[Leach, Sasaki, Wands, Liddle (2018)]
[Jackson, Assadullahi, Gow, Koyama, Vennin, Wands (2023)]

Generalised gradient expansion

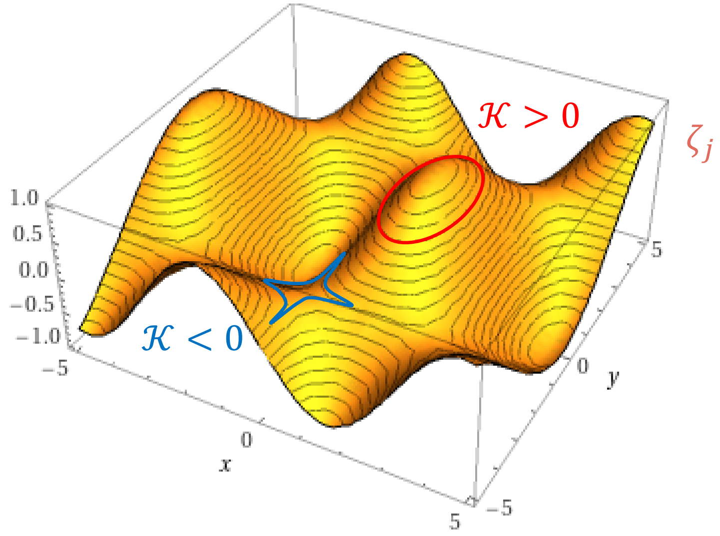
- Consider the Starobinsky model



- $N_j \equiv$ start using gradient expansion
- Consider curved FLRW patches

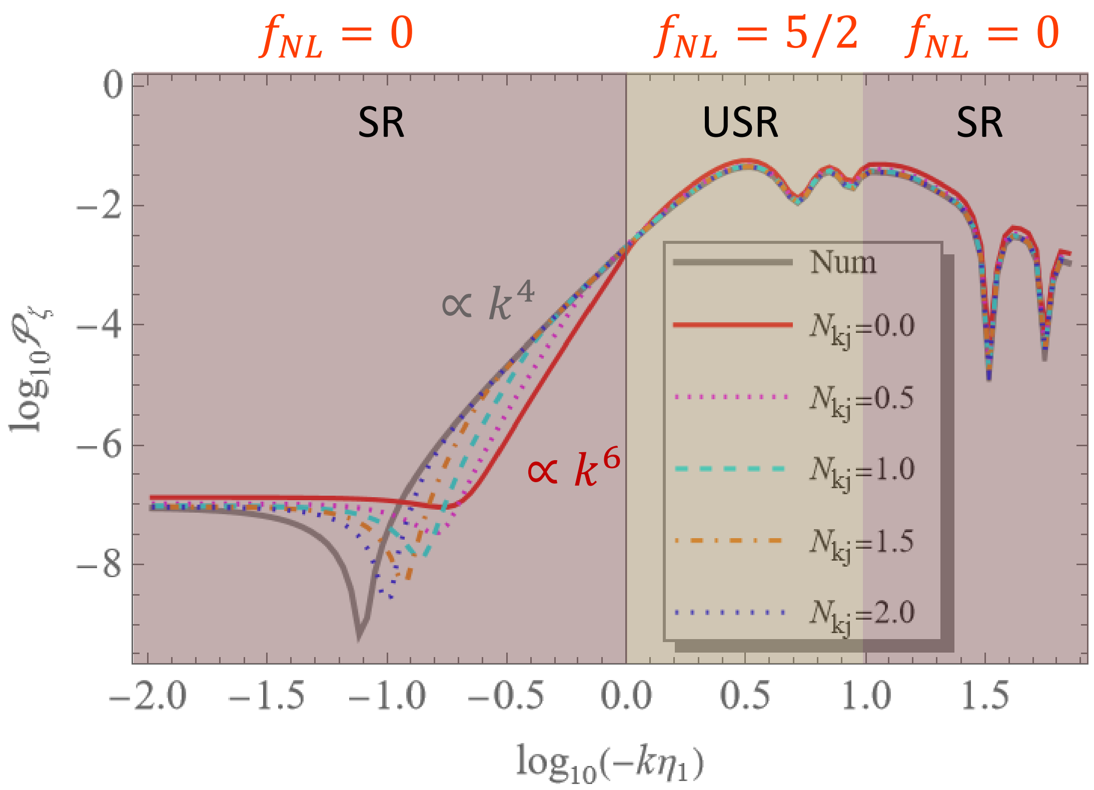
$$H^2 = H_0^2 - \mathcal{K}e^{-2(N-N_j)}$$

and initially $\mathcal{K}e^{2N_j} \equiv \frac{2k^2}{3}\zeta_j$



[DA, Pi, Tanaka (2024)]

Generalised gradient expansion



$N_k \equiv$ horizon-crossing time

$N_j \equiv$ start using gradient expansion

$$N_{kj} \coloneqq N_j - N_k > 0$$

- If N_{kj} is long, all trajectories align on the phase-space attractor: u_{nad} is negligible and the usual separate-universe approach matches perturbation theory.
- If not, the contribution from $\phi_{\mathcal{N}}$ at the transition must be taken into account.

[Pi, J. Wang (2022)]
[**DA**, Pi, Tanaka (2024)]

Generalised gradient expansion

- The scalar field obeys non-linearly to

$$\left[\partial_{\mathcal{N}}^2 + \left(3 + \frac{\mathcal{K}}{H_0^2} e^{-2(\mathcal{N}-\mathcal{N}_j)}\right)\partial_{\mathcal{N}}\right]\phi + \frac{V_{,\phi}}{H_0^2}\left(1 + \frac{\mathcal{K}}{H_0^2} e^{-2(\mathcal{N}-\mathcal{N}_j)}\right) = \mathcal{O}(\mathcal{K}^2)$$

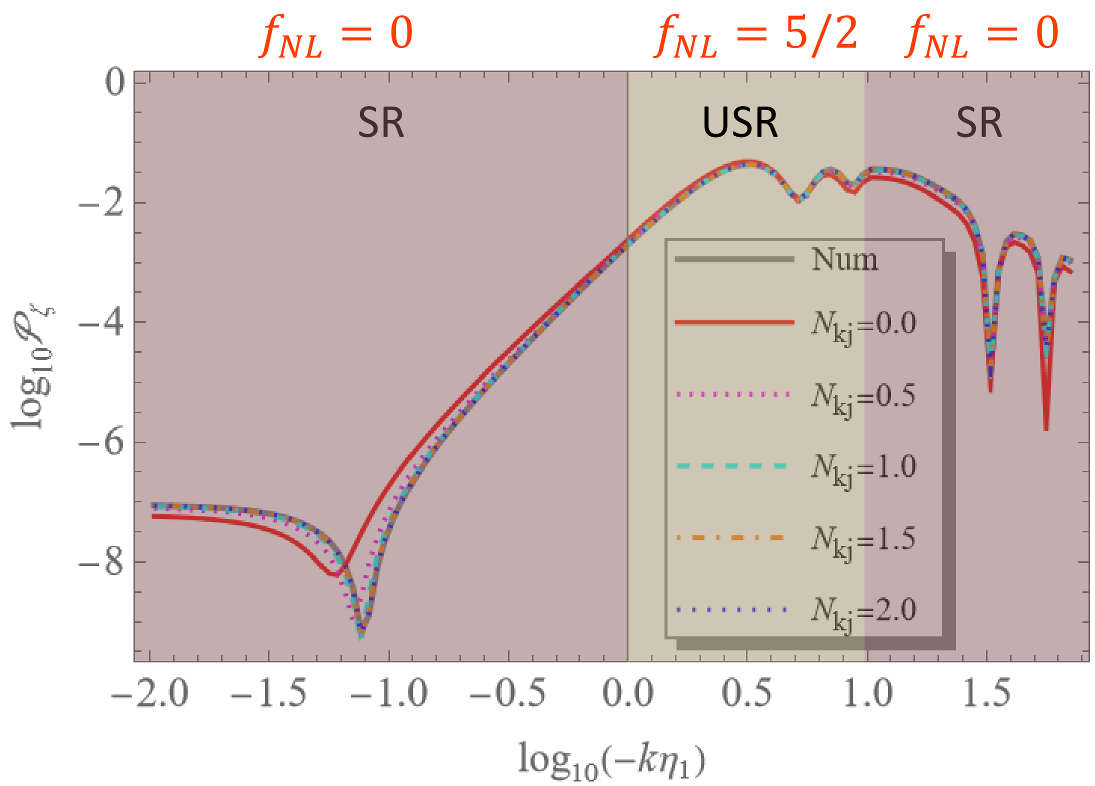
- It is easy to find an analytical solution for ϕ which can then be inverted to find the *e*-folding number for each phase.

$$\mathcal{N}_{j1} = \underbrace{\mathcal{W}}_{\mathcal{O}(\mathcal{K})} e^{-2\mathcal{N}_{j1}} + \underbrace{\mathcal{X}}_{\delta\phi_{\mathcal{N}}(\mathcal{N}_j)} e^{-3\mathcal{N}_{j1}} + \underbrace{\mathcal{Z}}_{\delta\phi_{\mathcal{N}}(\mathcal{N}_j)}$$

Solutions: Lambert function.

[**DA**, Pi, Tanaka (2024)]

Generalised gradient expansion



$N_k \equiv$ horizon-crossing time
$N_j \equiv$ start using gradient expansion

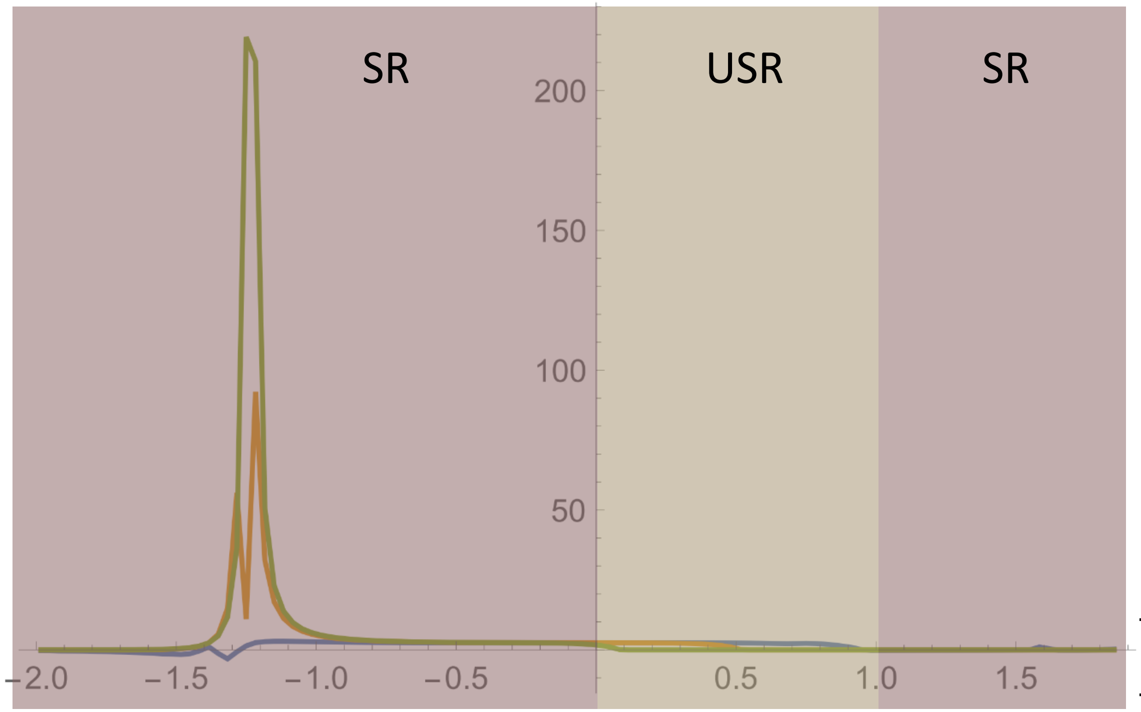
$$N_{kj} := N_j - N_k > 0$$

- The generalised gradient expansion is consistent with linear perturbation theory during slow roll.
- We can use it to track non-linearities (such as f_{NL}) during the transition.

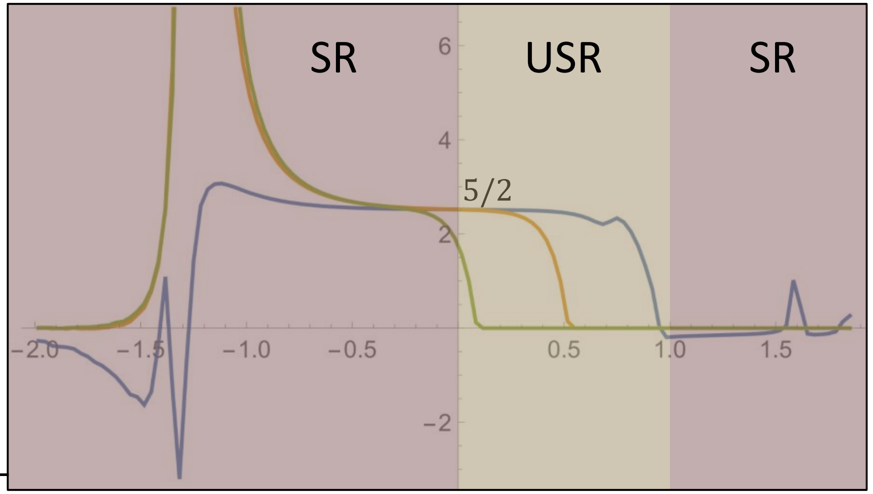
[**DA**, Pi, Tanaka (2024)]

Generalised gradient expansion

$\Delta \mathrm{N}_{kj} = 0$
$\Delta \mathrm{N}_{kj} = 1$
$\Delta \mathrm{N}_{kj} = 2$



[Preliminary results]



- The f_{NL} can be obtained from $\mathcal{N}$. If $\mathcal{N}$ doesn't depend on $\phi_{\mathcal{N}}$, then

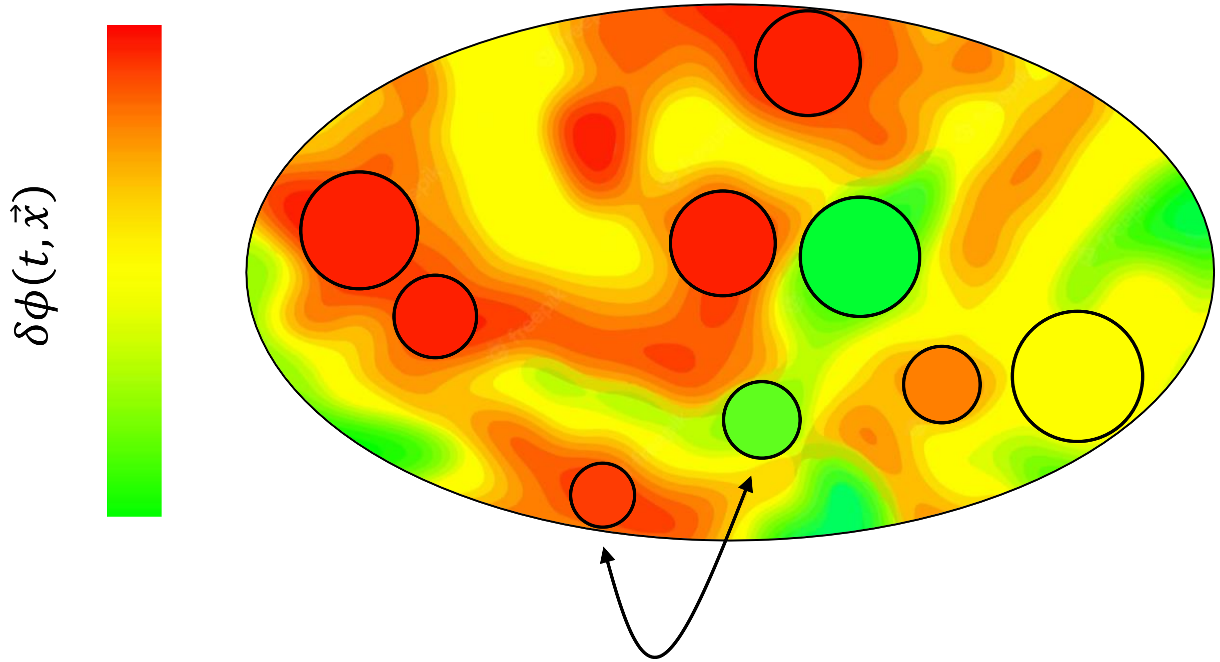
$$f_{NL} = \frac{5}{6}\frac{\mathcal{N}_{\phi\phi}}{\mathcal{N}_{\phi}^2}$$

[Maldacena (2002)]
[Bartolo et al. (2004)]
[Yokoyama, Suyama, Tanaka (2007)]

- The f_{NL} transits continuously from $0 \rightarrow 5/2 \rightarrow 0$ as expected.

[**DA**, Pi, Tanaka (2024)]

Generalised gradient expansion



- The different patches emerge when amplifying quantum fluctuations
  $\Rightarrow$ they should be correlated.

- One should take into account some initial f_{NL} from subhorizon modes.

[Maldacena (2002)]
[Bartolo et al. (2004)]
[Yokoyama, Suyama, Tanaka (2007)]

[**DA**, Pi, Tanaka (2024)]

Conclusion

- To constrain inflationary models, non-linear effects may be important.

- The gradient expansion describes non-linear effects during inflation.

  Describe a set of flat FLRW patches. $\zeta = \delta\mathcal{N}$.

  Well understood for the case of slow roll.

- Extended gradient expansion: curved FLRW patches.

  Captures the k^2-correction of ζ. Relevant e.g. in ultra-slow roll.

- The f_{NL} evolves continuously from slow roll to ultra-slow roll $0 \to 5/2$. PBHs may be created even from modes that exited the horizon during the slow-roll phase. [work in progress]