

BMS particles

Unitary irreducible representations of the BMS group

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From joint works with

- **L. Donnay** and **Y. Herfray**, Phys. Rev. Lett. **135** (2025)
[arXiv:2412.06002 [hep-th]]
- **Y. Herfray**, Commun. Math. Phys. **407** (2026)
[arXiv:2505.05368 [hep-th]]

Motivations

Recent investigations in the infrared structure of gauge theories and gravitation (such as the discovery of the infrared triangle, the quest of flat holography, etc) suggest to take Bondi-Metzner-Sachs (BMS) group as the proper symmetry group of gravitational scattering processes.

e.g. infrared triangle

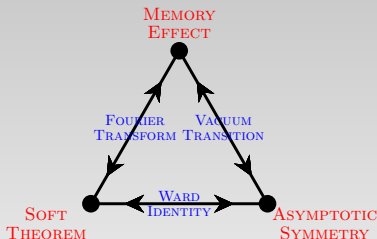


Fig. 1 in [Strominger, arXiv:1703.05448 [hep-th]]

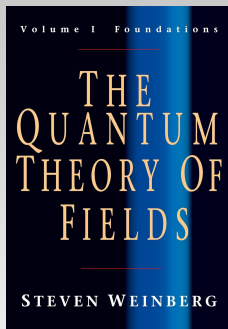
Tantalising goal

Dream: build a BMS-invariant S-matrix

- *Tentative program:* look for a streamlined approach.

Open Weinberg's book
and

Try to replace everywhere “Poincaré” by “BMS”



Tantalising goal

Main goal: write BMS-invariant scattering processes

- *Tentative program:*

Look for a streamlined approach based on the representation theory of the BMS group.

- *First step:*

A natural candidate for the Hilbert space of asymptotic states is the Fock space of (suitably) symmetrised tensor products of one-particles states spanning a unitary irreducible representation (UIR) of the BMS group.

- *Preliminary step:*

Revisit the unitary dual of BMS group with modern physical insights in mind.

BMS representation theory

Classification of (induced) unitary irreducible representations

Milestones:

- $ISO(3,1) = SO(3,1) \ltimes \mathbb{R}^{3,1}$
Wigner (1939)
- Semisimple $\ltimes$ Abelian
Mackey (1968)
- $BMS_4 = SO(3,1) \ltimes C^\infty(S^2)$
 - Early papers by Cantoni (1966-1967)
 - Seminal series of paper by McCarthy (1972-1978)
 - Correct choice of topology by Girardello & Parravicini (1974)
 - Complete proof (exhaustivity) by Piard (1977)
 - Hard representations
 - Massless: $s=0$ (Sachs, 1962), $s=1,2$ (Ashtekar, 1980s), ...
 - Massive: Longhi & Materassi (1999), ...
- $BMS_3 = SO(2,1) \ltimes C^\infty(S^1)$ or $Diff(S^1) \ltimes C^\infty(S^1)$
Barnich & Oblak (2014-2016), Melas (2017), ...

1 Supertranslations & Supermomenta

- Supertranslations
- Supermomenta
- Hard/Soft decomposition

2 BMS particles

- UIRs of BMS
- Hard vs Soft representations

3 Conclusion

- Take aways
- Future directions

Supertranslations & Supermomenta

Cuts of Scri

The natural locus to discuss the BMS group is:

Null infinity $\mathcal{I} \cong \mathbb{R} \times S^2$

This cone is a bundle of null rays over the sphere (an $\mathbb{R}$ -principal bundle).

$\implies$ The sections $S^2 \hookrightarrow \mathcal{I}$ of this ray bundle $\mathcal{I} \twoheadrightarrow S^2$ are called **cuts**.

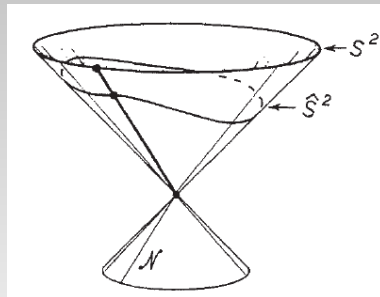


Fig. 2 in [R. Penrose, 1974]

Cuts as simultaneity slices on Scri

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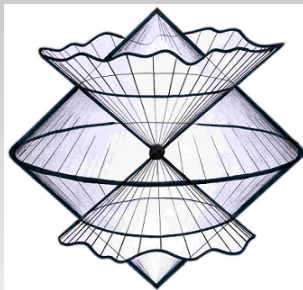
They can be thought as simultaneity slices $u = f(z, \bar{z})$ on $\mathcal{I}$ with respect to the retarded time (which is an affine parameter along the null rays generating $\mathcal{I}$).

Cuts as celestial spheres

Each cut is preserved by a Lorentz subgroup $SO(3,1) \subset BMS_4$ of the BMS group.

$SO(3,1)$ acts on S^2 as conformal transformations of the **celestial sphere**.

$\implies$ Many objects in BMS group theory have a natural formulation in terms of conformal geometry on the celestial sphere.



Cover of B. Oblak, arXiv:1610.08526

Space of cuts

Each cut is preserved by a Lorentz subgroup $SO(3,1) \subset BMS_4$ of the BMS group.

$SO(3,1)$ acts on S^2 as conformal transformations of the **celestial sphere**.

Remark: As will be argued later on, wavefunctions of BMS particles are fields on the infinite-dimensional space

$$\Gamma(\mathcal{I}) \cong \frac{BMS_4}{SO(3,1)}$$

of all cuts, which is the BMS analogue of Minkowski spacetime

$$\mathbb{R}^{3,1} \cong \frac{ISO(3,1)}{SO(3,1)}.$$

Conformal densities on the celestial sphere

Definition

Conformal densities: A conformal density of weight $w \in \mathbb{R}$ on the celestial sphere is a scalar field on S^2 transforming under Weyl transformations as follows

$$\tilde{f}(z, \bar{z}) = \Omega(z, \bar{z})^w f(z, \bar{z}), \quad \tilde{g}(z, \bar{z}) = \Omega(z, \bar{z})^2 g(z, \bar{z}),$$

where $\Omega(z, \bar{z}) > 0$ is a nowhere-vanishing conformal factor. The vector space of weight- w conformal densities will be denoted $\mathcal{E}[w]$.

With a standard abuse of notation, the equivalence class $[\tilde{g}, \tilde{f}]$ will be denoted by its representative $f(z, \bar{z}) \in C^\infty(S^2)$ for the flat metric $dz d\bar{z}$, with its transformation law under Weyl transformations implied:

$$\tilde{f}(z, \bar{z}) = \Omega(z, \bar{z})^w f(z, \bar{z}), \quad \tilde{g}(z, \bar{z}) = \Omega(z, \bar{z})^2 dz d\bar{z},$$

Conformal densities on the celestial sphere

More concretely, a conformal density on the celestial sphere is an object that transforms under a Möbius transformation of the stereographic coordinates

$$z \rightarrow z' = \frac{az + b}{cz + d}, \quad \text{with} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}),$$

as follows:

$$f(z, \bar{z}) \mapsto f'(z, \bar{z}) = |cz + d|^{2w} f\left(\frac{az + b}{cz + d}, \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}\right).$$

Conformal densities are assumed to be smooth and globally defined on the whole sphere S^2 .

$\implies$ The transformation under inversion implies that the representative $f(z, \bar{z})$ has a constrained behavior as $|z| \rightarrow \infty$:

$$f(z, \bar{z}) = |z|^{2w} \hat{f}(z^{-1}, \bar{z}^{-1}) \stackrel{|z| \rightarrow \infty}{\sim} |z|^{2w} \hat{f}(0, 0).$$

Supertranslations

Definition (point of view at null infinity)

Supertranslations: vertical automorphisms

$$u \rightarrow u' = u + \mathcal{T}(z, \bar{z})$$

of the fibre bundle $\mathcal{I}$ over S^2 .

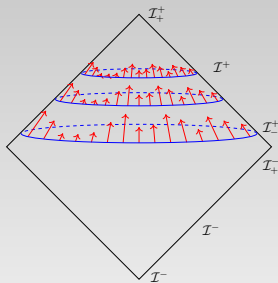


Fig. 17 in [Strominger, arXiv:1703.05448 [hep-th]]

Supertranslations

Definition (point of view at null infinity)

Supertranslations: vertical automorphisms

$$u \rightarrow u' = u + \mathcal{T}(z, \bar{z})$$

of the fibre bundle $\mathcal{I}$ over S^2 .

The shifts $\mathcal{T}(z, \bar{z})$ have a specific transformation law under $SL(2, \mathbb{C}) \implies$

Definition (point of view on celestial sphere)

Supertranslations: A supertranslation $\mathcal{T}(z, \bar{z}) \in \mathcal{E}[1]$ is a conformal density of weight 1 on the celestial sphere.

BMS group

Definition

BMS group: semi-direct product

$$BMS_4 = SO(3, 1) \ltimes \mathcal{E}[1]$$

of the Lorentz group acting on the additive group of supertranslations.

PHYSICAL REVIEW

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Asymptotic Symmetries in Gravitational Theory*

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U. S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey

(Received July 23, 1962)

It is pointed out that the definition of the inhomogeneous Lorentz group as a symmetry group breaks down in the presence of gravitational fields even when the dynamical effects of gravitational forces are completely negligible. An attempt is made to rederive the Lorentz group as an "asymptotic symmetry group" which leaves invariant the form of the boundary conditions appropriate for asymptotically flat gravitational fields. By analyzing recent work of Bondi and others on gravitational radiation it is shown that, with apparently reasonable boundary conditions, one obtains not the Lorentz group but a larger group. The name "generalized Bondi-Metzner group" ("GBM group") is suggested for this larger group.

GBM group contains an infinite number of different subgroups isomorphic to the inhomogeneous orthochronous Lorentz group; that the infinitesimal GBM group algebra permits at least one nontrivial representation, which is directly analogous to the rest-mass-zero and spin-zero representation of the Lorentz group; that in any representation of the infinitesimal GBM group algebra there is a "rest mass" operator which commutes with all the other operations; and that no similar "spin" operator appears to exist. It is argued that the GBM group is so similar to the inhomogeneous Lorentz group that the former may be compatible as a symmetry group with present microphysics.

Two applications are given: Certain quantum commutation

Translations as supertranslations

Fun fact: There is a canonical, $SL(2, \mathbb{C})$ -equivariant, linear injection

$$q_\mu \left| \begin{array}{ll} \mathbb{R}^{3,1} & \hookrightarrow \mathcal{E}[1] \\ T^\mu & \mapsto \mathcal{T}(z, \bar{z}) \end{array} \right.$$

of translations into supertranslations given by

$$q_\mu(z, \bar{z}) = (1 + |z|^2, \quad z + \bar{z}, \quad -i(z - \bar{z}), \quad 1 - |z|^2)$$

i.e.

$$\begin{aligned} \mathcal{T}(z, \bar{z}) &= T^\mu q_\mu(z, \bar{z}) \\ &= (T^0 + T^3) + (T^1 - iT^2)z + (T^1 + iT^2)\bar{z} + (T^0 - T^3)|z|^2 \end{aligned}$$

Supermomenta: motivation

For BMS symmetries, it is natural to generalise:

- the momentum basis of one-particle states, which are eigenvectors of the translation generators,

$$\hat{P}_\mu |p\rangle = p_\mu |p\rangle,$$

- to the supermomentum basis of eigenvectors of supertranslations

$$\hat{\mathcal{P}}(z, \bar{z}) |\mathcal{P}\rangle = \mathcal{P}(z, \bar{z}) |\mathcal{P}\rangle.$$

Supermomenta: definition

Definition

Supermomentum: conformal density $\mathcal{P}(z, \bar{z}) \in \mathcal{E}[-3]$ of weight -3 or, more generally, distribution $\mathcal{P}(z, \bar{z}) \in \mathcal{E}[1]'$ on the space of supertranslations.

Supermomenta $\mathcal{P}$ are dual to supertranslations $\mathcal{T}$:

$$\langle \mathcal{P}, \mathcal{T} \rangle = \int_{S^2} dz d\bar{z} \mathcal{P}(z, \bar{z}) \mathcal{T}(z, \bar{z})$$

Supermomenta: another motivation

Finite (super)translations act simply on (super)momentum basis states, merely as phase shifts:

$$e^{iT \cdot \hat{P}} |p\rangle = e^{iT \cdot p} |p\rangle, \quad e^{i\langle T, \hat{\mathcal{P}} \rangle} |\mathcal{P}\rangle = e^{i\langle T, \mathcal{P} \rangle} |\mathcal{P}\rangle$$

This provides a unitary representation of the BMS group, but it is very far from being irreducible.

Remark: Via (infinite-dimensional) Fourier transform of wave functions $\Psi[\mathcal{P}]$ on the space of supermomenta $\mathcal{E}[1]'$, one obtains wave functions on the space of cuts $\Gamma(\mathcal{I})$:

$$\Psi[\mathcal{T}] = \int \mathcal{D}\mathcal{P} \, e^{i\langle \mathcal{P}, \mathcal{T} \rangle} \Psi[\mathcal{P}]$$

They can be thought as “off-shell” BMS particles.

Momenta from supermomenta

Fun fact*: There is a canonical, $SL(2, \mathbb{C})$ -equivariant, linear surjection

$$\pi_\mu \left| \begin{array}{ll} \mathcal{E}[-3] & \twoheadrightarrow (\mathbb{R}^{3,1})^* \\ \mathcal{P}(z, \bar{z}) & \mapsto p_\mu \end{array} \right.$$

from supermomenta onto momenta. It is given by

$$p_\mu = \int_{S^2} dz d\bar{z} \mathcal{P}(z, \bar{z}) q_\mu(z, \bar{z})$$

i.e.

$$\begin{aligned} p_0 &= \int_{S^2} dz d\bar{z} (1 + |z|^2) \mathcal{P}(z, \bar{z}), & p_1 &= \int_{S^2} dz d\bar{z} (z + \bar{z}) \mathcal{P}(z, \bar{z}), \\ p_2 &= i \int_{S^2} dz d\bar{z} (z - \bar{z}) \mathcal{P}(z, \bar{z}), & p_3 &= \int_{S^2} dz d\bar{z} (1 - |z|^2) \mathcal{P}(z, \bar{z}). \end{aligned}$$

Momenta from supermomenta

Proof: For $\mathcal{T}(z, \bar{z}) = T^\mu q_\mu(z, \bar{z})$,

$$\langle \mathcal{P}, \mathcal{T} \rangle = \int_{S^2} dz d\bar{z} \mathcal{P}(z, \bar{z}) T^\mu q_\mu(z, \bar{z}) = T^\mu \pi_\mu(\mathcal{P}) = p \cdot T$$

where $p_\mu = \pi_\mu(\mathcal{P})$.



Physically, this implies that translations act as usual in the supermomentum basis:

$$e^{i \langle T \cdot q, \hat{\mathcal{P}} \rangle} | \mathcal{P} \rangle = e^{i T \cdot p} | \mathcal{P} \rangle$$

Soft supermomenta: definition

Definition

We will call $p_\mu = \pi_\mu(\mathcal{P})$ the **momentum associated to the supermomentum** $\mathcal{P}(z, \bar{z})$. In particular, supermomenta with zero momentum will be called **soft supermomenta**.

Remark: Soft supermomenta annihilate translations and vice versa,

$$\langle \text{soft supermomentum} \mid \text{translation} \rangle = 0$$

Physically, this implies that soft supermomentum states do not feel translations.

Soft supermomenta

Theorem (particular results of [Gelfand et al, 1966] revisited)
The Paneitz operator (conformal completion of Laplacian squared) on the celestial sphere

$$\bar{\partial}^2 \bar{\partial}^2 : \mathcal{E}[1]/\mathbb{R}^{3,1} \xrightarrow{\sim} \text{Ker}(\pi) : \mathcal{N}(z, \bar{z}) \mapsto \mathcal{P}(z, \bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z})$$

is an $SL(2, \mathbb{C})$ -equivariant isomorphism between the quotient $\mathcal{E}[1]/\mathbb{R}^{3,1}$ and the space of soft supermomenta:

$$\pi_\mu(\mathcal{P}) = 0, \quad \mathcal{P}(z, \bar{z}) \in \mathcal{E}[-3] \iff$$

$$\exists \mathcal{N}(z, \bar{z}) \in \mathcal{E}[1]/\mathbb{R}^{3,1}, \quad \text{such that} \quad \mathcal{P}(z, \bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z}).$$

In particular,

$$\partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z}) = 0, \quad \mathcal{N}(z, \bar{z}) \in \mathcal{E}[1] \iff$$

$$\exists T^\mu \in \mathbb{R}^{3,1}, \quad \text{such that} \quad \mathcal{N}(z, \bar{z}) = T^\mu q_\mu(z, \bar{z}).$$

Supermomenta: hard vs soft

There is no $SL(2, \mathbb{C})$ -equivariant linear injection of momenta into supermomenta.

Supermomenta: hard vs soft

There is no $SL(2, \mathbb{C})$ -equivariant linear injection of momenta into supermomenta. However,

Proposition (XB, L. Donnay, Y. Herfray)

A supermomentum

$$\mathcal{P}(z, \bar{z}) \in \mathcal{E}[-3]$$

is canonically equivalent to a pair

$$(p_\mu, \mathcal{N}(z, \bar{z})) \in (\mathbb{R}^{3,1})^* \times (\mathcal{E}[1]/\mathbb{R}^{3,1})$$

where $p_\mu = \pi_\mu(\mathcal{P}) \in (\mathbb{R}^{3,1})^*$ is the momentum and $\mathcal{N}(z, \bar{z}) \in \mathcal{E}[1]/\mathbb{R}^{3,1}$ is the soft charge of the supermomentum $\mathcal{P}(z, \bar{z})$.

The supermomentum decomposes uniquely into hard/soft parts as

$$\mathcal{P}(z, \bar{z}) = P(z, \bar{z}) + \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z})$$

Supermomenta: hard vs soft

Definition (soft)

The equivalence class $\mathcal{N}(z, \bar{z}) \in \mathcal{E}[1]/\mathbb{R}^{3,1}$ of weight-one densities was called the **soft charge of the supermomentum** $\mathcal{P}(z, \bar{z})$.

Definition (hard)

Hard supermomentum: supermomentum with vanishing soft charge.

Hard supermomenta: explicit definition

Definition (Explicit)

Hard supermomentum: element $P(z, \bar{z}) \in \mathcal{E}[1]'$ such that $p_\mu = \pi_\mu(P)$ and given by

- 1 $P^2 < 0$ (massive case) for any mass $m > 0$:

$$p_\mu(\vec{k}) = \pm m \left(-\sqrt{1 + \vec{k}^2}, \vec{k} \right), \quad P(z, \bar{z}) = \pm \frac{m}{\pi} \left(p^\mu(\vec{k}) q_\mu(z, \bar{z}) \right)^{-3}$$

- 2 $P^2 = 0$ (massless case) for any non-vanishing energy $\omega \neq 0$:

$$p_\mu = \omega \left(1 + |\zeta|^2, \zeta + \bar{\zeta}, i(\zeta - \bar{\zeta}), 1 - |\zeta|^2 \right), \quad P(z, \bar{z}) = \omega \delta^{(2)}(z - \zeta, \bar{z} - \bar{\zeta})$$

Hard supermomenta: explicit definition

The rationale behind this decomposition is the prescription:

Asymptotic states with hard charge $\leftrightarrow$ bdy data of usual particles

Accordingly, one reads off the hard charge from the explicit action of asymptotic symmetries at the boundary of Minkowski spacetime on boundary data of usual fields.

Example: Massless case

A supertranslation

$$u' = u + \mathcal{T}(z, \bar{z})$$

acts on supermomentum states as

$$e^{i\langle \mathcal{P}, \mathcal{T} \rangle} | \omega, \zeta, \bar{\zeta}; Q \rangle = e^{i\mathcal{T}(\zeta, \bar{\zeta})\omega} | \omega, \zeta, \bar{\zeta}; Q \rangle$$

for

$$p_\mu = \omega q_\mu(\zeta, \bar{\zeta}), \quad P(z, \bar{z}) = \omega \delta^{(2)}(z - \zeta, \bar{z} - \bar{\zeta}).$$

Hard/Soft decomposition: some subtleties

Summary (so far): $\exists$ Lorentz-invariant decomposition of supermomenta into sums

$$\mathcal{P}(z, \bar{z}) = P(z, \bar{z}) + \Sigma(z, \bar{z})$$

of a hard part $P(z, \bar{z})$ and a soft part $\Sigma(z, \bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z})$.

- However, this decomposition is *not* linear in the supermomentum.
 $\implies$ *The above decompositions are not stable under the addition of supermomenta.*

So the addition of two (or more) generic hard supermomenta, in general, includes a soft part.

$$\sum_i p_i^\mu = 0 \implies \sum_i P_i(z, \bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}(z, \bar{z}) \neq 0$$

- **Physical corollary:** If BMS group is a symmetry of the S-matrix, then one cannot truncate the spectrum of asymptotic states to hard supermomenta.

What are BMS particles?

What are BMS particles?

Faddeev-Kulish states?

(= hard particles surrounded by a cloud of soft particles)

[Kapec-Perry-Raclariu-Strominger, 2017]

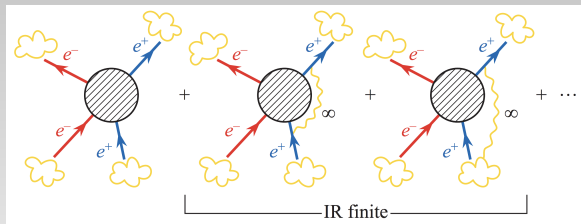


Fig. 14 in [Strominger, arXiv:1703.05448 [hep-th]]

UIRs of BMS group

What are BMS particles?

Agnostic answer: BMS particles are UIRs of BMS group.

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In turn, this raises the question:

What becomes of the usual (Poincaré) particles?

UIRs of BMS group

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Agnostic answer: BMS particles are UIRs of BMS group.

In turn, this raises the question:

What becomes of the usual (Poincaré) particles?

Agnostic answer:

Poincaré particles are UIRs of a Poincaré subgroup of BMS group.

UIRs of Poincaré subgroup: some subtleties

In usual QFT, the vacuum state is the unique state annihilated by the Poincaré group, i.e. it carries the trivial representation of the Poincaré group. In this sense, each Poincaré subgroup of BMS group defines a vacuum. This is also natural from the point of view of asymptotically flat gravity since:

Gravity vacua $\longleftrightarrow$ Poincaré subgroups of BMS group

Definition (Ashtekar, 1984)

Space of gravity vacua: homogeneous space

$$\mathbb{V} \simeq \frac{BMS_4}{ISO_0(3,1)} = \frac{SO_0(3,1) \ltimes \mathcal{E}[1]}{SO_0(3,1) \ltimes \mathbb{R}^{3,1}}.$$

It is an affine space modeled on the quotient vector space $\mathcal{E}[1]/\mathbb{R}^{3,1}$ whose elements are shifts of vacua.

UIRs of Poincaré subgroup: some subtleties

Mathematically, there are infinitely-many Poincaré subgroups (or gravity vacua).

Physically, the usual concept of particle is vacuum (or observer) dependent.

Remarks:

- This is usual in QFT on curved spacetimes, or even on Minkowski spacetime, e.g.

Unruh effect: the vacuum state appears as thermal state for a uniformly accelerated observer.

- The usual concept of particle is vacuum (or observer) dependent, also in asymptotically flat spacetimes (but in a different sense than in QFT on curved spacetime) due to

Memory effect: gravitational waves modify the vacuum at null infinity.

UIRs of Poincaré subgroup: some subtleties

The branching rules for the restriction of BMS group to any Poincaré subgroup imply the following feature.

For generic BMS representations, a usual particle is to be thought as an element of a BMS multiplet: in the presence of a soft charge, a state is tied up to a given gravity vacuum in the sense that, in any other vacuum, it will appear as a quantum superposition of particles with distinct spins.

Classification of unitary irreducible representations

Wigner's classification: a reminder

ANNALS OF MATHEMATICS
 Vol. 40, No. 1, January, 1939

ON UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP*

BY E. WIGNER

(Received December 22, 1937)

1. ORIGIN AND CHARACTERIZATION OF THE PROBLEM

It is perhaps the most fundamental principle of Quantum Mechanics that the system of states forms a *linear manifold*,¹ in which a unitary *scalar product* is defined.² The states are generally represented by wave functions³ in such a way that φ and constant multiples of φ represent the same physical state. It is possible, therefore, to normalize the wave function, i.e., to multiply it by a constant factor such that its scalar product with itself becomes 1. Then, only a constant factor of modulus 1, the so-called phase, will be left undetermined in the wave function. The linear character of the wave function is called the superposition principle. The square of the modulus of the unitary scalar product (ψ, φ) of two normalized wave functions ψ and φ is called the transition probability from the state ψ into φ , or conversely. This is supposed to give the probability that an experiment performed on a system in the state φ , to see whether or not the state is ψ , gives the result that it is ψ . If there are two or more different experiments to decide this (e.g., essentially the same experiment,

* Parts of the present paper were presented at the Pittsburgh Symposium on Group Theory and Quantum Mechanics. Cf. Bull. Amer. Math. Soc., 41, p. 306, 1935.

¹ The possibility of a future non linear character of the quantum mechanics must be admitted, of course. An indication in this direction is given by the theory of the positron, as developed by P. A. M. Dirac (Proc. Camb. Phil. Soc. 30, 150, 1934, cf. also W. Heisenberg, Zeits. f. Phys. 30, 209, 1934; 32, 623, 1934; W. Heisenberg and H. Euler, ibid. 38, 714, 1936 and R. Serber, Phys. Rev. 48, 49, 1935; 49, 545, 1936) which does not use wave functions and is a non linear theory.

² Cf. P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford 1935, Chapters I and II. Cf. also N. Bohr, *Mathematisk Grundlæggelse af Quantummekanik*, Berlin 1932, pages 1-10.

Wigner's classification: a reminder

Unitary irreducible representations of the Poincaré group

- The Poincaré group $ISO(3,1) = SO(3,1) \ltimes \mathbb{R}^{3,1}$ is the semidirect product of a semisimple Lie group acting on an Abelian normal subgroup.
- **Method of induced representation**
 - ➊ Consider the UIRs of the translation subalgebra $\mathbb{R}^{3,1}$:
 labelled by real eigenvalues (unitary & irreducible) of generators $\hat{P}_\mu$
 $\Rightarrow$ momenta p_μ
 - ➋ Identify the orbit and stabiliser of these eigenvalues
 $\Rightarrow$ “mass-shell” and “little group” ℓ_p
 - ➌ Induce the UIR of the Poincaré group from an UIR
 of the little group ℓ_p
 $\Rightarrow$ “spinning” degrees of freedom (or “physical components”)

The wave function of a particle has support on the orbit $\mathcal{O}_p$ of the momentum p_μ under the action of Lorentz group and takes values in the UIR of the little group ℓ_p .

Wigner's classification: a reminder

UIR	Orbit (as mass-shell)	Little group	Orbit (as coset)
Massive	2-sheeted hyp $P^2 = +m^2$	$SO(3)$	$\frac{SO(3,1)}{SO(3)} \cong H_3$
Massless	light-cone $P^2 = 0$	$ISO(2)$	$\frac{SO(3,1)}{ISO(2)} \cong \mathcal{I}_3$
Tachyonic	1-sheeted hyp $P^2 = +m^2$	$SO(2, 1)$	$\frac{SO(3,1)}{SO(2,1)} \cong dS_3$
Zero-momentum	origin $p_\mu = 0$	$SO(3, 1)$	point

Remark: The corresponding data sits at infinity.

UIR	Asymptotic data	Stabiliser	∞ as coset
Massive	temporal infinity	$SO(3)$	$\frac{SO(3,1)}{SO(3)} \cong H_3^\pm$
Massless	null infinity	$ISO(2)$	$\frac{SO(3,1)}{ISO(2)} \cong \mathcal{I}_3^\pm$
Tachyonic	spatial infinity	$SO(2, 1)$	$\frac{SO(3,1)}{SO(2,1)} \cong dS_3^0$

Asymptotic data

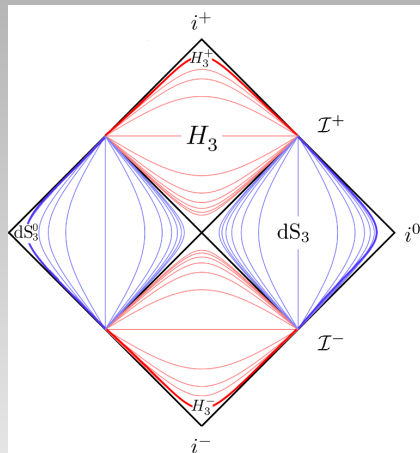


Fig. 1 in [A. Ball, E. Himwich, S. A. Narayanan, S. Pasterski and A. Strominger, arXiv:1905.09809 [hep-th]]

McCarthy's classification: a glimpse

Unitary irreducible representations of the BMS group

- The BMS group $BMS_4 = SO(3, 1) \ltimes \mathcal{E}[1]$ is the semidirect product of a semisimple Lie group acting on an Abelian normal subgroup.
- **Method of induced representation**
 - ➊ Consider the UIRs of the supertranslation subalgebra $\mathcal{E}[1]$:
 labelled by real eigenvalues of generators $\hat{T}(z, \bar{z})$
 $\Rightarrow$ supermomenta $\mathcal{P}(z, \bar{z})$
 - ➋ Identify the orbit and stabiliser of these eigenvalues
 $\Rightarrow$ BMS little group $\ell_{\mathcal{P}}$ ($\subset$ Poincaré little group $\ell_{\pi(\mathcal{P})}$)
 - ➌ Induce the UIR of the BMS group from an UIR of the BMS little group $\ell_{\mathcal{P}}$.

The wave function of a BMS particle has support on the orbit $\mathcal{O}_{\mathcal{P}}$ of the supermomentum $\mathcal{P}(z, \bar{z})$ under the action of the Lorentz group and takes values in the UIR of the BMS little group $\ell_{\mathcal{P}}$.

BMS shell

- The orbit $\mathcal{O}_{\mathcal{P}}$ of a supermomentum $\mathcal{P}$ is merely a coset $SO(3,1)/\ell_{\mathcal{P}}$, so all orbits are **finite-dimensional**.
- The **hard/soft distinction** is Lorentz-invariant, hence all supermomenta inside the orbit of a hard (resp. soft) supermomentum are hard (resp. soft).

Definition

Hard/Soft representation:

UIR of BMS group with hard/soft supermomenta.

BMS shell

- The orbit $\mathcal{O}_{\mathcal{P}}$ of a supermomentum $\mathcal{P}$ is a merely a coset $SO(3,1)/\ell_{\mathcal{P}}$, so all orbits are **finite-dimensional**.
- The **hard/soft distinction** is Lorentz-invariant, hence all supermomenta in the orbit of a hard (resp. soft) supermomentum are hard (resp. soft).
- **Hard momenta** $\mathcal{P}(z, \bar{z}) = P(z, \bar{z})$ are such that their BMS and Poincaré little groups coincide: $\ell_{\mathcal{P}} = \ell_{\pi(\mathcal{P})}$. Therefore, the orbits of hard supermomenta coincide with usual mass shells: $\mathcal{O}_{\mathcal{P}} \cong \mathcal{O}_P$.
- **Soft momenta** $\mathcal{P}(z, \bar{z}) = \partial^2 \bar{\partial}^2 \mathcal{N}(z, \bar{z})$ are such that their associated momentum is zero, $\pi(\mathcal{P}) = 0$, so their Poincaré little group is the whole Lorentz group. Therefore, the dimensions of the orbits of soft supermomenta range from 0 to 6.
- For **generic supermomenta** $\mathcal{P}(z, \bar{z}) = P(z, \bar{z}) + \Sigma(z, \bar{z})$:

$$\ell_{\mathcal{P}} = \ell_P \cap \ell_{\Sigma}$$

where $P(z, \bar{z})$ is the hard part and $\Sigma(z, \bar{z})$ is the soft part.

Hard representations of BMS

Correspondence principle: one-to-one correspondence between
hard UIRs of BMS $\longleftrightarrow$ non-zero momentum UIRs of Poincaré

In more physical terms:

Hard BMS particles = Poincaré particles (non-zero momentum)

Zero-momentum representations of Poincaré

By definition, the momentum associated to soft supermomenta is zero.

Definition (Lorentz as quotient)

Lorentz group: quotient group

$$SO(3,1) \cong ISO(3,1)/\mathbb{R}^{3,1}$$

of Poincaré group by normal subgroup $SO(3,1) \triangleleft ISO(3,1)$

Corollary: *zero-momentum representations of Poincaré group are unfaithful UIRs of Poincaré but they faithful UIRs of Lorentz group.*

Zero momentum UIRs of Poincaré = UIRs of Lorentz

Soft representations of BMS

Definition

Komar group: quotient group

$$BMS_4/\mathbb{R}^{3,1} \cong SO(3,1) \ltimes (\mathcal{E}[1]/\mathbb{R}^{3,1})$$

soft UIRs of BMS group (non-zero supermomentum)



(non-trivial) faithful UIRs of Komar group

Generic representations of BMS

Definition

Generic representations:

UIR of BMS group whose hard and soft parts of supermomenta are both non-vanishing ($P(z, \bar{z}) \neq 0$ and $\Sigma(z, \bar{z}) \neq 0$).

Classification of generic supermomenta via their little groups

Super momentum	BMS little group	BMS shell dimension	Momentum	Poincaré little group	Mass shell dimension
Generic	Trivial	6	Massive	$SO(3)$	3
Generic	Trivial	6	Massless	$ISO(2)$	3
Generic	Trivial	6	Tachyonic	$SO(2,1)$	3
Hard	$SO(3)$	3	Massive	$SO(3)$	3
Hard	$ISO(2)$	3	Massless	$ISO(2)$	3
Hard	$SO(2,1)$	3	Tachyonic	$SO(2,1)$	3

Remark: The possible dimensions of BMS shells for supermomenta with non-zero momenta range from 3 to 6.

Classification of generic supermomenta via their little groups

Super momentum	BMS little group	BMS shell dimension	Momentum	Poincaré little group	Mass shell dimension
Generic	Trivial	6	Massive	$SO(3)$	3
Generic	$SO(2)$	5	Massive	$SO(3)$	3
Hard	$SO(3)$	3	Massive	$SO(3)$	3

BMS little groups for *massive* momenta

Classification of generic supermomenta via their little groups

Super momentum	BMS little group	BMS shell dimension	Momentum	Poincaré little group	Mass shell dimension
Generic	Trivial	6	Massless	$ISO(2)$	3
Generic	$SO(2)$	5	Massless	$ISO(2)$	3
Generic	$\mathbb{R}$	5	Massless	$ISO(2)$	3
Generic	$\mathbb{R}^2$	4	Massless	$ISO(2)$	3
Generic	$ISO(2)$	3	Massless	$ISO(2)$	3
Hard	$ISO(2)$	3	Massless	$ISO(2)$	3

BMS little groups for *massless* momenta

Wavefunction of BMS particles

- Wave functions of BMS particles have support on the finite-dimensional orbit $\mathcal{O}_{\mathcal{P}}$ of a supermomentum $\mathcal{P}$ under the action of the Lorentz group.
- Via (infinite-dimensional) Fourier transform, one obtains wave functions depending on both positions **and** vacua.
 $\implies$ This allows to make contact with Strominger phase space (eigenstates of the “Goldstone boson” of “spontaneously broken” supertranslation invariance).

Take aways and future directions

Tawe aways

- Although BMS representation theory has been known since the 1970s, some physical interpretation of generic UIRs was somewhat missing and modern investigations on infrared issues in QFT provide a fresh source of inspiration for proposing one.
- Main lessons:
 - If the S-matrix is BMS-invariant, then supermomentum conservation suggests to take seriously generic BMS UIRs (i.e. with generic supermomenta)
 - Supermomenta can be split into hard and soft parts
This splitting is Lorentz-invariant but *not* linear (hence sums of hard supermomenta generically contain a soft part).
 - Wave functions of BMS particles admit various bases:
 - Eigenstates of supertranslation generators:
supermomentum basis (momentum & soft charge)
 - Eigenstates of “position operator” and “Goldstone mode”:
dual basis of supermomentum basis (position & gravity vacuum)
 - Mixed versions (e.g. momentum & gravity vacuum)

Some future directions

Tantalising goal: build a BMS-invariant QFT

- *Tentative program:*

Open Weinberg's book, replace everywhere "Poincaré" by "BMS"
and

See what happens...

- It suggests in particular to consider as Hilbert space of states, the Fock space of (suitably) symmetrised tensor products of one-particle states spanning BMS UIRs.
- *Some natural questions to try answering:*
 - 1 Does BMS-invariant QFT exist?
 - 2 And if it exists, does this BMS-invariant QFT:
 - 1 satisfy some minimal physical requirements, such as
Correspondence principle: $\text{BMS QFT} \xrightarrow{?} \text{Poincaré QFT}$
 e.g. generalised cluster decomposition principle?
 - 2 keep its promises? e.g.
 - IR-finite S-matrix?
 - Flat holography?